THE THEORY OF LINEAR OPERATORS

FROM THE STANDPOINT OF DIFFERENTIAL EQUATIONS OF INFINITE ORDER

By

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To Agnes,
who endured so patiently the writing of it,
this book is affectionately
dedicated.

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PREFACE

In writing the present volume the author has had in mind the preparation of an outline of the theory of linear operators and the applications of this theory to the study of general types of linear functional equations. He has sought to trace the development of the subject from its origin in the symbols of integral and differential calculus down to the present time when the invention and exploration of new operational symbolism has become one of the important methods of extending the doman of analysis. Throughout the book the author has attempted to indicate the many problems, in essentially all domains of science, which lead to operators of the type studied here. A number of typical examples of these applications have been given.

During the last half century we have witnessed an intensive activity in the study of types of functional equations. This activity was accelerated at the beginning of the present century by the remarkable discoveries of Volterra and Fredholm in integral equations and the concurrent investigations of Pincherle and others in the inversion of general types of linear operators. These problems, as soon as they were proposed, were seen to have much in common with the problem of solving infinite systems of linear equations in an infinite number of variables, a study which was initiated by Hill in his classical investigation of the motion of the lunar perigee and which was systematically developed by Poincaré, von Koch, and numerous others. These studies naturally led in turn to an extension of the domain of quadratic and bilinear forms, an extension which was first made by Hilbert and which in the last quarter of a century has been the source of numerous novel applications. This sequence of ideas culminated in the theory of spectra, an exposition of which is given in the concluding chapter of the present work.

As the author now looks back upon the arduous task of assembling so many divergent theories and of coordinating them with one another, he is fully aware of the many omissions of which he is guilty. It would require a far larger volume than the present one to do full justice to this noble subject. The theories of ordinary linear differential equations, partial linear differential equations, linear difference equations, and integral equations are themselves sufficiently extensive to fill a number of volumes. However, there is much to be gained in surveying an expansive terrain, and although one misses many of the attractive details of the picture, the large prominences stand more clearly before the eye.

In attempting a survey of the theory of linear operators it is necessary to adopt some general point of view. In the present instance this coordinating principle has been found in the theory of differential equations of infinite order. Others have adopted the point of view of a general formulation of Fourier integrals or that of linear systems in infinitely many variables. While both of these aspects of the general problem have been presented in the present volume, the development in the main has centered around the generalized differential operator. As one may readily apprehend, however, these various aspects of the problem of linear operators have an intimate relation with one another; they may be regarded as the facets of a gem which has been cut in a many sided pattern.

The theory of differential equations of infinite order has long fascinated the author because of its unifying power on the one hand and its easy translation of specific problems into domains that have been extensively explored on the other. One may feel, perhaps, that this theory imposes an unusual restriction in the tacit assumption that the functions to which it may be applied are infinitely differentiable. This restriction, however, is more apparent than real, since one may regard an operator of the form $(1 - e^{-xz})/z$, $z = d/dx$, as equivalent to the operator $1/z$, where $1/z$ has the full generality of integration. It is in this sense that the formal aspects of so broad a discipline as that of integral equations may be brought within the scope of this theory.

While the present volume was going through proof a survey of differential equations of infinite order was published by Dean R. D. Carmichael of the University of Illinois [see *Bibliography*: Carmichael (15)], in which he characterized the theory as a "relatively unexplored domain, the importance of which will certainly be more fully recognized as the subject is further developed in the next two or three decades". If the present volume can help to stimulate this development by setting forth the present status of the theory of these equations, by tracing the connection between them and other more thoroughly explored types of equations, and perhaps by indicating the most promising fields of investigation, then the aims of the author will have been fully achieved.

The author owes a heavy debt to a number of people in the preparation of the manuscript of the present volume. Professor J. D. Tamarkin of Brown University has examined part of the proof and his suggestions have been very helpful. Dean R. D. Carmichael furnished the author with a bibliography of differential equations of infinite order, and another on the related subject of Appell polynomials. These were very useful in rounding out the author's original compilation

and through their help the bibliography at the end of this work should be reasonably complete as it pertains to this particular phase of the subject of linear operators. Professor R. C. Archibald of Brown University, Professor C. H. Sisam of Colorado College, and Professor C. F. Roos of Colorado College and the Cowles Commission for Research in Economics were consulted on several technical points in the manuscript. Dr. M. M. Flood of Princeton University read chapter 3 with some care and his suggestions were very valuable.

During the preparation of the manuscript the author had the good fortune to see a photostat copy of a manual on linear operators, mainly from the formal point of view, prepared by Professor E. Stephens of Washington University. The latter's extensive knowledge of the historical sources, particularly as they relate to the formal development which took place in England around 1850, was of much help in various parts of the present work.

From his colleague Professor K. P. Williams the author has received constant encouragement in the preparation of the present volume and considerable technical advice at certain points. His colleague Professor J. R. Kantor has maintained a zealous interest in the progress of the work. To the many students who have taken an active part in the development of this volume and who have given invaluable help in many places the author wishes to express his real appreciation.

One should also not be unmindful of the inspiration which led to the inception of the present undertaking. Courses taken at Harvard University under Dean G. D. Birkoff of Harvard and Professor I. A. Barnett of the University of Cincinnati first introduced the general theory to the author's attention. This interest was later encouraged by Professor E. B. Van Vleck at the University of Wisconsin. The latter's stimulating lectures on the classical problems of linear differential equations, both ordinary and partial, have furnished continued inspiration and knowledge.

The author also owes a debt of gratitude to the Waterman Institute of Indiana University and to Dean Fernandus Payne, its director, and to President W. L. Bryan and the trustees, a debt which he now hopes partially to discharge through the publication of the present work. During the years 1927 to 1931 he held a fellowship in the Institute which greatly reduced his teaching load over this period. The last year was spent in research study at both Harvard and Princeton Universities. Throughout these years and later Dean Payne has been a constant source of inspiration and this book could scarcely have been brought to its completion without his kindly advice and encouragement.

Finally, the author wishes to express his perennial debt to the members of the staff of the Dentan Printing Company, who gave lavishly of their time to make the difficult typography of this work conform to the highest printing standards.

Indiana University, 1936. H. T. DAVIS.

ERRATA

Page 7, line 31. Read *fonctionnelles* for *fonctionelles*.

Page 54, line 11. Absolute value signs are to be understood around $f(x)$ and x^k.

Page 70. Replace sentence in lines 27 and 28 by the following: "But if μ is a fraction, the coefficient of A_n becomes zero only if n is a negative integer."

Page 95, line 30. Read "principle" for "principal."

Page 130, line 27. Change subscript of x from i to k.

Page 263, line 19. Change "Weiner" to "Wiener".

CHAPTER I

Linear Operators

1. The Nature of Operators. A few years ago it was the fashion in mathematical physics to seek mechanical explanations of natural phenomena. "I never satisfy myself until I can make a mechanical model of a thing," said Lord Kelvin. "If I can make a mechanical model, I can understand it. As long as I cannot make a mechanical model all the way through, I cannot understand it." The history of the physics of the nineteenth century is bound up with the history of the light-bearing ether, an invention designed to give a mechanical picture of the transfer of electro-magnetic radiation through space.

The mysterious behaviour of light and electrons, which in some experiments behave as discrete entities, and in others as undulations, has given a powerful weapon into the hands of the opponents of mechanistic philosophy. Models seem to fail in attempts to explain nature by the epistemology of Lord Kelvin. Sir James Jeans says of the situation: "We have already seen that radiation cannot be adequately portrayed either as waves or as particles, or in terms of anything that we can imagine, and we shall soon find that the same is true also of matter."* As a more general thesis, which applies to all phenomena, this philosopher avers: "*A priori,* as we have seen, there are very great odds against our being able to form any kind of visual picture of the fundamental processes of nature."†

What, then, shall be the approach to knowledge? What dictum of epistemology shall we oppose to the mechanism of Lord Kelvin? Perhaps the clearest statement of a position acceptable to modern physical philosophy is found in *The Logic of Modern Physics,* by P. W. Bridgman. "In general, we mean by any concept nothing more than a set of operations," says the author; "*the concept is synonymous with the corresponding set of operations.* If the concept is physical, as of length, the operations are actually physical operations, namely, those by which length is measured; or if the concept is mental, as of mathematical continuity, the operations are mental operations, namely those by which we determine whether a given aggregate of magnitudes is continuous."

If this shall be the ultimate refuge of those who seek to avoid the dilemmas of modern science, then the *operator* has become a

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**The New Background of Science*, New York (1933), viii + 301 p.; in particular p. 65.

†*Ibid.*, p. 171.

supreme tool in all exploration. Knowledge is confined entirely to what we know. Data obtained by the processes of measurement, numbers constructed by definite algorithms, are the basis of knowledge. *Pi,* as the idealization of a limiting process, is forever beyond our reach, but, as the 707-place approximation attained by Shanks, it is within the range of our knowledge.

It does not seem to be fully apprehended by writers on natural philosophy that this operational aspect of the problem of knowledge has confronted mathematicians for a long time. The paradoxes associated with the mathematical continuum antedate by some years the paradoxes of the electron and the photon. In the contemplation of an aggregate of points everywhere dense and non-denumerable, the mathematician was regarded by many as a metaphysician waging war with phantoms. The fact that $\sqrt{2}$ was not considered to be a number by L. Kronecker (1823-1891), seemed to argue to the mechanistic physicist that something was wrong with the definition of a point. Even the solution of R. Dedekind (1831-1916), that the irrational number was not a thing, but a "cut" defined by an operational process, did not seem a matter worthy of consideration.

This point of view merits further elucidation. Kronecker was unwilling to give any existential meaning to irrational numbers, as one gathers from his remark to F. Lindemann, the first to prove the transcendental character of π: "Of what use is your beautiful research on the number π? Why cogitate over such problems, when really there are no irrational numbers whatever?"* Opposed to this view is the general procedure of Dedekind. Affirming the position that "numbers are free creations of the human mind," he focused his attention upon the rational points, m/n, where m and n are integers, $m \leq n$, in the interval between 0 and 1. Then it is to be seen that the total continuum of points is divided into two classes by any one, x, of these rational points, the number x being assigned at pleasure to either class. We thus have the two classes of points, X_1 and X_2, such that all the points in X_1, lie to the right of x and the points in X_2 lie to the left of (and perhaps include) the point x. We have thus obtained what is called a *Dedekind cut* (X_1, X_2). Now in order to define an irrational number such, for example, as $\sqrt{2} - 1$, Dedekind affirmed his *principle of continuity*:

"If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."

*F. Cajori: *A History of Mathematics*, 2nd ed. (1929), p. 362.

Our purpose in introducing this well-known concept here is to show that the "existence" of the irrational number which is thus attained rests essentially upon an operational process. The cut is *constructed* by an algorithm and the unique irrational number at the end of the ever-narrowing tunnel of approximations over the rational numbers owes its existence to the underlying operation.

2. Definition of an Operator. Before proceeding further in our discussion let us define, if possible, what we shall mean by an operator.

The existence of an operator implies the existence of a *law of transformation* by means of which one or more members of a class of objects in a given domain of definition are transformed unambiguously into one or more members of a second set of objects of the same or of a different class in the same or in a different domain. To reduce this to symbols let us denote by A a class of objects, by a a subset of this class, by S an operator belonging to A, by R the class of objects obtained when S operates upon the class A, and by r the specific subject of R which corresponds to the transformed subject a. Thus we shall have

$$S \rightarrow a = r.$$

This definition is seen to be one of very wide application since it embraces most if not all phenomena, both physical and mental, which contain the element of change or transformation. The operational character of both $+$ and $\times$ is easily revealed by writing

$$\Sigma x_i \ , \quad \Pi x_i \ ,$$

in which the symbols Σ and Π, denoting *summation* and *product* respectively, are the operators upon the class $\{x_i\}$. We should note that these operators are not commutative since, in general,

$$\Sigma_j (\Pi_i x_{ij}) \neq \Pi_i (\Sigma_j x_{ij}) \ .$$

The operational character of addition and multiplication can be exhibited in an interesting manner in algebras different from the common one. Let us take as an example Boolian logic in which the region of the definition of the operands is between 0, the null class, and 1, the universe. If we designate by a and b two members of the class, then the sum,

$$a + b \ ,$$

denotes the class which includes the elements of both a and b and which is contained in all other classes which contain both of them. Also the product,

denotes the class which contains the elements common to both a and b and only these elements.

This example was suggested by a beautiful application made by D. N. Lehmer in his study of methods for factoring prime numbers. Briefly stated, Lehmer desired to construct a set of stencils, each with 50 columns and 100 rows to correspond to the first 5000 prime numbers beginning with 1 and ending with 48,593, which are listed on the first page of his *List of Prime Numbers*.* Each stencil was designed to show the numbers among the 5000 primes which had a given R (the number of the stencil) as a quadratic residue. We recall the definition that R is a quadratic residue of N, where N is prime to R, provided the congruence,

$$X^2 \equiv R \bmod N \ ,$$

has an integral solution. Each stencil was punched with holes corresponding to every prime number that had the number of the stencil for a quadratic residue. This heroic task was undertaken for residues from —238 to +238. Fortunately, it was discovered that a stencil could be cut for composite residues from the stencils for prime residues by means of a simple application of Boolian algebra.

A stencil conjugate to A and designated by A' was defined to be one which contained all the holes not cut in A. That is to say,

$$A + A' = 1.$$

Also the sum of two stencils A and B was defined to be that stencil, C, which contained all the holes found in both A and B; i.e.,

$$C = A + B \ .$$

In similar manner the product of two stencils, A and B, was the stencil D, which contained only the holes common to both A and B; that is to say,

$$D = A\,B \ .$$

From these definitions we derive at once the familiar equations

$$AA' = 0, \quad AA = A^2 = A \ .$$

Lehmer then combined these ideas with the following theorem:

If A and B are residues common to a prime number p, then the product residue $A \times B$ will also be a residue of the prime; moreover if neither A nor B is a residue of a prime number p, then the product $A \times B$ will be a residue of p.

It is at once evident that the Boolian product, $A\,B$, will contain all primes in A and B which have the numerical product $A \times B$ as a

Carnegie Inst. Wash. Publication No 165 (1914).

residue. Similarly the Boolian product, $A'B'$, will contain the remainder of the primes which have the numerical product $A \times B$ as a residue. If we designate the holes in the final product stencil by $(A\ B)$, it is clear that we can write

$$(A\ B) = A\ B + A'B' \ .$$

In this manner the addition and multiplication processes have been translated directly into the *mechanical operation* of superimposing one stencil upon another.

We have indicated in the preceding discussion how intimately mechanical operations are connected with their mental constructs. The theory which we propose to develop in the following pages is designed to exhibit this useful dualism, which lies at the heart of all applications of mathematics to the interpretation of natural phenomena. We are therefore endeavoring to paint upon a large canvass, and the details which delight us in smaller and more perfect pictures must be left in many instances to the imagination of the reader.

We shall proceed by defining three classes of things which are fundamental in the concept of operations.

First: There must be a class of objects upon which to operate. These objects may be numbers as in the case of arithmetic, integers and rational fractions as in the theory of numbers, functions as in analysis, combinations as in the theory of groups, biological organisms as in psychology, matter and energy as in physics, etc.

Special subclasses of the class of all things constitute the subject matter of the different fields of science, but this subject matter is naturally not a complete delineation of a science. The behavior of human beings is fundamental in such different domains as physiology, psychology, biology, and economics, but the attributes of this behavior which are isolated for particular investigation by each of these fields characterize a narrower delineation. To put the matter more precisely, we specialize our sciences in terms of the operations which we propose to apply to the objects of our study. We thus reach the conclusion:

Second: We must define in some explicit manner the operations that are to be performed upon the objects of the class. For example, we have addition and multiplication in arithmetic, differentiation and integration in calculus, substitutions in the theory of groups, stimuli in psychology, the mixing of compounds in chemistry, etc.

But the operation and the subject of the operation are not the entire story. That which comes out of the test tube, the results of the experiment, the responses to the stimuli, the sum obtained by addition, the function which emerges from an integration, are the rewards

sought by those who employ the operators. These results constitute the novelties of each science and we may thus say:

Third: We must investigate the nature of the final element; that is to say, the result after the operator has been applied to the objects of the original class. These results, of course, may or may not belong to the original class of objects.

If it happens that the result of an operation belongs to the original class of objects, we may operate again and hence obtain a result of second order. Symbolically this may be described as follows:

Let us designate the operator by S, an object of a class A by a, and the result of the operation by r_1, which we shall assume is a member of the class A. Hence we get

$$r_1 = S \to a, \qquad r_2 = S \to (S \to a) = S^2 \to a \ .$$

If r_2 now belongs to the class A, we may again repeat the operation to define r_3, and thus continue the process so long as the r_i remain within the operational class.

In this way we arrive very naturally at the symbol S^n, which is known as the *power of an operation.*

If a class of objects has two operators belonging to it — for example, S and T — and if $r_1 = S \to a$ belongs to A, then

$$T \to (S \to a)$$

is known as the product of S by T and may be denoted symbolically by TS. Similarly, if $R_1 = T \to a$ also belongs to A, then $S \to (T \to a)$ is the product of T by S.

If it happens that

$$TS = ST$$

for all members of the class A, the multiplication is called *commutative*, and if for a third operator Q we have

$$T(SQ) = (TS)Q \ ,$$

the multiplication is *associative*.

This last property leads at once to the *index law*,

$$S^m S^n = S^{m+n}$$

for positive integral exponents greater than one. That it is not always necessary to apply the last restriction will become apparent from the examples of later sections.

The operators with which we shall be particularly concerned in this book are those which satisfy the two conditions

$$S \rightarrow (a+b) = S \rightarrow a + S \rightarrow b$$

$$S \rightarrow ka = kS \rightarrow a \ ,$$

where k is any quantity which belongs to what is called a *scalar* class, and a and b are members of the class A. In the field of functions and functional operators the scalar class comprises all complex numbers.

Operators which satisfy the two conditions stated above are said to be *linear*, or *distributive.*

3. A Classification of Operational Methods. Surveying the field of operational methods from the heights attained by modern analysis, one might perhaps classify the general theory and its historical development into five main divisions.

The first may be called the *formal theory of operators,* which, beginning with G. W. Leibnitz (1646-1716) and J. Lagrange (1736-1813), was largely developed in England under the stimulus of George Boole (1815-1864), R. Murphy (1806-1843), R. Carmichael (1828-1861), George Peacock (1791-1858), D. F. Gregory (1813-1844), A. De Morgan (1806-1871), and numerous others.

The second is the theory of the *generatrix calculus,* which was created by P. S. Laplace (1749-1827) and enshrined in immortality by being made the principal method of his *Théorie analytique des probabilités.* The first edition of this great treatise appeared in 1812.

Strangely enough, the third movement in the theory of operators was initiated by important researches in electrical communication. The protagonist of this dramatic story was Oliver Heaviside (1850-1925), a self-taught scientist, scorned by the mathematicians of his day, who saw only yawning chasms of unrigor behind his magic formulas. These methods which have proved so useful to engineers are now collected under the name of the *Heaviside operational calculus.*

To the theory of integral equations we are indebted for the fourth division of our subject, which we shall call generically the problem of *functionals* (*fonctionelles*). The modern theory of integral equations was initiated almost simultaneously by E. I. Fredholm (1866-1927), a native of Stockholm, and Vito Volterra of Rome whose fundamental and searching papers set the mathematical world to the development of one of its richest fields. The first work of Volterra was published in 1896 and that of Fredholm four years later.*
Volterra subsequently showed that integral equations are included in

*More accurately, Volterra had considered the problem of integral equations in a paper on electrostatics published in 1884. His ideas did not reach maturity before 1896. For a more extended account of the history of integral equations the reader is referred to a report prepared by H. Bateman for the British Association for the Advancement of Science in 1910 and to a study by the author: *The Present Status of Integral Equations.* Indiana University Studies, No. 70 (1926).

a broader discipline where functions are characterized by their dependence upon other functions, a discipline which he developed under the engaging title of the theory of *functions of lines*. This has led in the sequal to what is now commonly referred to as the *theory of functionals*, a calculus of broad generality which is, however, included in the general calculus of operators.

The fifth phase of the theory of operations depends also for its origin upon the theory of integral equations and may be referred to as the *calculus of forms in infinitely many variables*. This broad field of modern study originated with David Hilbert and E. Schmidt, who in a series of classic memoirs reviewed the results of Fredholm's inversion of definite integrals from the standpoint of infinite matrices and the theory of elementary divisors. Recent investigations into the mysteries of quantum mechanics and the properties of electrons have given new impetus to the study of this type of operator and have led to the formulation of the *matrix calculus*. The theory must be regarded as a highly important union of a special case of the calculus of forms with physical ideas.

4. The Formal Theory of Operators. The history of operators begins most properly with some observations made by G. W. Leibnitz (1646-1716) in which he noticed certain striking analogies between algebraic laws and the behavior of differential and integral operators. One of these analogies he formulated in what is now known in mathematical literature as the *rule of Leibnitz*, which states that the nth derivative of the product $u(x)\ v(x)$ can be expressed by the symbolic binomial expansion $[u(x)+v(x)]^{(n)}$. This formula is found in *Symbolismus memorabilis Calculi Algebraici etc.* published in 1790. (See *Bibliography.*) In a letter written to G. F. A. l'Hospital (1661-1704), September 30, 1695,* Leibnitz commented upon the algebraic analogy and expressed the symbol $\int^n$ as d^{-n}. He then continued with the following prophetic observations:

"Vous voyés par là, Monsieur, qu'on peut exprimer par une serie infinie une grandeur comme $d^{\frac{1}{2}}\overline{xy}$ ou $d^{1:2}\overline{xy}$, quoyque cela paroisse eloigné de la Geometrie, qui ne connoist ordinairement que les differences à exposans entiers affirmatifs, ou les negatifs à l'égard des sommes, et pas encor celles, dont les exposans sont rompus. Il est vray, qu'il s'agit encor de donner $d^{1:2}x$ pro illa serie; mais encor cella se peut expliquer en quelque façon. Car soyent les ordonnées x en progression Géometrique en sort que prenant une constante $d\beta$ soit $dx = x\,d\beta:a$, ou (prenant a pour l'unité) $dx = x\,d\beta$, alors $d\,dx$ sera $x\,\overline{d\beta}^2$, et d^3x sera $= x\,\overline{d\beta}^3$ etc. et $d^ex = x\,d\beta^e$. Et par cette addresse l'exposant differentiel est changé en exposant potentiel et remetant

*See *Leibnizen's Mathematische Schriften*, vol. 2 (1850), pp. 301-302.

$dx:x$ pour $d\beta$, il y aura $d^e x = \overline{dx:x}^e \cdot e$. Ainsi il s'ensuit que $d^{1:2} x$ sera egal à $x\sqrt[2]{dx:x}$. Il y a de l'apparence qu'on tirera un jour des consequences bien utiles de ces paradoxes, car il n'y a quères de paradoxes sans utilité. Vous estes de ceux qui peuvent aller le plus loin dans les decouvertes, et je seray bientost obligé *ad lampadem aliis tradendam* (to surrender the torch to others). Je voudrois avoir beaucoup à communiquer, car ce vers: *Scire tuum nihil est nisi te scire hoc sciat alter,* (unless another knows what you know, you know nothing) est le plus vray en ce que des pensées qui estoient peu de chose en elles mêmes peuvent donner occasion à des bien plus belles."

In another letter, written to J. Bernoulli in 1695, we find Leibnitz affirming:

"There are yet many things latent in these progressions of summation and differentiation, which will gradually appear. There is thus notably agreement between the numerical powers of binomial and differential expansions; and I believe that I do not know what is hidden there."*

And in reply Bernoulli concurred: "Nothing is more elegant than the agreement which you have observed between the numerical power of the binomial and differential expansions; there is no doubt that something is hidden there."†

We may pause a moment to comment on the curious fact that this same air of mystery has enveloped the subject of operational methods down to the present time. Thus we find the following statement made by George Boole in the preface to his *Differential Equations* published in 1859:

"This question of the true value and proper place of symbolical methods is undoubtedly of great importance. Their convenient simplicity — their condensed power — must ever constitute their first claim upon attention. I believe, however, that in order to form a just estimate, we must consider them in another aspect; viz., *as in some sort the visible manifestation of truths relating to the intimate and vital connection of language with thought — truths of which it may be presumed that we do not yet see the entire scheme and connection.* But while this consideration vindicates to them a high position, it seems to me clearly to define that position. As discussions about words can never remove the difficulties that exist in things, so no skill in the use of those aids to thought which language furnishes can relieve us from the necessity of a prior and more direct study of the things which are the subject of our reasonings. And the more exact and the

*Multa adhuc in istis summarum & differentiarum progressionibus latent, quae paulatim prodibunt. Ita notabilis est consensus inter numeros postestam a binomio, & differentiarum rectanguli; — puto nescio quid arcani subesse.

†Nihil elegantius est quam consensus quem observasti inter numeros potestam a binomio — differentiarum rectangulo; haud dubie aliquid arcani subest.

more complete the study of things has been, the more likely shall we be to employ with advantage all instrumental aids and appliances."

The secret of this mystery resides, perhaps, in the efficacy of analogy and generalization. For example, an equation like that of Laplace,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 ,$$

is found to be applicable to the *a priori* science of analytic functions and to the *experimental* sciences of electricity, magnetism, heat conduction, and the flow of underground waters. We shall find it appearing sooner or later in the theory of the elasticities of economic variables and in other fields remote from its mathematical origin. What strange analogy is here? What hidden root leads to so many branches?

Laplace himself begins his immortal treatise on probabilities with these words:

"Magnitudes considered in general are commonly expressed by the letters of the alphabet, and it is to Vieta that is due this convenient notation which transfers to analytical language, the alphabets of known tongues. The application which Vieta made of this notation, to geometry, to the theory of equations, and to angular sections, forms one of the remarkable epochs of the history of mathematics. Some very simple signs express the correlations of magnitudes. The position of one magnitude following another is sufficient to express their product. If these magnitudes are the same, this product is the square or the second power of this magnitude. But in place of writing it twice, Descartes conceived of writing it only once, giving it the number 2 as an exponent; and he expressed successive powers by increasing successively this exponent by unity. This notation, considering it only as an abridged way of representing powers, seemed scarcely anything; but such is the advantage of a well constructed language that these most simple notations have often become the source of most profound theories; it is this which happened for the exponents of Descartes. Wallis, who had set himself specially to follow the thread of induction and analogy, has been led by this means to express radical powers by fractional exponents; and just as Descartes expressed by the exponents 2, 3, etc., the second, third, etc., powers of a magnitude, he represents square roots, cube roots, etc., by the fractional exponents 1/2, 1/3, etc. In general he expresses by the exponent m/n the nth root of a magnitude raised to the power m. In fact, following the notation of Descartes, this expression holds in the case where m is divisible by n; and Wallis, by analogy, extends it to every case."

Here again we find *analogy* the open sesame to useful and almost mysterious generalization. The curious reader who wishes to pursue

this inquiry further will find an illuminating discussion held between H. S. Carslaw, H. Jeffreys, and T. J. I'a. Bromwich in vol. 14 (1928-29) of the *Mathematical Gazette*, pp. 216-228. The point of controversy was the use of the Heaviside operational methods in mathematical physics. Dr. Carslaw was arguing for the use of classical contour integration as the principal tool to be employed in the solution of the differential equations of mathematical physics. "There is no room for mystery in mathematics. If we can be clear, let us be so. And for my part I consider the best way of attacking many of these questions is to use contour integrals."

The challenge of the mysterious element is not fully accepted by either Jeffreys or Bromwich, who seem to contend that the heuristic methods of Heaviside still demand verification after each application. Bromwich makes the following interesting admission: "It is true that in 1914 (when the work was more or less completed for my paper published in 1916) I felt that I had only *'established an analogy.'* But my experiences during the subsequent fourteen years would justify me in stating that Heaviside's method has never led me astray — except in that kind of mistake which is due to human fallibility."

Leaving these esoteric thoughts, which the reader, however, may find interesting to keep in mind as he follows the future development of the subject in this book, let us return to the historical narrative.

The torch laid down by Leibnitz came into the hands of J. Lagrange (1736-1813), who in a notable memoir (see *Bibliography*) greatly accelerated the study of operational methods. In the introduction to his paper he makes the following remarks:

"Mais ni lui (he refers to Leibnitz) ni aucun autre que je sache n'a poussé plus loin ces sortes de recherches, si l' on en excepte seulement M. Jean Bernoulli, qui, dans la Lettre XIV du *Commercium epistolicum*, a montré comment on pouvait dans certains cas trouver l' intégrale d'une différentielle donnée en cherchant la troisième proportionnelle à la différence de la quantité donnée et à cette même quantité, et changeant ensuite les puissances positives en différences, et les négatives en sommes ou intégrales. Quoique le principe de cette analogie entre les puissances positives et les différentielles, et les puissances négatives et les intégrales, ne soit pas évident par lui-même, cependant, comme les conclusions qu'on en tire n'en sont pas moins exactes, ainsi qu'on peut s'en convaincre *à posteriori*, je vais en faire usage dans ce Mémorie pour découvrir différents Théorèmes généraux concernant les différentiations et les intégrations des fonctions de pleusieurs variables, Théorèmes dont la plupart sont nouveaux, et auxquels il serait d'ailleurs très-difficile de parvenir par d' autres voies."

In this paper Lagrange demonstrated the operational validity of the two formulas

$$\Delta^\lambda u = \left(e^{\frac{\partial u}{\partial x}\xi + \frac{\partial u}{\partial y}\eta + \frac{\partial u}{\partial z}\zeta} - 1\right)^\lambda ,$$

where

$$\Delta u = u(x+\xi, y+\eta, z+\zeta) - u(x,y,z), \quad \Delta^\lambda u = \Delta\, \Delta^{\lambda-1} u, \quad \text{and}$$

$$\Sigma^\lambda u = 1/\left(e^{\frac{\partial u}{\partial x}\xi + \frac{\partial u}{\partial y}\eta + \frac{\partial u}{\partial z}\zeta} - 1\right)^\lambda .$$

Under the hands of his successors, these equivalents were to form the fundamental structure for the *calculus of finite differences.*

The formal theory of operators was mainly concerned with three problems: (a) the interpretation of symbols, particularly the formal inverses of well known operators; (b) the interpretation of symbolic products; (c) the problem of the factorization of operators.

We may cite as an instance of the first problem the interrelationship of the symbols Δ, E and $D = d/dx$, the first two defined by the equations

$$\Delta\, u(x) = u(x+1) - u(x) ,$$

$$Eu(x) = u(x+1) = (1+\Delta)u(x) = e^D u(x) .$$

When the formal expansions,

$$\begin{aligned} \Delta^n u(x) &= (E-1)^n u(x) = [E^n - {}_nC_1 E^{n-1} + {}_nC_2 E^{n-2} \\ &\qquad - \cdots + (-1)^n] u(x) \\ &= u(x+n) - {}_nC_1 u(x+n-1) + {}_nC_2 u(x+n-2) \\ &\qquad - \cdots + (-1)^n u(x) , \end{aligned}$$

$$\begin{aligned} E^n u(x) &= (1+\Delta)^n u(x) = u(x+n) \\ &= u(x) + n\,\Delta\, u(x) + \frac{n(n-1)}{2!}\Delta^2 u(x) + \cdots , \end{aligned}$$

were found to yield correct numerical results, it was natural to inquire how far the algebraic analogy might be carried. The symbol $\Delta^{-1}u(x)$ was interpreted as a summation and hence the formal expansion

$$\begin{aligned} \Delta^{-1} u(x) &= \frac{1}{E-1} u(x) = \frac{1}{e^D - 1} u(x) \\ &= (1/D - \tfrac{1}{2} + B_1 D/2! - B_2 D^3/4! + B_3 D^5/6! \\ &\qquad - \cdots)\, u(x) \end{aligned}$$

where $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, etc. are the Bernoulli num-

bers, should give a formula for the summation of series by integration and differentiation. The success of this interpretation led to intensive exploration and many ingenious variants were discovered. The well known Euler-Maclaurin formula in its various forms is easily derived by this means.*

As an example of the second problem we may cite the numerous functions of D which were examined by the investigators of the early part of the nineteenth century. A typical investigation is the inquiry into the meaning to be attached to the symbol.

$$
\begin{aligned}
f(x) &= e^{g(x)D} \to u(x) \ , \\
&= [1 + (gD) + (gD)^{(2)}/2! + (gD)^{(3)}/3! + \cdots] \to u(x) \ .
\end{aligned}
$$

This problem was studied by C. Graves [See *Bibliography*: Graves (3)] and generalized to n variables by Robert Carmichael in *The Calculus of Operations* (1855). The reader will find it instructive to verify the interpretation,

$$f(x) = u\{G^{-1}[G(x) + 1]\} \ ,$$

where $G(x) = \int [1/g(x)]dx$, and $G^{-1}(x)$ is the inverse of $G(x)$. Thus we get for $g(x) = 1$, $G(x) = x$, $f(x) = u(x+1)$; $g(x) = x$, $f(x) = u(ex)$; $g(x) = x^2$, $f(x) = u\{x/(1-x)\}$; $g(x) = x^3$, $f(x) = u\{x/(1-2x^2)^{\frac{1}{2}}\}$, and in general, for $g(x) = x^m$,

$$f(x) = u\{x/[1-(m-1)x^{m-1}]^{1/(m-1)}\} \ .$$

In more recent mathematics a similar interest in the formal interpretation of symbolic processes has been aroused through the demands of the cabalistic equation,

$$pq - qp = I \ ,$$

which has developed around the postulates of the quantum theory. In several instructive papers, N. H. McCoy (see *Bibliography*) has examined the meaning of functions of p and q, where the operators p and q obey the non-commutative law stated above.

This operational equation was apparently first studied by Charles Graves as early as 1857. Interest was revived in it when W. Heisenberg introduced it into the theory of quantum mechanics. P. A. M. Dirac, in his desire to preserve the formal features of classical mechanics in the theory of quanta, exhibited an essential analogy between non-commutative operators and the Poisson bracket symbols

*Consult Boole: *Bibliography*, Boole (11); Whittaker and Robinson: *The Calculus of Observations* (see *Bibliography*); H. T. Davis: *Tables of the Higher Mathematical Functions*, vol. 2, Bloomington (1935), section on Bernoulli numbers.

of ordinary dynamics. In a trilogy of related monographs published in 1931 E. T. Whittaker, W. O. Kermack, and W. H. McCrea (see *Bibliography*) have investigated the significance of such operators from the point of view of contact transformations and have derived numerous solutions of differential equations associated with such transformations.

The third problem mentioned above was essentially that of solving linear differential and difference equations by reducing the operators to linear factors. For example, the equation,

$$[z^2 + a(x)z + b(x)] \rightarrow u(x) = f(x) \quad ,$$

is formally equivalent to the equation,

$$[z + a_1(x)] \rightarrow [z + a_2(x)] \rightarrow u(x) = f(x) \quad ,$$

where

$$a(x) = a_1(x) + a_2(x), \; b(x) = a_1(x)a_2(x) + a_2'(x) \quad , \quad z = d/dx \, .$$

The principal contribution made by G. Boole, Robert Carmichael, D. F. Gregory and others who studied this problem in a large number of memoirs published for the most part around or before 1850 was the assembling of a calculus of ingenious formulas for factoring linear differential and difference operators and interpreting their inversions. An excellent account of these methods will be found in chapters 16 and 17 of Boole's *Differential Equations* (1859) and in chapter 13 of his Calculus of *Finite Differences* (1860). *The Calculus of Operations* (1855) by Robert Carmichael is also rich in examples.

Closely related to this thread of ideas is the concept of an algebraic theory of linear differential operators. This problem furnished the basis for the dissertation of H. Blumberg published in 1912 (see *Bibliography*). This author considered the question of the factorization of differential operators. He obtained conditions under which operators are relatively prime and developed algorithms for the determination of the least common multiple and the highest common factor of two or more operators. The problem of the commutativity of differential operators is naturally suggested by this algebraic analogy and it has furnished the basis for a paper by J. L. Burchnall and T. W. Chaundy (see *Bibliography*) published in 1922. If P and Q are linear differential operators, then they are commutative provided,

$$(P - hI) \rightarrow Q = Q \rightarrow (P - hI) \quad ,$$

where h is any constant and I is the identical operator. The principal result attained by Burchnall and T. W. Chaundy was the theorem that *if P and Q are permutable operators of orders m and n respec-*

tively, they satisfy identically an operational-algebraic identity of the form

$$F(P,Q) = 0 ,$$

of degree n in P and of degree m in Q.

Returning now to the historical narrative, we find among those who first intensively explored the territory opened by Lagrange the names of L. F. A. Arbogast (1759-1803), J. F. Français, J. P. Grüson (1768-1857), and A. M. de Lorgna (1735-1796). These authors confined their attention mainly to the symbols, D, Δ, Σ, and $\int$, which they regarded as quantities that between the initial and terminal operations could be manipulated formally by the rules of algebra. The principal contribution of Arbogast was the *Calcul des Dérivations* published in 1800 from which we quote:

"I apply to differentials, to general derivatives, to relations between differentials and (finite) differences, a method of calculation which one may name the *méthode de séparation des échelles d'opération*: this provides the means for presenting complicated formulas under a very simple form and for arriving at important results with extreme ease. Considered generally, this method consists in detaching from the function of the variables, when this is possible, the signs of operations which affect this function, and in treating the expression formed from these signs mixed with some of the quantities, an expression which I have called the *échelle d'opérations*, in treating it, I say, just as if the signs of operation which enter it were quantities; then in multiplying the result by the function."

F. J. Servois (1767-1847) made a notable advance over the ideas of his predecessors by showing in 1814 that the reason for the formal analogy between operational symbols and algebraic symbols had its roots in the distributive, commutative and associative laws obeyed by both sets of symbols. Servois introduced the names *distributive* and *commutative*, but the term *associative* seems to be due to W. R. Hamilton (1805-1865).*

A. L. Cauchy, under the stimulus of some researches of B. Brisson (1777-1828), a pupil of G. Monge (1746-1818), developed a number of formulas, deriving among other results the summation of Euler-Maclaurin. He inquired into the convergence of the series obtained by formal processes and considered methods for establishing the validity of results obtained by operational methods.

The development of these formal methods progressed rapidly, particularly with the English school of mathematicians. The main contributors of this period were G. Boole, B. Bronwin, R. Carmichael, W. Center, A. De Morgan, W. F. Donkin, J. T. Graves, S. S. Great-

*See F. Cajori's: *History of Mathematics*, 2nd ed. (1929), p. 273.

heed, H. S. Greer, D. F. Gregory, C. J. Hargreave, R. Murphy, G. Peacock, S. Roberts, W. H. L. Russell, and W. Spottiswoode.

One of the significant contributions of this period was the generalization of the Leibnitz rule, $(uv)^{(n)} = (u + v)^{(n)}$, which was achieved by C. J. Hargreave (1820-1866) [See *Bibliography*: Hargreave (2)] in 1848, who showed that

$$F(z) \to uv = uF(z) \to v + u'F'(z) \to v + \frac{1}{2!}u''F''(z) \to v + \cdots .$$

Here the letter z represents the operation d/dx, a use which will be frequently employed in the later pages of this book.

Closely related both to the Liouville theory of fractional operators and to the Laplace theory of generatrix functions (see sections 5 and 7) we find the symbolic calculus of G. Oltramare, (1816-1906), a student of Cauchy. The main points of the calculus were set forth in a memoir published in 1886* and the work reached final form in the *Calcul de généralisation* published in 1899. The calculus was further developed and applied by C. Cailler in his thesis published in 1887 and by D. Mirimanoff in 1900. (See *Bibliography*).

The principal objection to the calculus was the restrictive expansion,

$$g(x) = \sum_n f(n)e^{nx} ,$$

assumed for the development of functions to which the operations were applied. By forming a table of operations, Oltramare was able to invert a number of types of functional equations. A more extensive account of this calculus will be found in (f) section 12, chapter 2.

The period of the formal development of operational methods may be regarded as having ended by 1900. The theory of integral equations was just beginning to stir the imagination of mathematicians and to reveal the possibilities in systems of equations in infinitely many variables. The rich researches of S. Pincherle on the analytic operator, researches carried out over a long period of time, served as a bridge between the older analysis and the new.

5. *Generalized Integration and Differentiation.* One of the most interesting phases of the formal theory was the development of generalized differentiation and integration. We have already indicated

*Sur la généralisation des identités. *Memoire de inst. nat. Geneve,* vol. 16 (1886), pp. 1-109.

how the idea of interpreting $d^{\frac{1}{2}}y/dx^{\frac{1}{2}}$ occurred to Leibnitz as soon as the formal analogies had been observed by him.

Not, however, until the mathematical development of the nineteenth century was well advanced did this suggestive concept gain headway and even then it appeared sporadically and unrelated in the literature. If we are to judge from the lack of reference to other work, fractional operators were discovered independently by at least P. S. Laplace (1749-1827), J. Fourier (1768-1830), N. H. Abel (1802-1829), J. Liouville (1809-1882), G. F. B. Riemann (1826-1866), H. Laurent (1841-1908), and O. Heaviside (1850-1925). It is rather a curious fact that the obvious power of these generalized operators and their intimate connection with the Cauchy integral formula in some of its more important applications have not succeeded even yet in securing for them a passing reference in standard treatises on the calculus and in the theory of analytic functions.*

It will thus be seen that the generalization to fractional exponents for the operators z^n and z^{-n}, $z = d/dx$, has not been attained with the same ease as the analogous generalization for algebraic quantities. The pathway seems to have been beset with errors. Riemann, approaching the subject from Taylor's series, found himself inextricably tangled in difficulties with the complementary function. This subject also led Liouville into error and resulted in curious difficulties encountered by G. Peacock (1791-1858) while attempting to apply his *principle of the permanence of equivalent forms*. (See section 8, chapter 2.) Much of the distrust encountered by Heaviside from the Cambridge rigorists† may be traced quite probably to the lack of any adequate theory of fractional processes.

Apparently no definite attempt was made to form a theory of fraction-operators until L. E. Euler devoted a few pages to the subject in 1731.‡ This idea lay fallow for nearly a century until we encounter it again in the work of Laplace and Fourier. Thus we find Laplace employing the formulas:§

For negative values of n,

$$\frac{d^n y_x}{dx^n} = \int_c T(t)\, t^{-x} [\log(1/t)]^n dt$$

and

$$\nabla^n y_x = \int_c T(t)\, t^{-x} (a + b/t + \cdots + q/t^m)^n dt \; ,$$

*We note, however, an extended account of these operators in the recently published work of A. Zygmund: *Trigonometrical Series*, Warsaw (1935), 331 p., in particular, pp. 222-233.

†Let us add that this term is not used disparagingly. The author has the deepest sympathy for a completely rigorous scrutiny of all heuristic processes.

‡Vol. 5, *Commentaires de St. Petersbourg* (1730-31), p. 55.

§*Théorie analytique des Probabilités.* 3rd ed. (1820), pp. 85 and 156.

where we define

$$y_x = \int_C T(t)\, t^{-x} dt\ , \qquad \nabla y_x = a y_x + b y_{x+1} + \cdots + q y_{x+m}\ .$$

For positive values of n,

$$\frac{d^n y_x}{dx^n} = \int_C \varphi(t)\, t^x (\log t)^n dt$$

and

$$\Delta^n y_x = \int_C \varphi(t)\, t^x (t-1)^n dt\ ,$$

where we define

$$y_x = \int_C \varphi(t)\, t^x dt\ .$$

Fourier's approach was through operations upon the integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda i x}\, d\lambda \int_{-\infty}^{\infty} e^{-\mu\lambda i} f(\mu)\, d\mu\ ,$$

and we find him writing, for general values of n,*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(\mu) \cos \lambda (x - \mu)\, d\mu\ ,$$

$$\frac{d^n}{dx^n} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda^n\, d\lambda \int_{-\infty}^{\infty} f(\mu) \cos(\mu x - \mu\lambda + \tfrac{1}{2} n\pi)\, d\mu\ .$$

Five years later we find A. L. Cauchy (1789-1857) making use of the Fourier integral for operators of the form $F(z)$. $z = d/dx$, but apparently he considered only the case where $F(z)$ is a power series about the origin. Other suggestions are also found in the work of S. F. Lacroix (1765-1842)† and G. Peacock to whom we have already referred. The latter constructed his generalization on a formal extension of the nth derivative and integral of x^m. In 1823 N. H. Abel (1802-1829)‡ attracted attention to the subject by applying fractional operators to the problem of the tautochrone. He employed both the symbol $d^{-\frac{1}{2}} \psi(x)/dx^{-\frac{1}{2}} = (1/\sqrt{\pi}) \int^{\frac{1}{2}} \psi(x)\, dx$ and the symbol $\psi(x) = \sqrt{\pi}\, d^{\frac{1}{2}} s/dx^{\frac{1}{2}}$ and his attainment of the solution in terms of

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**Théorie de la Chaleur* (1822), sec. 422.

†*Traité du calcul différentiel et du calcul intégral.* 2nd ed. (1819), vol. 3, p. 409.

‡Solution de quelques problèmes à l'aide d'intégrales définies. *Werke,* vol. 1 (1881), pp. 10-27.

them was simple and elegant. An account of this application is given in problem 1, section 7, chapter 6.

The most extensive development of the calculus of fractional operators during this initial period was made by J. Liouville (1809-1882), who between 1832 and 1836 devoted eight memoirs totalling about three hundred pages to the subject.* He gave a number of applications to problems in geometry and mechanics, a few of which will be found in section 7, chapter 6.

The definition employed by Liouville was rather restrictive, however, and its usefulness considerably limited by considerations of convergence. Thus he assumed that the derivative of a function $f(x)$, which can be expressed as the series,

$$f(x) = \sum_{n=0}^{\infty} c_n \, e^{a_n x} \quad ,$$

$$D_x^{\nu} f(x) = \sum_{n=0}^{\infty} c_n \, a_n^{\nu} \, e^{a_n x} \quad .$$

It thus appears that application of the Liouville definition is limited to the class of functions which can be represented by a Dirichlet series. The problem of imposing conditions for such expansions has been extensively studied in recent years.†

Under the stimulus of the activity which centered around operational methods in England during the middle years of the nineteenth century, a number of papers were prepared on fractional operations. P. Kelland (1810-1879) published an extensive memoir on the subject in 1839 and a second in 1846. S. S. Greatheed in 1840, W. Center in 1848 and H. S. Greer in 1860 contributed to the discussion and A. De Morgan (1806-1871) devoted several pages in his *Differential and Integral Calculus* (1842) to the subject.‡ De Morgan, however, regarded the matter as being in confusion. Thus he pointed out that the system advocated by Peacock leads to the formula, (see section 7, chapter 2)

$$D_x^n x^{-m} = x^{-m-n} \, \Gamma(1-m)/\Gamma(1-n-m) \quad ,$$

while the system of Liouville derives

$$D_x^n x^{-m} = (-1)^n x^{-m-n} \, \Gamma(m+n)/\Gamma(m) \quad .$$

But if n is an integer then these systems will be found to give the same result which leads De Morgan to the following remark:

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*See *Bibliography.*

†See G. H. Hardy and M. Riesz: The General Theory of Dirichlet's Series. *Cambridge Tracts in Math. and Math. Physics,* No. 18, Cambridge (1915), 78 p.

‡Pages 597-600.

"Now as both these expressions are certainly true when n is a whole number, the one becomes the other after multiplication by a factor . . . which becomes unity when n is a whole number. Both these systems, then, may very possibly be parts of a more general system; but at present I incline (and incline only, in deference to the well known ability of the supporters of the opposed systems), to the conclusion that neither system has any claim to be considered as giving *the* form of $D^n x^n$, though either may be *a* form".

The situation complained of by De Morgan is now thoroughly cleared and no real ambiguity remains. It is probably fair to state that the main aim of the writers of that period was to find a plausible generalization for fractional operators without attempting an investigation of the consequences of the definitions in the complex plane. Mention should be made, however, of a paper of considerable penetration published by M. Wastchenxo-Zachartchenxo in the *Quarterly Journal of Mathematics* in 1861.

New impetus was given to the study of fractional operators by a paper written by B. Riemann in 1847 while he was still a student but which was not published until 1876, ten years after his death. His approach to the subject was through a generalization of Taylor's series, but as has already been stated, he found himself in difficulties over the interpretation of the complementary function. An account of the Riemann theory will be found in section 9, chapter 2. The editors of Riemann's works, who are responsible for the appearance of this paper, remark that the manuscript was probably never intended for publication since the author would not have recognized in his later work the validity of the principles upon which it rested. A. Cayley (1821-1895),* however, in a brief note in 1880 considered "the idea . . . a noticeable one", but made the following comment: "Riemann deduces a theory of fractional differentiation: but without considering the question which has always appeared to me to be the greatest difficulty in such a theory: what is the real meaning of a complementary function containing an infinity of arbitrary constants? or, in other words, what is the arbitrariness of the complementary function of this nature which presents itself in the theory?" H. Holmgren in 1863 took the same integral representation arrived at by Riemann as his point of departure for a long memoir on the subject and in 1867 applied his theory to the integration of a linear differential equation of second order.

It was not, however, until H. Laurent published an account of the subject in 1884 that we find a broadening of the point of view with regard to these operators which might make them palatable to

Mathematische Annalen, vol. 16 (1880), pp. 81-82.

modern mathematicians. Laurent proceeded from the Cauchy formula,

$$u^{(n)}(x) = \frac{n!}{2\pi i} \int_C \frac{u(t)}{(x-t)^{n+1}} dt ,$$

and showed how the contour C could be chosen so that the Cauchy formula might be generalized for fractional values of n.

The slow introduction of these operators into analysis was accelerated in another direction by the publications of Oliver Heaviside in 1893, who made a brilliant and useful application of them to the theory of the transmission of electrical currents in cables. He introduced the operators $p^{\frac{1}{2}}$ and $p^{-\frac{1}{2}}$, $p = d/dt$, which appear naturally in any attempt to solve the equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} ,$$

by operational methods. For the technical application to this problem and to the more general problem of the *equation of telegraphy,* the reader is referred to section 6, chapter 7.

Since the beginning of the present century the number of papers devoted to the subject of fractional operators has rapidly increased. V. Volterra founded his theory of *functions of composition,* and his class of *permutable functions* of the *closed cycle type* (see section 9) includes what are essentially functions obtained from fractional operations of the kind described here. N. Wiener in a paper published in 1926 [See *Bibliography,* Wiener (2)] appraised the theory of *branch point* operators from the standpoint of Fourier series and gave it a more rigorous foundation than it had hitherto possessed. Other papers devoted both to the fundamental concept and to applications have been published by P. Levy, E. L. Post, the author and others. Naturally a great deal is said about these operators in all the papers collectively enumerated under the title of the Heaviside calculus.

Inaugurated through a paper published by H. Weyl in 1917, a series of recent memoirs have been devoted to fractional operators from the standpoint of the existence and function-theoretic character of the transformation,

$$F_a(x) = \frac{1}{\Gamma(a)} \int_a^x (x-t)^{a-1} f(t) dt .$$

G. H. Hardy and J. E. Littlewood in particular proved the fol-

lowing theorems:* (i) that if $f(x)$ belongs to the Lebesgue class L^p, $p > 0$, and if $0 < a < 1/p$, then F_a belongs to L^r, where $r = p/(1-pa)$; (ii) if $f(x)$ belongs to L^p in (a,b), $a < b < \infty$, then F_a belongs to Lip.* $(a-1/p)$; (iii), if $p \geqq 1$, and if $0 < k < 1/p$, then any function which belongs to Lip. (k,p) also belongs to Lip. $(k-1/p+1/p', p')$, where p' is any number subject to the condition $p < p' < p/(1-kp)$; (iv), if $k > 1/p$, then $f(x)$ belongs to the class just stated for $p' > p$ and is equivalent to a function of the class Lip. $(k-1/p)$.

Weyl showed that if $f(x)$ is a function of unit period with Fourier coefficients c_n and belongs to Lip. a, then the series $\Sigma|n^\beta c_n|^2$ converges provided $\beta < a$. J. D. Tamarkin, under the stimulus of these ideas, published in 1930 a careful investigation of the validity of the Abel inversion problem when applied to functions belonging to Lebesgue classes. His bibliography enumerates some of the work of the Russian school on this problem, particularly memoirs by A. V. Letnikov, P. A. Nekrassov, and N. Sonine.

The first appearance of the logarithmic operator, $z^{\nu} \log z$, is due to V. Volterra who formulated it in a theory of *logarithms of composition,* which he applied effectively in the solution of the integral equation

$$f(x) = \int_0^x \{\log(x-t) + C\}\, u(t)\,dt, \; f(0) = 0 \;,$$

where $C = 0.5772157 \cdots$ is Euler's constant. Later F. Sbrana [see *Bibliography*: Sbrana (2)] and the author examined both the operational-theoretic properties and the application of this new transformation. For other details the reader is referred to section 11, chapter 2 and section 8, chapter 6.

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*We shall need the following definitions:

I. $f(x)$ belongs to Lebesgue class, L^p, $p > 1$, provided $f(x)$ and $|f(x)|^p$ are integrable in the sense of Lebesgue in (a,b).

II. $f(x)$ belongs to Lipschitz class, Lip. k, $0 \leqq k \leqq 1$, in (a,b) provided

$$f(x) - f(x-h) = O(h^k) \;,$$

uniformly for $a \leqq x-h < x \leqq b$.

III. $f(x)$ belongs to Lipschitz class, Lip.* k, $0 \leqq k \leqq 1$, in (a,b), provided

$$f(x) - f(x-h) = o(|h|^k) \;,$$

when h approaches zero uniformly in x.

IV. $f(x)$ belongs to Lipschitz class, Lip.(k,p), $p \geqq 1$, $0 \leqq k \leqq 1$, in (a,b), provided $f(x)$ belongs to L in (a,b) and

$$\int_a^b |f(x) - f(x-h)|^p\,dx = O(|h|^{kp}) \;.$$

6. *Differential and Integral Equations of Infinite Order.* Another and more recent aspect of the formal theory has been the development of differential and integral equations of infinite order; i. e., equations of the form

$$a_0(x)u(x) + a_1(x)u'(x) + a_2(x)u''(x) + \cdots = f(x) \qquad (6.1)$$

$$b_0(x)u(x) + b_1(x)u^{(-1)}(x) + b_2(x)u^{(-2)}(x) + \cdots = F(x) \ .$$

The largest amount of attention has been given to differential equations of infinite order and particularly to the case of constant coefficients, $a_i(x) = a_i$. Among the first to recognize the significance of such equations was S. Pincherle who introduced them in discussing the solution of the difference equation

$$\sum_{n=1}^{m} h_n \varphi(x + a_n) = f(x) \ .$$

This memoir was published in 1886, but its anticipation of much of the work in functional operators of the *closed cycle class* was not generally recognized until its republication in *Acta Mathematica* in 1926.* These studies culminated in a treatise on distributive operations published in 1901 in collaboration with U. Amaldi.

Perhaps the first to give a general impetus to the investigation of differential equations of infinite order was C. Bourlet (1866-1913), who in 1897 contributed fundamental results in a paper entitled: *Sur les opérations en général et les équations différentielles linéaires d'ordre infini.* This paper was amplified in certain important details in 1899.†

In 1917 J. F. Ritt considered the infinite product,

$$(1 - z/a_1)(1 - z/a_2)(1 - z/a_3) \cdots (1 - z/a_n) \cdots \ ,$$
$$z = d/dx \ ,$$

under the assumption that $\sum_{n=1}^{\infty} 1/|a_n|$ converges, and derived important properties of its inversion. This approach has been extended by G. Valiron and G. Pólya.‡

The problem presented by the differential equation with constant coefficients was studied in 1918 by F. Schürer under general conditions. I. M. Sheffer in 1929 devoted two memoirs to the subject, imposing the condition that

*See *Bibliography*: Pincherle (1).
†See *Bibliography*: Bourlet (1) and (2).
‡See *Bibliography*: Ritt, Valiron and Pólya.

$$\lim_{n} |f^{(n)}(x)|^{1/n} = \text{constant.}\ *$$

This limitation was removed by the author,† who extended the domain of admissible solutions to functions summable in the sense of Borel. The relationship between operators regarded as Laurent expansions, valid in different annuli, of the same analytic function was also interpreted.

An extended domain of validity was given to these operators by N. Wiener through the use of the Fourier transform. The work of Wiener is especially notable because of the highly rigorous treatment of a subject, which for all of its formal power has been regarded as having suspicious origins.‡

Next to the case of constant coefficients, the differential equation of infinite order of Laplace type has received special attention. In this equation the coefficients are polynomials of bounded degree, $a_n(x) = a_{n0} + a_{n1}x + a_{n2}x^2 + \cdots + a_{np}x^p$, where p is a positive integer and not all the quantities a_{np} are zero. It is obvious that the case of constant coefficients is included by setting $p = 0$.

The study of the Laplace equation, apart from the *calculus of differences* in which it may be regarded as having had its origin, is due to T. Lalesco in 1908.§ Incidental to a consideration of the problem of the inversion of Volterra integrals, he applied to the homogeneous case [in which $f(x) = 0$] of the Laplace equation the transformation

$$u(x) = \int_c e^{xt} v(t)\, dt\ ,$$

in which the path is a conveniently chosen one depending upon the coefficients of the equation. He made special application to the expanded form of the difference equation, $u(x+1) - x\, u(x) = 0$, which defines the gamma function.

E. Hilb‖ in 1921 was the first to solve the non-homogeneous case of the Laplace equation by means of an unlimited differentiation of (6.1). The system thus obtained was found to come under the general Hilbert theory of linear equations in an infinite number of unknowns as applied to Laurent forms (see section 2, chapter 3) and a solution of the system was thus explicitly obtained. Hilb also developed in more detail the method which Lalesco had used in the homogeneous case.

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*See *Bibliography*: Schürer and Sheffer (2) and (4).

†See *Bibliography*: Davis (4).

‡See *Bibliography*: Wiener (2).

§See *Bibliography*: Lalesco (1).

‖See *Bibliography*: Hilb (1).

At the same time O. Perron* considered the general Laplace equation and derived a limitation upon the number of solutions by means of an application of known results concerning the solution of the linear system

$$\sum_{n=0}^{r} (a_n + b_{mn}) x_{m+n} = c_m \ , \quad m = 0, 1, 2, \cdots \ .$$

I. M. Sheffer in two papers published in 1929† considered the details of solution for the cases $p = 0$ and $p = 1$. In the first of these, writing the equation in the form $\{A_0(z) + xA_1(z)\} \rightarrow u(x) = f(x)$, $z = d/dx$, he discussed the cases (a) $A_1(z) = z-a$, and (b) $A_1(z) = (z-a)(z-b)$. He further showed that if $A_1(z)$ has r zeros of multiplicities $p_1, p_2, \cdots p_r$, then the equation of infinite order can be replaced by an equation of finite degree m, where $m = p_1 + p_2 + \cdots + p_r$.

In the second paper Sheffer employed methods similar to those used by S. Pincherle, reducing equation (6.1) by means of a Laplace transformation to a contour integral equation and expressing the resolvent kernel by means of a second contour integral. He further proved that if $f(x)$ is expansible in a series of Appell polynomials, then a solution $u(x)$ can be expressed simply in terms of the coefficients of the expansion.

Since Appell polynomials have been important in the theory of differential equations of infinite order we shall give a brief account of them. By an *Appell polynomial*, $A_n(x)$, we mean a polynomial such that

$$d\, A_n(x)/dx = n\, A_{n-1}(x) \ .$$

The most general set of such polynomials has the form

$$A_n(x) = \sum_{p=0}^{n} p_r \, {}_nC_r \, x^{n-r} \ ,$$

where ${}_nC_r$ is the rth binomial coefficient and the coefficients p_r are arbitrary.

Moreover, if we have

$$a(h) = \sum_{r=0}^{\infty} p_r \, h^r/r! \ ,$$

then it is easily proved that

$$a(h)\, e^{hx} = \sum_{n=0}^{\infty} A_n(x)\, h^n/n! \ .$$

The function $a(h)$ is called the *generatrix* of the polynomials.

Such polynomials were first defined by P. Appell in 1880 and have been the subject of a number of investigations. The following bibliography has been furnished the author by R. D. Carmichael:

A. Angelesco: Sur une classe de polynômes et une extension des séries de Taylor et de Laurent. *Comptes Rendus*, vol. 176 (1923), pp. 275-278. Sur des polynômes qui se rattachent á ceux de M. Appell. *Ibid.*, vol. 180 (1925), p. 489.

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*See *Bibliography*: Perron.

†See *Bibliography*: Sheffer (1) and (3).

P. Appell: Sur une classe de polynômes. *Ann. Sci. Norm. Sup.*, vol. 9 (2nd ser.), (1880), pp. 119-144.

E. T. Bell: On Generalizations of the Bernoullian Functions and Numbers. *American Journal of Math.*, vol. 47 (1925), pp. 277-288. Invariant Sequences. *Proc. Nat. Acad. Sci.*, vol. 14 (1928), pp. 901-904. Certain Invariant Sequences of Polynomials. *Trans. American Math. Soc.*, vol. 31 (1929), pp. 405-421. Exponential Polynomials. *Annals of Math.*, vol. 35 (1934), pp. 258-277.

S. Bochner: See *Bibliography*: Bochner (1).

G. H. Halphen: Sur certaines séries pour le dévelopment des fonctions d'une variable. *Comptes Rendus*, vol. 93 (1881), pp. 781-783. Sur quelques séries pour le développement des fonctions à une seule variable. *Bull. des Sc. Math.*, vol. 5 (2nd ser.) (1881), pp. 462-488. Sur une série d'Abel. *Bull. de la Soc. Math.* vol. 10 (1881-1882), pp. 67-87.

P. Humbert: Sur une classe de polynômes. *Comptes Rendus*, vol. 178 (1924), pp. 366-367.

R. Lagrange: Sur un algorithme des suites. *Comptes Rendus*, vol. 184 (1927), pp. 1405-1407. Sur certaines suites de polynômes. *Ibid.*, vol. 185 (1927), pp. 175-178; 444-446. Mémoire sur les suites de polynômes. *Acta Mathematica*, vol. 51 (1928), pp. 201-309.

H. Léauté: Développement d'une fonction à un seule variable. *Journal de Math.*, vol. 7 (3rd ser.) (1881), pp. 185-200.

W. T. Martin: On Expansions in Terms of a Certain General Class of Functions. *American Journal of Math.*, vol. 58 (1936), pp. 407-420.

L. M. Milne-Thomson: Two Classes of Generalized Polynomials. *Proc. London Math. Soc.*, vol. 35 (2nd ser.) (1933), pp. 514-522.

N. Nielsen. *Traité élémentaire des nombres de Bernoulli.* Paris, (1923), 398 p.

N. E. Nörlund: Mémoire sur les polynômes de Bernoulli. *Acta Mathematica*, vol. 43 (1922), pp. 121-196.

S. Pincherle: Alcune osservazioni sui polinomi del prof. Appell. *Atti dei Lincei*, vol. 2 (4th ser.) (1886), pp. 214-217. Sulle serie procedenti secondo le derivate successive di una funzione. *Rendiconti di Palermo*, vol. 11 (1897), pp. 165-175. See also *Bibliography*: Pincherle (7), (10), (15).

I. M. Sheffer: See *Bibliography*: Sheffer (3), (5).

J. Touchard: Sur le calcul symbolique et sur l'opération d'Appell. *Rendiconti di Palermo*, vol. 51 (1927), pp. 321-368.

M. Ward: A Certain Class of Polynomials. *Annals of Math.*, vol. 31 (2nd ser.) (1930), pp. 43-51.

The general equation of Laplace type was considered by the author* in 1931 from the operational point of view and the resolvent generatrix determined for it. The domain of solutions was extended to include formal expansions summable by the method of Borel.

Comparably little attention has been given to integral equations of infinite order, due most probably to the fact that the methods of Volterra have proved both powerful and satisfactory. T. Lalesco in 1910† considered the questions invoked when a differential equation

$$\sum_{p=0}^{n} a_p(x)\,u^{(p)} = 0\ ,$$

is replaced by an integral equation,

$$\sum_{p=0}^{n} b_p(x)\,y^{-(p)} = 0\ ,$$

*See *Bibliography*: Davis (5).

†See *Bibliography*: Lalesco (2).

where $b_p(x) = a_{n-p}(x)$ and $y = u^{(n)}$. His main result showed that if the coefficients $b_p(x)$ are bounded with n within a common domain R, and if a set of constants C_m exist such that $\sum_{m=1}^{\infty} C_m B_m$ converges where the B_m dominate the coefficients in R, then a unique solution of the integral equation exists for which the integrals of different orders take the preassigned values C_m.

The author in 1930* reduced the Volterra integral equation

$$b_0(x)u(x) + \int_a^x K(x,t)u(t)\,dt = f(x)$$

to an integral equation of infinite order and obtained its inversion by the method of generatrix functions and the operational product of C. Bourlet. Special application was made to the case of the *closed cycle*, i. e., for kernels of the form $K(x-t)$.

Closely related to the problem of differential and integral equations of infinite order is the problem of infinite systems of differential equations, that is to say, systems of the form

$$\frac{du_i}{dx} - \sum_{j=1}^{\infty} a_{ij}(x)\, u_j(x) = f_i(x), \quad (i = 1, 2, \cdots, \infty) \ .$$

where the functions $a_{ij}(x)$ and $f_i(x)$ are given.

Such systems were first studied by H. von Koch in 1899, who obtained a general existence theorem by means of majorant functions and applied his theory to the solution of certain types of partial differential equations. Although T. Lalesco discussed the problem of such systems in connection with his solution of Volterra integral equations in 1908, the next memoir on the subject was published by F. R. Moulton in 1915. Incidental to his development of his theory of general analysis, E. H. Moore in 1906 had included the abstract theory of infinite differential systems, but he made no attempt to study such systems independently. Under the stimulus of the work of Moulton and Moore, W. L. Hart from 1917 to 1922, T. H. Hildebrandt in 1917, and I. A. Barnett in 1922 published a series of papers extending the general theory in several ways. W. T. Reid in 1930 and D. C. Lewis in 1933 again attacked the problem and applications were made to the theory of partial differential equations. Other contributions were made by A. Wintner in a series of papers published between 1925 and 1931, by L. Lichtenstein in 1927, by W. Sternberg in 1920, and by M. R. Siddiqi in 1932.†

The demands of the Heaviside calculus naturally focused attention upon systems of equations with constant coefficients and the lit-

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**The Theory of the Volterra Integral Equation of Second Kind.* Indiana University Studies (1930), 76 p.; in particular, pp. 27-35.

†For the specific titles of these contributions see the *Bibliography*.

erature of this subject contains numerous discussions of this subject. I. M. Sheffer in two papers published in 1929 stated existence theorems for infinite systems of differential equations with constant coefficients.

7. *The Generatrix Calculus.* In his article on "Probability" in the 13th edition of the *Encyclopedia Britannica*, F. Y. Edgeworth (1845-1926) makes the following remark:

"It has been said that there is no book equal to Laplace's '*Théorie des Probabilités*' for a comprehensive and masterly treatment of probability, but this '*ne plus ultra* of mathematical skill and power' as it is called is not easy reading. Much of its difficulty is connected with the use of a mathematical method which is now almost superseded, namely 'Generatrix Functions.' "

With the first sentence no one who has looked into this great treatise would quarrel; the second sentence, however, is open to doubt since the *generatrix calculus* is essentially the calculus of the Laplace transformation. Notations for this transformation may change, but the method remains today one of the most effective tools in the solution of numerous theoretical and applied problems. We may aver, for example, that the *method of saddle points* in the theory of asymptotic series on the one hand and the *Heaviside calculus* on the other are at heart applications of the generatrix calculus.

This calculus as developed by Laplace depends essentially upon two operators defined as follows:*

If $f(x)$ is a function represented by the series

$$f(x) = \sum_{n=-\infty}^{\infty} a(n) x^n ,$$

where $a(n)$ is a function of n, then we shall have as a definition of the operators G and D the relations

$$Ga(n) = f(x) \quad \text{and} \quad Df(x) = a(n) .$$

Obviously we can replace summation by integration and thus write

$$f(x) = \int_s a(n) x^n dn , \tag{7.1}$$

from which we seek the inversion

$$Df(x) = a(n) .$$

*An account of this subject with examples showing its application to the problem of interpolation, the expansion of functions in series, the asymptotic development of the probability integral and the solution of difference equations will be found in a paper by Dr. Irene Price: Laplace's Calculus of Generatrix Functions, *American Mathematical Monthly*, vol. 35 (1928), pp. 228-235.

Both the discrete and continuous problem can obviously be united into a single one by means of the Stieltjes-Lebesgue integral

$$f(x) = \int_s x^n da(n) \ .$$

Equation (7.1) was probably the first example of an integral equation and has been made the basis of many investigations. Replacing x by e^{-t} and $f(x)$ by $g(t)$ the equation is translated into what is generally called the *Laplace integral equation,*

$$g(t) = \int_s e^{-tn} a(n) dn \ .$$

We shall make a brief résumé of some of the most important contributions to this subject.

N. H. Abel (1802-1829) employed the equation in the more general form*

$$f(x,y,z,\cdots) = \int_s e^{xu+yv+zp+\cdots} \varphi(u,v,p,\cdots) du\, dv\, dp \cdots \ .$$

He developed a number of fundamental properties of the transformation, but discovered no general method for finding its inversion.

M. Lerch (1860-1922) in 1892 discussed the homogeneous case, $g(t) = 0$, where the path of integration is the real axis from 0 to ∞.† He showed that if $g(t)$ is zero for an infinity of values such that $x = b + km$ $(m = 0, 1, 2, \cdots)$, then in general $g(t)$ will be identically zero, except over a set of measure zero. The possibility of expressing $\sin kt$, $\cos kt$, $1/\Gamma(kt)$, etc., as an integral of Laplace type is thus excluded.

A. L. Cauchy (1789-1857) was familiar with the inversion

$$f(x) = \int_0^\infty e^{-xt} g(t) dt \ , \quad g(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{xt} f(x) dx \ ,$$

provided a is chosen sufficiently large, and this formula was effectively employed by G. F. B. Riemann (1826-1866) in his celebrated memoir: *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, published in 1859.‡

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*Sur les fonctions génératrices et leur determinantes. *Oeuvres*, vol. 2, pp. 67-81.

†Sur un point de la Théorie des Fonctions Génératrices d'Abel. *Acta Mathematica,* vol. 27 (1903), pp. 339-351. First published, *Rozpravy ceské Akadamie,* 2nd class, vol. 1, no. 33 (1892), and vol. 2, no. 9 (1893).

‡*Monatsberichte der Berliner Akademie* (1859). *Werke,* 2nd ed., Leipzig (1892), pp. 145-153.

This problem was further studied by H. Poincaré (1854-1912) who in a paper published in 1912 employed it in discussing the, at that time, novel theory of quanta.* H. Hamburger in 1920† discussed the inversion for certain types of discontinuities in connection with the Riemann problem of prime numbers and this was again extended by J. D. Tamarkin in 1926.‡ V. Romanowsky§ found the inversion of use in connection with sampling problems in statistics and contributed the inversion

$$f(z) = A^{\frac{1}{2}(s-1)} z^{\frac{1}{2}(s-3)} e^{-Az} / \Gamma[\tfrac{1}{2}(s-1)] \ ,$$

$$\int_0^\infty f(z) e^{az} dz = A^{\frac{1}{2}(s-1)} (A - a)^{-\frac{1}{2}(s-1)} \ ,$$

under the restriction

$$\int_0^\infty f(z) \, dz = 1 \ .$$

An extensive discussion of the inversion of the Laplace integral has been given by D. V. Widder|| in a long memoir published in 1934. In this he considered the Stieltjes-Lebesgue integral

$$f(x) = \int_0^\infty e^{-xt} \, d\,a(t) \ .$$

Obtaining the inversion of the integral by various limiting processes, Widder considered the type of function thus represented. The zeros of both $f(x)$ and its inverse were investigated and the results extended to the complex plane. The paper concluded with the establishment of general conditions for the solution of the moment problem discussed below.

The problem of the *self-reciprocal function,* i. e., the function which satisfies the equation

$$f(x) = \lambda \int_0^\infty e^{-xt} f(t) \, dt \ ,$$

was first discussed by H. Weyl** and later became the subject of in-

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*Sur la theorie des quanta. *Journal de Physique,* ser. 5, vol. 2 (1912), pp. 5-34; in particular, pp. 23-24.

†Über eine Riemannsche Formel aus der Theorie der Dirichletschen Reihen. *Math. Zeitschrift,* vol. 6 (1920), pp. 1-10.

‡On Laplace's Integral Equations. *Trans. Amer. Math. Soc.,* vol. 28 (1926), pp. 417-425.

§On the moments of standard deviation and of correlation coefficients in samples from normal. *Metron,* vol. 5 (1925), No. 4, pp. 3-54; in particular, p. 9.

||The Inversion of the Laplace Integral and the Related Moment Problem. *Trans. American Math. Soc.,* vol. 36 (1934), pp. 107-200; see also: A Generalization of Dirichlet's series and of Laplace's integrals by means of a Stieltjes integral. *Trans. American Math. Soc.,* vol 31 (1929), pp. 694-743.

**Singuläre Integralgleichungen. *Dissertation,* Göttingen, (1908), 86 p.

vestigations by T. Carleman,* J. Hyslop,† G. H. Hardy, and E. C. Titchmarsh.‡ The solution appears in the form

$$f(x) = Ax^{-a} + Bx^{a-1} ,$$

where A, B, λ, and a, are subject to the restrictions

$$A^2\Gamma(1-a) = B^2\Gamma(a) \quad , \quad \sin a\pi = \lambda^2\pi \; .$$

The use of the Laplace transformation as a means of solving certain types of difference and differential equations is probably too well known to require more than a passing reference here. This subject will be extensively developed in chapter 10, where a brief historical résumé will be found. P. Humbert in 1914 considered the problem of inverting the Laplace transformation by means of known solutions of the associated differential equation.§ His suggestive paper generalized the problem by considering the inversion of integral equations of the form

$$f(x) = \int_0^\infty K(xt)u(t)dt \; .$$

Closely associated with the differential equation problem, we find the theory of asymptotic expansions from the standpoint of E. Borel. This theory culminates in what is referred to as the *method of saddle points,* or the *method of steepest descent.* Some demands upon knowledge of this powerful tool of analysis will be made in the ensuing pages and the reader is referred for a description and bibliography to the author's *Tables of the Higher Mathematical Functions,* vol. 1 (1933), part 2, pp. 41-64. Fundamental references include the work of Borel,‖ P. Debye,** E. W. Barnes†† and G.

*Sur les équations intégrales singulières à noyau réel et symétrique. *Uppsala Universitets Arsskrift* (1923), 228 p.

†The integral expansion of arbitrary functions connected with integral equations. *Proc. Cambridge Phil. Soc.*, vol. 22 (1925) pp. 169-185.

‡Solutions of Some Integral Equations Considered by Bateman, Kapteyn, Littlewood and Milne. *Proc. London Math. Soc.*, 2nd ser., vol. 23, pp. 1-26; also: Solution of an Integral Equation. *Journal of the London Math. Soc.*, vol. 4 (1929), pp. 300-304.

§On Some Results Concerning Integral Equations. *Proc. of the Edinburgh Math. Soc.*, vol. 32 (1914), pp. 19-29.

‖*Leçons sur les Séries Divergentes.* Paris (1901); in particular, chap. 4.

**Näherungsformeln für die Zylinderfunktionen für grosse Werte des Arguments und unbeschränkt veränderliche Werte des Index. *Mathematische Annalen,* vol. 67 (1909), pp. 535-558.

††A Memoir on Integral Functions. *Phil Trans. of the Royal Soc.,* vol. 199(A), (1902), pp. 411-500. The Asymptotic Expansion of Functions Defined by Taylor's Series. *Phil. Trans. of the Royal Soc.,* vol. 206(A), (1906), pp. 249-297.

N. Watson.* A résumé of the method will be found in section 4, chapter 5.

The famous *moment* problem of analysis is essentially an outgrowth of the generatrix calculus. In the Stieltjes-Lebesgue form we seek the inversion of the infinite system of equations

$$\mu_n = \int_0^1 t^n \, d\,a(t) \quad , \quad n = 0, 1, 2, 3, \cdots , \tag{7.1}$$

where the quantities μ_n are given. The transformation $t = e^{-s}$ brings the problem within the scope of the generatrix calculus.

J. Liouville (1809-1882) in 1837 considered the existence theorem for the homogeneous case,

$$\int_a^b t^n \varphi(t)\, dt = 0 \quad , \quad n = 0, 1, 2, 3, \cdots ,$$

and showed that if $\varphi(t)$ is analytic in the interval, then $\varphi(t)$ must be identically zero.† This problem was further studied by C. Severini,‡ who later used it in connection with the closure properties of orthogonal functions.§

One of the most instructive papers on the formal problem was published by H. Laurent (1841-1908) in 1878, who attained many results of great elegance.|| Considering the finite problem,

$$\int_a^b t^n \varphi(t)\, dt = 0 \quad , \quad n = 0, 1, 2, \cdots , m-1 ,$$

he obtained the solution

$$\varphi(x) = \frac{d^m}{dx^m} [(x-a)^m (x-b)^m \psi(x)] ,$$

where $\psi(x)$ is a function which does not vanish at either a or b. Laurent obtained the differential equation satisfied by $\varphi(x)$ and developed the theory of the Legendre and Laguerre polynomials.

**Theory of Bessel Functions.* Cambridge (1922). Chapter 7. Also: An Expansion Related to Stirling's Formula, derived by the Method of Steepest Descents. *Quarterly Journal of Math.*, vol. 48 (1920), pp. 1-18.

†Solution d'un problème d'analyse. *Journal de Math.*, vol. 2 (1837), pp. 1-2.

‡Sulle equazioni integrali $\int_a^b \theta(x)\, dx = 0 \quad , \quad n = 0, 1, 2, \cdots .$ *Atti dei Lincei,* vol. 30 (1), (1921), pp. 17-19.

§Sulla theoria di chiusura dei sistemi di funzioni ortogonali. *Rendiconti di Palermo,* vol. 36 (1913), pp. 177-202.

||Sur le calcul inverse des intégrales definies. *Journal de Math.*, vol. 4, series 3 (1878), pp. 225-246.

The most extensive researches on the moment problem are due to T. J. Stieltjes (1856-1894), who proved the following fundamental result: *

Consider the moment problem in the form

$$\mu_n = \int_0^\infty t^n \, d\,a(t) \quad , \quad n = 0, 1, 2, 3, \cdots .$$

If $a(t)$ is a solution, then we have formally

$$\int_0^\infty d\,a(t) / (x+t) = \mu_0/x - \mu_1/x^2 + \mu_2/x^3 - \cdots .$$

Hence there exists a corresponding continued fraction:

$$\frac{a_0|}{|x} + \frac{a_1|}{|1} + \frac{a_2|}{|x} + \frac{a_3|}{|1} + \cdots = \frac{1|}{|b_0x} + \frac{1|}{|b_1x} + \frac{1|}{|b_2x} + \frac{1|}{|b_3x} + \cdots .$$

Stieltjes then proved the theorem: The continued fraction has infinitely many corresponding integrals or only one, according as the series $\Sigma\, b_n$ converges or diverges. If $\Sigma\, b_n$ diverges, then the fraction for all values of x which do not belong to the negative real axis, including zero, converges and equals the corresponding integral.

F. Hausdorff in 1923 proved the theorem: † A necessary and sufficient condition that equation (7.1) have a bounded non-decreasing solution $a(t)$ is that the sequence of moments shall be completely monotonic. Researches on this subject have also been made by H. Hamburger,‡ E. Stridsberg,§ M. Riesz,|| R. Nevalinna,** T. Carle-

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*Recherches sur les fractions continues. *Annales de Toulouse*, vol. 8 (1894), pp. J, 1-122; in particular, p. 71 et seq.; vol. 9 (1895), pp. A, 1-47.

†Momentprobleme für ein endliches intervall. *Mathematische Zeitschrift*, vol. 16 (1923), pp. 220-248.

‡Über eine Erweiterung des Stieltjeschen Momentenproblems. I, II, and III. *Mathematische Annalen*, vol. 81 (1920), pp. 235-319; vol. 82 (1921), pp. 120-164, 168-187.

§Nagra aritmetiska undersökningar rörande fakulteter och vissa allmännare koefficientsviter. Notes 2 and 3. *Arkiv för Mat.*, vol. 13, No. 25 (1918), pp. 1-70; vol. 15, No. 22 (1921), pp. 126.

||Sur le problème des moments. Notes 1, 2 and 3. *Arkiv för Mat.*, vol. 16, Nos. 12 (1921), pp. 1-23 and 19 (1922), pp. 1-21; vol. 17, No. 16 (1923), pp. 1-52. Also: Sur le problème des moments et le théorème de Parseval correspondant. *Acta Literarum Ac. Scientiarum Regiae Universitatis Hungaricae Francisco- Josephinae*, vol. 1 (1922-1923), pp. 209-225.

**Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjesche Momentenproblem. *Ann. Ac. Scient. Fenn.*, vol. 18, No. 5 (1922), pp. 1-53.

man,* D. V. Widder,† I. J. Schoenberg,‡ T. H. Hildebrandt,§ E. Hille and J. D. Tamarkin‖ and others. An extensive account of the problem will be found in Oskar Perron's *Die Lehre von den Kettenbrüchen*, Leipzig and Berlin (1913), chapter 9.

8. The Heaviside Operational Calculus. We have already spoken in preceding sections of the controversy which has centered around the question of the validity of the methods initiated by Oliver Heaviside in the theory of electrical communication. Since the problem of the *Heaviside calculus*, as it pertains to the general theory of operators, will be systematically treated in chapter 7 we shall content ourselves at this place with a meager description and a brief outline of its history.

The first problem attacked by the Heaviside calculus is the determination of functions which represent the charges, Q_i, in an electrical network. Since the charges are connected with the currents by the formula, $I_i = dQ_i/dt$, the problem is equivalent to finding the currents in the various circuits which comprise the network. If we represent a derivative with respect to time by p, $p = d/dt$, as is customary in this theory, and if we write $Z_{ij}(p) = R_{ij}p^2 + L_{ij}p + 1/C_{ij}$, where R, L and C represent resistence, inductance and capacity respectively, then the Heaviside calculus is concerned with the solution of the system,

$$\begin{array}{l} Z_{11}(p) \to Q_1 + Z_{12}(p) \to Q_2 + \cdots \ + Z_{1n}(p) \to Q_n = f_1(t) \ , \\ Z_{21}(p) \to Q_1 + Z_{22}(p) \to Q_2 + \cdots \ + Z_{2n}(p) \to Q_n = f_2(t) \ , \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ Z_{n1}(p) \to Q_1 + Z_{n2}(p) \to Q_2 + \cdots \ + Z_{nn}(p) \to Q_n = f_n(t) \ . \end{array} \tag{8.1}$$

The functions $f_i(t)$ are known electromotive forces (e. m. f. s.) which are regarded as continuous (except at the origin) and specified functions of time. These equations are to be solved subject to initial conditions, which may be imposed without loss of generality at $t = 0$.

*Sur le problème des Moments. *Comptes Rendus*, vol. 174 (1922), pp. 1680-1682; *Sur les équations intégrales singulières a noyau réel et symétrique.* Upsala (1923), 228 p., in particular pp. 204-220.

†*Loc. cit.*, p. 191 et seq.

‡On Finite and Infinite Completely Monotonic Sequences. *Bulletin Amer. Math. Soc.*, vol. 38 (1932), pp. 72-76.

§On the Moment Problem for a Finite Interval. *Bulletin Amer. Math. Soc.*, vol. 38 (1932), pp. 269-270.

‖Questions of Relative Inclusion in the Domain of Hausdorff Means. *Proc. of National Academy of Sciences*, vol. 19 (1933), pp. 573-577; On Moment Functions. *Ibid.*, pp. 902-908; On the Theory of Laplace Integrals. *Ibid.*, pp. 908-912.

The second problem of the Heaviside calculus is concerned with the solution of a class of linear partial differential equations of which typical members are the equation of wave motion, the equation of heat conduction, and the *equation of telegraphy.* (See section 6, chapter 7).

To those familiar with the history of differential equations it seems curious to regard system (8.1) as presenting a problem that is sufficiently novel and difficult to warrant the invention of a special calculus for its solution. Classical methods of great power were in existence to attack it long before the time of Heaviside. The system appears in many types of dynamical problems and is found as early as 1788 in the *Mécanique Analytique* of Lagrange (vol. 1, p. 390). The novelty consisted in two facts. In the first place the functions $f_i(t)$ in (8.1), representing known electromotive forces, are discontinuous at $t = 0$. One of the simplest of these, for example, is the unit e. m. f., $f(t) = 1$, $t \geqq 0$, $f(t) = 0$, $t < 0$. In the second place a problem of this kind is actually solved by a single operational process which yields not only particular integrals of the system, but those particular integrals which satisfy as nearly as possible the discontinuous boundary conditions imposed upon the system.

It should be particularly noted at this time that integrals with the specified discontinuity cannot be constructed from the analytic functions which satisfy the differential system. All that can be expected is to find integrals which vanish at $t = 0$ to as high an order as possible. This simple fact is essential in interpreting the Heaviside calculus and differentiates the problem of electrical networks from other dynamical systems which employ identical differential equations but other boundary conditions.

The source of information regarding the network problem is to be found first of all in the original papers of Heaviside and particularly in his classical work on *Electromagnetic Theory* (see *Bibliography*). This work should be consulted not only for its deep insight into the problems of electrical theory, but also for its numerous excursions into philosophy. One may not be expected to agree in all details with the thrusts made by Heaviside at mathematical rigor, but the evidences of a sparkling intellect are to be found on every page.

Probably the principal result of Heaviside as it pertains to the circuits defined by (8.1) is what is called the *Expansion Theorem.* In the second volume (p. 127) of his *Electromagnetic Theory,* Heaviside describes the efficacy of this theorem in the following words: "It does not require special investigations of the properties of normal functions. It is very direct and uniform of application. It

avoids, in general, a large amount of unnecessary work. The investigation of the conjugate property, and of the terminal apparatus in detail in order to apply it to the determination of the coefficients, is wholly avoided. It applies to all kinds of series of normal functions, as well as Fourier series. And it applies generally in electromagnetic problems, with a finite or infinite number of variables; or more generally, to the system of dynamical equations used by Lord Rayleigh in the first volume of his treatise on Sound, which covers the rest of the work, and upon which he bases his discussion of general properties."

The methods of Heaviside were unappreciated for some years, but by 1916 it began to appear that they were destined to play a leading rôle in modern electrical research. Among those who have been most influential in investigating and interpreting the Heaviside calculus should be mentioned K. W. Wagner, T. J. Bromwich, J. R. Carson, V. Bush, N. Wiener, H. Jeffreys, G. Giorgi, H. W. March, L. Cohen, E. Berg, T. C. Fry, W. H. Eccles, W. O. Pennell, J. J. Smith, H. Salinger, H. Pleijel and R. Liljeblad, F. Sbrana, J. B. Pomey, B. Van der Pol and P. Levy.*

It would seem fair to say that four methods have been used prominently in discussing the Heaviside calculus. The first of these is a direct use of formal operators, the actual expansion being that of the outer Laurent annulus of the inverse operator. The significance of this statement will be discussed in chapter 6.

The second method was initiated by Bromwich and forms the basis of the exposition published by Jeffreys. It is founded upon the use of complex circuits of the form,

$$Q(t) = (1/2\pi i) \int_c e^{ts} V(s)\, ds \ .$$

This method goes back essentially to A. Cauchy and has the advantage not possessed by other methods of furnishing a solution for general boundary conditions as well as the special conditions of the Heaviside problem. The work of March and Fry in particular bear upon this type of approach.

The third method is due essentially to Carson and consists in reducing the Heaviside problem to the inversion of the Laplace integral,

$$1/[p\, Z(p)] = \int_0^\infty A(t)\, e^{-pt} dt \ .$$

*The reader is referred to the bibliography for an account of the contributions made by these men.

This method has the advantage of stating the problem in a form which can make use of all the accumulated knowledge classified under the heading of the *generatrix calculus*. An extensive evaluation of integrals of the Laplace type has been made by Carson and others and this furnishes a simple tabular solution of problems which would otherwise present great formal difficulties.

The fourth method is associated with the Fourier integral and possesses the usual power of this great analytical tool. Among its chief exponents have been Wiener, Giorgi, Sbrana and Bush, although, as one might expect, there is a very close connection between it and the methods which depend upon the Cauchy integral and the Laplace equation. Probably the highest rigor has been attained by the use of this method. It also possesses the advantage of simple application, particularly since the publication in 1931 by G. A. Campbell and R. M. Foster of a table of Fourier integrals.*

We turn next to a consideration of the second Heaviside problem which is devoted to the solution of problems pertaining to wave propagation in cables. As we have previously stated these problems are formulated in terms of partial differential equations and are characterized, as in the first problem, by the discontinuities at the time origin imposed by the instantaneous introduction of electromotive forces.

A characteristic problem is that of a non-inductive cable with distributed resistance R and capacity C per unit length subject to an impressed voltage $V_0(t)$ at the point $x = 0$. This problem leads to the differential equations

$$R\,I = -\,(\partial V/\partial x) \ ,$$

$$C(\partial V/\partial t) = -\,(\partial I/\partial x) \ ,$$

It will be seen in chapter 7 that this problem presents two very interesting questions. The first and simpler of these is the interpretation of the symbol $p^{\frac{1}{2}}$, where $p = d/dt$. We have already discussed this fractional operator and will give a much more extended account of it in chapter 2. It may be added that the special interpretation arrived at by Heaviside is entirely justified by the general theory of such operators.

The second question is much more profound and concerns the interpretation of certain divergent series which result from the formal application of operational symbols. Although such series have appeared from time to time since the days of Euler and have been the subject of extensive investigations in the present century, it is doubtful whether they are yet admitted without some suspicion

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*Fourier Integrals for Practical Applications. *The Bell Telephone System, Technical Publications* (1931), Monograph B-584, 177 p.

into mathematical literature. F. Cajori comments:* "It is a strange vicissitude that divergent series, which early in the nineteenth century were supposed to have been banished once for all from rigorous mathematics, should at its close be invited to return." We shall not comment further at this point on the subject of divergent and asymptotic series.

It was natural that the novel and empirical methods of Heaviside, which were designed primarily to reach practical results with a minimum of computational labor, should arouse objections among the mathematicians who were then deeply engrossed in the problems of rigor which were being vigorously pursued by Weierstrass and the German school. The controversy was unfortunate, but probably salutary in the end. At any rate, one reads with great interest the following appraisal made by E. T. Whittaker in an article which anyone should consult who wishes to interpret the significance of Oliver Heaviside in modern electrical theory:†

"Looking back on the controversy after thirty years, we should now place the Operational Calculus with Poincaré's discovery of automorphic functions and Ricci's discovery of the Tensor Calculus as the three most important mathematical advances of the last quarter of the nineteenth century. Applications, extensions and justifications of it constitute a considerable part of the mathematical activity of today".

9. The Theory of Functionals. Proceeding to the fourth phase of the historical development we can do no better than to quote from Volterra, who above all others has vigorously pursued this important subject:‡

"Now, if we consider the isoperimetric problem and if we regard a plane area as dependent upon the curve which encloses it, we have a quantity which depends upon the shape of a curve, or as we say today, a *function of a line.* Since a line can be represented by an ordinary function, the area may be regarded as a quantity which depends upon all values of a function. It is evidently a function of an infinite number of variables. In fact, we may regard it as a limiting case of a function of several variables by supposing that their number increases without limit in the same manner as a curve may be regarded as the limiting case of a polygon the number of whose sides increases to infinity.

"But the area is only a special case. On all sides we are able to

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**History of Mathematics.* New York (1919), p. 375.

†Oliver Heaviside. *Bulletin of the Calcutta Mathematical Society,* vol. 20 (1928-1929), pp. 199-220.

‡*Leçons sur les fonctions de lignes,* p. 14. (See Bibliography).

find other examples of functions of lines. Thus the action exerted by a flexible filiform electric current upon a magnetized needle, depends upon the shape which we give to the circuit and consequently is a function of a line.

"In order to unite in a general concept all the different particular cases, it is sufficient to imagine a quantity which depends in a given arbitrary manner upon the shape of a curve. A general function of a line will be one which corresponds to a quantity depending upon all the values of one or more functions and would always be regarded as a function of an infinite number of variables."

The general concept thus presented by Volterra is sufficiently broad, it will be observed, to bring most of the problems of analysis within its domain. A principal consideration is the determination of the order of generality which will lead to the most fruitful specialization. As is easily apprehended, the notion of definite integration is the first example encountered in mathematics of a function of a line since an integral, as contrasted with a derivative, depends upon all the values of a function within an interval. It is not, however, until we encounter the integral,

$$I = \int_a^b f(x, y, y')\, dx \ ,$$

which is the concern of the *calculus of variations*, that we are able to grasp the rich possibilities inherent in the idea. It is natural, therefore, in the generalization of the concepts of continuity, differentiation, integration, and the analytic expansion of functions, that the notion of variation in the sense in which it is found in the calculus of variations should be extensively employed in the theory of functions of lines.

The calculus of functionals in the modern sense began in 1887 with a series of papers by Volterra published in the *Rendiconti de la R. Accademia dei Lincei*. The term *functions of lines* was used to designate these researches, but it was J. Hadamard who first employed the term *functional* (*fonctionnelle*).* Since there is some confusion as to the precise use of these two designations it may be illuminating to quote the following from Volterra [See *Bibliography*: Volterra (1), p. 74]: "The name *function of a line* was initially taken to mean what in general is now called a *functional*. In this sense the term 'function of a line' has been used by many writers, and in particular by Volterra, who was the first to introduce this concept, in his Paris

*Sur les opérations fonctionnelles. *Comptes Rendus*, vol. 136 (1903), pp. 351-354.

lectures (1913) and in many earlier works. At present, however, having adopted the term 'functional' for this general concept as being more convenient because less specific, we shall reserve the name 'functions of lines' for those particular functions, of a more strictly geometrical nature."

The first treatise on the subject of functionals was the Paris lectures of Volterra published in 1913 under the title: *Leçons sur les fonctions de lignes,* although a work by J. Hadamard: *Leçons sur le calcul des variations,* which appeared in Paris in 1910, had adopted the point of view of the theory of functionals as a foundation for the calculus of variations.* The first comprehensive treatment of the subject, embracing on the one hand concepts of *general analysis* and on the other the theory of *permutable functions,* was due to G. C. Evans who published *Functionals and their Applications: Selected Topics, including Integral Equations,* in 1918. The well known work of P. Levy: *Leçons d'analyse fonctionnelle,* was published in Paris in 1922. The volume of research stimulated by the fruitful concept of fonctionals is now very great and for a more comprehensive survey of the subject than is possible in this brief historical sketch the reader is referred to Volterra's *Theory of Functionals and of Integral and Integro-differential Equations* published in 1930, a translation of lectures delivered at the University of Madrid in 1925 and published in Spanish in 1927.

In order to orient ourselves in the material of this field, we shall begin with a definition of a functional. Let us designate by the symbol,

$$z = F[x(t)] \ ,$$

a functional in the interval $(a \leq t \leq b)$. P. G. L. Dirichlet's well known definition of *function* in the ordinary sense is then extended as follows:

If a law is given by means of which to every function $x(t)$ defined within an interval $(a \leq t \leq b)$, there can be made to correspond one and only one quantity z, then z is said to be a functional of $x(t)$.

This general concept has been further generalized by E. H. Moore and M. Fréchet who applied it to *abstract aggregates,* the elements of which may be any quantities whatever, A, B, $\cdots$ etc. Definitions of continuity both for functionals and abstract aggregates depended upon the primary concept of length, which is made to depend upon the following postulates: The length between a pair of elements, A, B, is a number (A, B) such that (1) $(A, B) = (B, A) > 0$, $A \neq B$; (2) $(A, A) = (B, B) = 0$; (3) $(A, B) + (B, C) \geqq$

*See in particular chapter 7, book 2, pp. 281-312, of that work.

(A, C). If a sequence of elements, $A_1, A_2, \cdots, A_n$ is given, then an element A is the limit of the sequence provided

$$\lim_{n=\infty} (A, A_n) = 0 \ .$$

In terms of these concepts the continuity of a functional, $F[A]$, at an element A then follows provided,

$$\lim_{n=\infty} F[A_n] = F[A] \ ,$$

whenever

$$\lim_{n=\infty} A_n = A \ .$$

Uniform continuity then implies the existence of an arbitrarily small positive value, δ, to match an arbitrary ε, such that

$$| F[A] - F[A'] | < \varepsilon \ ,$$

whenever

$$(A, A') < \delta \ .$$

It is clear that a large measure of freedom is left in these definitions by the concept of length, which has been specialized in a number of ways. One of the most fruitful definitions has been the integral,

$$(X, Y) = \int_0^1 [X(t) - Y(t)]^2 \, dt \ ,$$

where $X(t)$ and $Y(t)$ are functions of integrable square in the sense of Lebesgue. Then $(X, Y) = 0$ implies that $X(t) - Y(t) = 0$, except over a set of points of measure zero. This concept reveals the fundamental significance of the definition of *Lebesgue measure* and necessitates the use of Lebesgue integration throughout arguments which depend upon this definition of length.* Functionals employing this concept are called *continuous in the mean* and limiting processes are referred to as *limits in the mean*. One also speaks of such functionals as being continuous *almost everywhere*, that is to say, except over sets of points of at most *measure zero*.

Another definition extensively employed is the following:

$$(X, Y) = \mathrm{Max}\, | X(t) - Y(t) | \ , \quad 0 \leq t \leq 1 \ .$$

When a functional, $F[x(t)]$ has continuity with this definition it is said to have *continuity of order* 0 at the element $x(t)$. If, moreover,

*This integral was discovered by H. Lebesgue in 1900. See Lebesgue: *Leçons sur l'Integration*. Paris (1904).

$$(X, Y) = \text{Max} \mid X(t) - Y(t) \mid \ , \ \mid X'(t) - Y'(t) \mid \ ,$$

and a functional $F[x(t)]$ is continuous with this definition, it is said to have *continuity of order 1.* In a similar way we may define continuity of any order.

The explicit representation of functionals was naturally one of the first problems to be studied in the new calculus and special attention was given to those obeying the postulates of linearity, that is to say, to functionals of first degree. They are of the type,

$$F[x(t)] = \int_a^b K(t)\, x(t)\, dt + \sum_i a_i\, x(t_i) \ ,$$

where $K(t)$ is a given function.

The general representation of linear functionals has occupied the attention of numerous writers, one of the first being given by Hadamard in 1903.* Another was given by F. Riesz in 1909† as follows:

Let $F[x(t)]$ be a linear functional, with continuity of order zero and limited in the field of functions that possess at most a finite number of finite discontinuities. Then define the function,

$$f(s) = F[X(t;s)] \ ,$$

where $X(t;s)$ is specified as follows:

$$\begin{aligned} X(t;s) &= 1 \text{ for } a \leq t \leq s \ , \\ &= 0 \text{ for } s < t \leq b \ . \end{aligned}$$

Then the most general functional of the specified kind is defined by the following Stieltjes integral:

$$F[x(t)] = \int_a^b x(t)\, df(x) \ .$$

In a similar way, more general linear functionals with continuity of order p have representations of the form:

$$\begin{aligned} F[x(t)] = \int_a^b K(t)\, x(t)\, dt + \sum_i a_i\, x(t_i) + \sum_i b_i\, x'(t_i) \\ + \cdots + \sum_i p_i\, x^{(p)}(t_i) \ . \end{aligned}$$

Functionals of first, second, third, etc. degree appear in the form,‡

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*Sur les opérations fonctionelles. *Comptes Rendus,* vol. 136 (1903) pp. 351-354. See also: *Leçons sur le calcul des variátions.* Paris (1910).

†Sur les opérations fonctionnelles lineáires. *Comptes Rendus,* vol. 149 (1909), pp. 974-977.

‡See P. Lévy: *Leçons d'analyse fonctionnelle.* Paris (1922), vi + 442 p.

$$F_1[x(t)] = \int_a^b K(s)x(s)ds \;,$$

$$F_2[x(t)] = \int_a^b \int_a^b K_2(s_1, s_2)x(s_1)x(s_2)ds_1\,ds_2 \;,$$

$$F_3[x(t)] = \int_a^b \int_a^b \int_a^b K_3(s_1,s_2,s_3)x(s_1)x(s_2)x(s_3)ds_1\,ds_2\,ds_3 \;,$$

. . . .

$$F_n[x(t)] = \int_a^b \cdots \int_a^b K_n(s_1,s_2,\cdots,s_n)x(s_1)x(s_2)\cdots \times x(s_n)\;ds_1\;ds_2\cdots ds_n \;.$$

The term regular functionals of degree n is applied to the series

$$G_n[x(t)] = \sum_{m=0}^{n} F_m[x(t)] \;,\; F_0 = \text{a constant},$$

and functional power series by the sum,

$$F[x(t)] = \lim_{n=\infty} G_n[x(t)] = \sum_{m=0}^{\infty} F_m[x(t)] \;,$$

provided the series is convergent for $|\,x(t)\,| < \varrho$.

The processes of differentiation and integration have been generalized for functionals, following closely the analogous operations in ordinary calculus. Thus for a continuous functional the variation

$$\delta F[x(t)] = |\,F[x(t) + \delta\,x(t)] - F[x(t)]\,| \;,$$

will be arbitrarily small with $|\,\delta\,x(t)\,| < \varepsilon$.

Now suppose that t varies between the limits s and $s + h$. The area under the variation function $\delta\,x(t)$ will then be

$$\delta A = \int_s^{s+h} \delta\,x(t)\;dt \;.$$

The derivative of the functional at the point $t = s$ is then defined to be

$$\lim_{\substack{h=0\\ \varepsilon=0}} \frac{\delta F}{\delta A} = F'[x(t)\,;s] \;.$$

In terms of this derivative the first variation of the functional can be written in the form,

$$\delta F[x(t)] = \int_a^b F'[x(t)\,;s]\;\delta\,x(s)ds \;.$$

If $\delta\, x(s)$ is replaced by $\varepsilon\, \varphi(s)$ and the limit of $\delta\, F/\varepsilon$ computed, we obtain,

$$\lim_{\varepsilon=0} [\delta\, F/\varepsilon] = \int_a^b F'[x(t)\,;s]\, \varphi(s)\, ds \ ,$$

and in general,

$$\lim_{\varepsilon=0} \{\delta\, F^{(n-1)}[x(t)\,; s_1, s_2, \cdots, s_{n-1}]/\varepsilon\}$$

$$= \int_a^b F^{(n)}[x(t)\,; s_1, s_2, \cdots, s_n]\varphi(s_1)\varphi(s_2) \cdots \varphi(s_n)\, ds_1 ds_2 \cdots ds_n$$

This idea, which is due primarily to Volterra,* has been generalized by Fréchet† who considered functionals of the form $F[x(t,a)]$, where $x(t,a)$ is differentiable with respect to the second parameter.

For the regular functional of degree n the derivative may be computed to be,

$$F'[x(t)\,;s] = K_1(s) + \sum_{m-2}^{n} m\, F_{m-1}[x(t), s] \ ,$$

where we abbreviate,

$$F_1 = \int_a^b K_2(t_1,s)\, x(t_1)\, dt \ ,$$

$$F_2 = \int_b^a \int_b^a K_3(t_1,t_2,s)\, x(t_1)\, x(t_2)\, dt_1 dt_2 \ , \text{ etc.}$$

The problem of constructing a logical theory of the integration of functionals was attacked in several ways. R. Gateau was the first to suggest such a theory which he did by generalizing the concept of *mean.*‡. For an ordinary function in n variables the integral of the function coincides with its mean provided the integration is taken over the unit cube. The reader is referred to the treatise of Lévy for a discussion of the difficulties involved and the success achieved by this generalization.

P. J. Daniell§ extended the general concept of the Stieltjes inte-

*Sopra le funzioni che dipendona da altre funzioni. *Rendiconti dei Lincei,* vol. 3 (4th series), (1887), pp. 97-105, 141-146, 153-158.

†Sur la notion de différentielle dans le calcul fonctionnel. *Comptes Rendus du Congress des Soc. Sav.,* (1912).

‡Sur la notion d'intégrale dans le domaine fonctionnel et sur la théorie du potentiel; avec note de Paul Lévy, *Bulletin de la Soc. de France,* vol 47 (1919), pp. 47-70. Also: Sur diverses questions de calcul fonctionnel. *Ibid.,* vol. 50 (1922), pp. 1-37.

§A General Form of Integral. *Annals of Mathematics,* vol. 19 (2nd series), (1917-1918), pp. 279-294; Integrals in an Infinite Number of Dimensions. *Ibid.,* vol. 20 (2nd series), (1918-1919), pp. 281-288.

grals to functional integration and important contributions were made to the theory by N. Wiener.* The concepts of integration in the generalized sense are also found in the work of É Borel† and M. Fréchet‡ as early as 1914. An extensive account of the origin of the generalizations and their subsequent development will be found in the third part (pp. 261-439) of Lévy's *Leçons d'analyse fonctionnelle.*

Specialization of these general ideas has led in a number of interesting directions. Foremost among these was naturally the theory of integral equations, which we shall treat in more detail in a later chapter. Closely associated with the development of the Volterra integral equation was the theory of *functions of composition* and the important sub-class of *permutable functions.* A technical treatment of these subjects will be found in chapter 4.

In order to discuss problems in magnetic hysteresis, elasticity, and other forms of hereditary physics, Volterra introduced the *principle of the closed cycle,* which may be described as follows:

Consider the operation, $g(x) = F \to u(x)$. Now let $u(x+T) = U(x)$ and $g(x+T) = G(x)$. Then if $G(x) = F \to U(x)$, the operator F is an *operator of the closed cycle.* The principle is simply illustrated in the case of elastic torsion. If ω represents the angle of torsion and P the torsion couple, the relationship between them is to a first approximation

$$\omega = kP \ ,$$

where k is a constant determined from physical considerations. But actually the relationship is more complicated than this since ω depends not only upon P but also upon the history of the elastic body the torsion of which is being studied. This second approximation is expressed in the form of an integral equation

$$\omega(t) = kP(t) + \int_{-\infty}^{t} K(t-s)P(s)\,ds \ , \qquad (9.1)$$

where $K(t-s)$ is the *coefficient of heredity.*

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*The Mean of a Functional of Arbitrary Elements. *Annals of Mathematics,* vol. 21 (2nd series), (1920), pp. 66-72; Differential Space. *Journal of Math. and Physics,* vol. 2 (1923), pp. 131-174; The Average of an Analytical Functional and the Brownian Movement. *Proceedings of the National Academy,* vol. 7 (1921), pp. 294-298; The Average Value of a Functional. *Proceedings of the London Math. Soc.,* vol. 22 (2nd series), (1922), 454-467.

†*Introduction géométrique à quelques théories physiques.* Paris (1914), vii + 137 p.

‡Les singularités des espaces à un très grand nombre de dimensions. *Congres de l'Association francaise pour l'Advancement des Sciences* (Le Havre). (1914), pp. 146- 147; Sur l'intégrale d'une fonctionelle étendue a un ensemble abstrait. *Bulletin de la Soc. Math de France,* vol. 43 (1915), pp. 248-265.

If $P(t)$ is periodic of period T, $P(t+T) = P(t)$, then $\omega(t)$ satisfies the criterion of the closed cycle and the operator which comprises the right member of (9.1) is an operator of the closed cycle for all periodic functions of period T.

As one may easily apprehend, operators of the closed cycle form a restrictive subclass of operators which embody the general concept of *hereditary mechanics*. This concept has been ably set forth by E. Picard, whom we quote:*

"In all this study (of classical mechanics) the laws which express our ideas on motion have been condensed into differential equations, that is to say, relations between variables and their derivatives. We must not forget that we have, in fact, formulated a principle of *non-heredity*. when we suppose that the future of a system depends at a given moment only on its actual state, or in a more general manner, if we regard the forces as depending also on velocities, that the future depends on the actual state and the infinitely neighboring state which precedes. This is a restrictive hypothesis and one which, in appearance at least, is contradicted by facts. Examples are numerous where the future of a system seems to depend on former states: here we have *heredity*. In some complex cases one sees that it is necessary, perhaps, to abandon differential equations and consider functional equations in which there appear integrals taken from a distant time to the present, integrals which will be, in fact, this hereditary part. The proponents of classical mechanics, however, are able to pretend that heredity is only apparent and that it amounts merely to this, that we have fixed our attention upon too small a number of variables. But the situation in this case is just as it was in the simpler one, only under conditions that are more complex."

The actual representation of hereditary mechanics in analytical form led to *integro-differential equations*. The case of torsion discussed above furnishes an elementary example if we consider the dynamical case and study the oscillations of the elastic body. Then $\omega(t)$ must be replaced by $\omega(t) - md^2P/dt^2$ and we obtain

$$\omega(t) - m\frac{d^2P}{dt^2} = kP(t) + \int_{-\infty}^{t} K(t-s)P(s)\,ds \ .$$

10. The Calculus of Forms in Infinitely Many Variables. The fifth stage of the theory of operators was ushered in by Fredholm's classical papers† on the solution of the integral equation

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*La mecánique classique et ses approximations successive. *Scientia (Rivista di Scienza)*, vol. 1 (1907), pp. 4-15; in particular, p. 15.

†Sur un nouvelle méthode pour la résolution du problème de Dirichlet. *Öfv. of Kong. Sv. Vetens kaps Akad. Föhr.*, vol. 57 (1900), pp. 39-46.
Sur une classe d'équations fonctionelles. *Acta Mathematica*, vol. 27 (1903), pp. 365-390.

$$u(x) + \lambda \int_a^b K(x,t) u(t)\, dt = f(x) \quad ,$$

a solution attained heuristically as the limiting form of a set of algebraic equations

$$u(t_j) + \lambda \sum_{i=1}^{n} K(t_j,t_i) u(t_i)\, \Delta_i = f(t_j) \quad , \quad j = 1, 2, \cdots, n \quad ,$$

where the t_i are n equally spaced points in the interval (a,b) and $\Delta_i = t_{i+1} - t_i$.

The solution appeared in the form

$$u(x) = f(x) + \int_a^b f(t)\ [D(x,t\,\lambda)/D\,\lambda)]dt \quad ,$$

where $D(x,t;\lambda)$, called *Fredholm's first minor,* and $D(\lambda)$, called *Fredholm's determinant,* are functions of λ .

However, the use of algebraic guides to obtain transcendental results was not original with Fredholm, this method having been employed effectively as early as 1836 by J. C. F. Sturm (1803-1855),* who investigated the properties of a differential equation of second order by means of the limiting form taken by the solution of the difference equation

$$L_i\, u_{i+1} + M_i\, u_i + N_i\, u_{i-1} = 0 \quad .$$

The solution of the integral equation

$$u(x) + \lambda \int_a^x K(x,t) u(t)\, dt = f(x)$$

was originally attained by Volterra in 1896 in this manner.† More recently R. D. Carmichael has indicated the scope and power of this heuristic guide by deriving oscillation, comparison, and expansion theorems for various types of functional equations.‡

The values of λ determined from the equation

$$D(\lambda) = 0$$

are called principal values (*Eigenwerte*) and as a set form the *spectrum* of the integral equation.

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*Sur les équations différentielles linéaires du second ordre. *Journal de Mathématiques*, vol. 1 (1836), pp. 106-186; see in particular, p. 186. See also: M. Bôcher: *Leçons sur les méthodes de Sturm.* Paris (1917), 118 p.

†Sulla inversione degli integrali definiti. *Atti de Torino,* vol. 31 (1895-96), pp. 311-323, 400-408, 557-567, 693-708; in particular, p. 311 et seq.

‡Algebraic Guides to Transcendental Problems. *Bulletin of the Amer. Math. Soc.*, vol. 28 (1922), pp. 179-210.

Under usual conditions the principal values form a discrete set and this set is called the *point spectrum* of the equation. Under singular conditions, solutions may exist for the homogeneous equation for a continuous set of values of λ, and this set is called the *continuous spectrum* of the equation. A simple example of such a spectrum has been given by E. Picard,* who considered the integral equation

$$u(x) = \lambda \int_{-\infty}^{\infty} e^{-|x-t|} u(t)\, dt \ ,$$

which has the spectrum

$$\lambda = \tfrac{1}{2}(1 + a^2) \ .$$

The general solution of this equation is

$$u(x) = A\, e^{aix} + B\, e^{-aix} \ .$$

Associated with the discrete spectrum we have a set of principal functions (*Eigenfunktionen*), $u_1(x)$, $u_2(x)$, $u_3(x)$, $\cdots$, which satisfy the homogeneous equation

$$u(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt = 0$$

when λ assumes the corresponding principal values.

If the kernel $K(x,t)$ is real and symmetric, $K(x,t) \equiv K(t,x)$, then the principal values are real and the kernel may be expanded in terms of the principal functions, as follows:

$$K(x,t) = \sum_{i=1}^{\infty} u_i(x)\, u_i(t)/\lambda_i \ ,$$

provided the series in the right hand member is uniformly convergent.

Many years before the introduction of the concepts of integral equations into analysis, special systems of functions called *orthogonal functions* had been employed. These functions have the property

$$\int_a^b G(t)\, u_i(t)\, u_j(t)\, dt = 0 \ , \ i \neq j \ ,$$

where $G(t)$ is a weighting function. It is obviously possible to set $G(t) = 1$ without affecting the generality of the situation since the system of orthogonal functions can be written

$$v_i(x) = \sqrt{G(x)}\, u_i(x) \ .$$

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*Sur une example simple d'une équation singulière de Fredholm. *Annales de l'école normale supéricure*, vol. 28, 3rd ser. (1911), pp. 313-324.

Well-known systems of such functions include sin mx, cos mx, $P_m(x)$ (the Legendre polynomials), $H_m(x)$ (the Hermite polynomials), $L_m(x)$ (the Laguerre polynomials), etc. (See section 5, chapter 12).

If an arbitrary function $f(x)$ is expanded in a series of *normalized orthogonal* functions

$$f(x) = \sum_{i=1}^{\infty} f_i u_i(x) \quad ,$$

where by normalized we mean that

$$\int_a^b u_i^2(t)\, dt = 1 \quad ,$$

then the coefficients

$$f_i = \int_a^b f(t)\, u_i(t)\, dt$$

satisfy what is known as *Bessel's inequality*:

$$f_1^2 + f_2^2 + f_3^2 + f_4^2 + \cdots + f_n^2 \leqq \int_a^b [f(t)]^2\, dt \quad ,$$

where n is any integer.

If the equality sign instead of the inequality holds for every function $f(x)$ of integrable square, then the set of orthogonal functions is called *closed;* otherwise it is said to be *incomplete* or *open.* The set of functions 1, sin mx, cos mx, has been shown by A. Liapounoff* and A. Hurwitz† to be closed.

In 1907 E. Fischer‡ and F. Riesz§ both demonstrated the following result now known as the Fischer-Riesz theorem:

If a series $\sum_{i=1}^{\infty} f_i^2$ converges, then there exists a function $f(x)$ of integrable square for which the f_i are the coefficients of the expansion

$$f(x) = \sum_{i=1}^{\infty} f_i\, u_i(x) \quad ,$$

*First published in 1896 in the *Proc. of the Math Soc. of the University of Kharkov* and discussed by W. Stekloff: Sur un problème de la théorie analytique de la chaleur. *Comptes Rendus,* vol. 126 (1898), pp. 1022-1025. See also E. T. Whittaker and G. N. Watson: *A Course of Modern Analysis.* 3rd ed. Cambridge (1920), pp. 180-182.

†Über die Fourierschen Konstanten integrierbarer Funktionen. *Math. Annalen,* vol. 57 (1903), pp. 425-446; in particular, p. 429.

‡Sur la convergence en moyenne. *Comptes Rendus,* vol. 144 (1907), pp. 1022-1024.

§Sur les systèmes orthogonaux et l'équation de Fredholm. *Comptes Rendus,* vol. 144 (1907), pp. 615-619, 734-736.

where the $u_i(x)$ form an infinite system of normalized orthogonal functions.

The situation which was so spectacularly revealed by the theory of integral equations immediately challenged generalization. This was forthcoming in 1908 in a series of monographs by D. Hilbert which were later collected into the treatise: *Grundzüge einer allgemeinen Theorie der Linearen Integralgleichungen,* Leipzig (1912). The pathway was laid open through the theory of systems in infinitely many variables. E. H. Moore (1862-1932) in his fruitful concept of *general analysis** saw a powerful unifying principle which he expressed as follows:

"The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features".

M. Fréchet in his thesis: *Sur quelques points du calcul fonctionnel,* Paris (1906), Red. Circ. Mat. Palermo, vol. 22 (1906), pp. 1-74, and in his more recent volume: *Les Espaces Abstraits*, Paris (1928) has explored the consequences of very general definitions. S. Pincherle in his numerous contributions and in particular in his book written in collaboration with U. Almaldi in 1901 (see *Bibliography*) gave an unrestricted view of the depth and power of the concepts of linear operations in the abstract.

These now classical memoirs have born much fruit, which we find in the more recent contributions of N. Wiener, A. Wintner, M. H. Stone, J. von Neumann, H. Weyl, S. Banach, and many others. (For specific reference, see *Bibliography*).

In view of the highly technical nature of these generalizations, it seems best to postpone further discussion to chapter 12. In the last two sections of that chapter a brief account of the history and present status of the theory of forms in infinitely many variables and of the abstract theory of linear operators is given.

**Introduction to a form of General Analysis.* Lectures delivered in 1906 in New Haven and published in 1910; On the Foundations of the Theory of Linear Integral Equations. *Bulletin Amer. Math. Soc.*, vol. 18 (1911-1912), pp. 334-362; *Proc. of the Cambridge International Congress* (1912), vol. 1, pp. 230-255.

CHAPTER II

Particular Operators

1. Introduction. In the first chapter we have attempted to set forth a general view of the concept of operator. We have traced the historical development from its early origins in algebraic analogy, through the formalism of the last century, to the deep and broad current of modern speculation. We have attempted to show the central position which the concept occupies in numerous applications.

In the present chapter we shall set forth in some detail a description and classification of several types of operators which will concern us in later pages. We shall mention somewhat lightly the definition attained by Fourier transforms and the inclusive generalization of the Stieltjes-Lebesgue integral. Not, let it be said, to minimize the importance of these matters, but because they have been treated elsewhere far more extensively than could be done in the compass of this volume.*

Our own approach to the subject will be made by means of expansions in terms of the elementary operator $z = d/dx$. In spite of what might be regarded as an unhappy formalism thus introduced, we shall find that a majority of the specific applications are easily attained in this manner. Linear differential and difference equations are naturally included and the theory of both Volterra and Fredholm integral equations, at least so far as their formal aspects are concerned, may be discussed without difficulty in terms of the elementary operator.

Since we propose in later pages to throw an unusual burden upon the operator z, it is not out of place to say a word about its generality. The derivative of a function is generally defined as

$$\lim_{h=0} [f(x+h) - f(x)]/h .$$

By pushing this limit to its logical conclusion, K. Weierstrass (1815-1897) was able to construct a function which is continuous over an interval and yet does not possess a derivative at any point of the interval.† The derivative is thus envisaged as a property *im kleinen* of a function. At a point it may inherit nothing from other portions of the function. It depends entirely upon the oscillatory character of the function within infinitesimal intervals. Let it be

*See, for example, references in the *Bibliography* under N. Wiener, M. H. Stone, and J. von Neumann.

†First published by P. du Bois-Reymond: *Journal für Mathematik*, vol. 79 (1874), p. 29.

added, however, that supplementary hypotheses may entirely alter the situation. If, for example, we consider the mode of approach of h to zero and assume that the limit is the same for every path, then the function is defined throughout the region of its analytic continuation.

An integral, on the other hand, as the limit of a sum avoids this restrictive *im kleinen* aspect of the derivative. It inherits from all parts of the interval. Its generality in the Stieltjes-Lebesgue form is a strange contrast to the narrow limitations of its inverse. It is a property *im grossen* of the function which defines it.

Thus to base a theory of operators upon z rather than upon $1/z$ would seem to impose an unfortunate blemish upon it. On the one hand, the use of z in general permits an easier formal manipulation as one may surmize from the fact that differential equations preceded integral equations by two centuries. On the other hand, its definition seems to limit application to a class of functions far more restricted than the class to which its inverse applies. One may reflect, however, upon the fact that $1/z$ may be expressed in terms of z by means of the formal equivalence

$$1/z = (1 - e^{-xz})/z \;. \; *$$

The first operator, which may be interpreted as an integral of Lebesgue type, has a very general application. The second operator, since it is non-singular and expansible only as a power series in z, is limited to the class of unlimitedly differentiable functions. However, in the domain of functions common to both operators the results obtained are identical. Is there any way in which the generality of the first can be wholly or partially restored to the second?

It is seen that this may be accomplished in large measure if the function to which the transformations are to be applied is defined by a Fourier integral,

$$f(x) = (1/2\pi) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\lambda i (x-\mu)} f(\mu)\, d\mu\, d\lambda \;. \qquad (1.1)$$

If then an operator of the form $F(x, z)$ be applied to $f(x)$ we can give meaning to the transformation by means of the following formal equivalence:

$$F(x,z) \to f(x) \qquad (1.2).$$

$$= (1/2\pi) \int_{-\infty}^{\infty} F(x, i\lambda)\, e^{\lambda i x}\, d\lambda \int_{-\infty}^{\infty} e^{-\mu\lambda i} f(\mu)\, d\mu \;.$$

It is clear that in this manner a certain measure of generality has been restored to the differential operator and that the limitation

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*See section 6.

upon the range of functions to which it applies has been transferred to the convergence of the integrals.

2. Polynomial Operators. In this and ensuing sections we shall lay the foundations of our study by considering a few common types of operators.

If we designate by the symbol z the differential operator d/dx, we may then define the elementary polynomial operators to be z, z^2, $z^3, \cdots, z^n$, from which the general polynomial operator is constructed through their linear combination:

$$F_p(x, z) = A_0(x) + A_1(x)z + A_2(x)z^2 + \cdots + A_p(x)z^p \ ,$$

where the $A_i(x)$ are functions of x which share a common region of definition.

It is obvious that this is the operator of ordinary linear differential equations.

3. The Fourier Definition of an Operator. As we have stated in the preceding chapter, the definition formulated in (1.2) goes back to J. Fourier, by whom it was stated in a slightly different form to apply to the case of fractional derivatives. An instructive application of it was made in 1895 by T. Levi-Civita in solving the Volterra integral equation,

$$f(x) = \int_c^x K(x-t)\, u(t)\, dt \ ,$$

with special reference to the inversion of the Abel integral in which the kernel is $(x-t)^a$.*

N. Wiener has suggested the following extended definition:†

Let us consider the function

$$F(x, z) \to f_\delta(x) \tag{3.1}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x, -i\lambda) S_\delta(\lambda) e^{-i\lambda x}\, d\lambda \int_{-\infty}^{\infty} e^{i\lambda\mu} f(\mu)\, d\mu \ ,$$

where $S_\delta(\lambda)$ is defined to be

$$S_\delta(\lambda) = 1 \ , \ [-\delta \leqq \lambda \leqq \delta] \ ,$$
$$S_\delta(\lambda) = \varphi(\lambda - \delta) \ , \ [\delta \leqq \lambda \leqq \delta + 1] \ ,$$
$$S_\delta(\lambda) = \varphi(\delta - \lambda) \ , \ [-\delta - 1 \leqq \lambda \leqq -\delta] \ ,$$
$$S_\delta(\lambda) = 0 \ , \ |\lambda| \geqq \delta + 1 \ ,$$

in which $\varphi(\lambda)$ is a function possessing the following properties:

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*Sull'inversione degli integrali definiti nel campo reale. *Atti di Torino*, vol. 31 (1895), pp. 25-51. See also the author's study: *A Survey of Methods for the Inversion of Integrals of Volterra Type.* Indiana University Studies, Nos. 76, 77 (1927), pp. 54-57.

†See *Bibliography*: Wiener: (2).

(a) $\varphi(\lambda)$ is defined over the interval (0,1) and has derivatives of all orders;

(b) $\varphi(0) = 1, \quad \varphi(1) = 0$;

(c) $\varphi^{(n)}(0) = \varphi^{(n)}(1) = 0$ for all positive integral values of n.

An example of such a function is given by

$$\varphi(\lambda) = \int_\lambda^1 e^{1/(t^2-t)}dt \Big/ \int_0^1 e^{1/(t^2-t)}dt \ .$$

The kernel, $S_\delta(\lambda)$, is thus seen to be a function of the type exhibited in the figure where the transition curve $\varphi(\lambda)$ makes infinite contact with the two branches A and B.

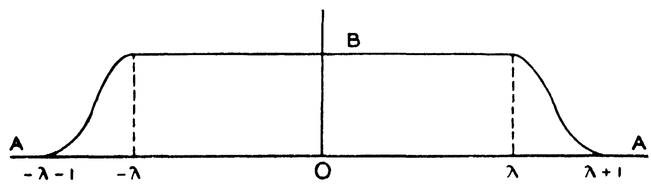

FIGURE 1

Then, if $f(x)$ is summable and of summable square over every finite range and if there are numbers A and k such that $f(x) \leqq A\, x^k$ for every x of sufficiently great magnitude, it can be proved that

$$\underset{\delta \to \infty}{\text{l. i. m.}}\, F(x, z) \to f_\delta(x)$$

exists, where l.i.m. is the abbreviation for *limit in the mean.* This limit is to be regarded as the operational equivalent of $F(x,z) \to f(x)$.

A sequence of functions of summable square, $f_1(x), f_2(x), \cdots, f_n(x)$, is said to *converge in the mean* toward a function $f(x)$ if

$$\lim_{n \to \infty} \int_a^b [f(x) - f_n(x)]^2\, dx = 0 \ .$$

A necessary and sufficient condition for convergence in the mean is that for every positive number ε, there exists a number n such that

$$\left| \int_a^b [f_{n'}(x) - f_n(x)]^2 \ dx \right| < \varepsilon \ ,$$

for all values of $n' > n$. This theorem is due to E. Fischer (*Comptes Rendus*, vol. 144 (1907), pp. 1022-1024) and F. Riesz (*Ibid.*, pp. 615-619; 734-736).

In order to prove the theorem let us first define

$$I_A(f, g) = \int_A |f - g|^2\, dx \ ,$$

where A is the measure of the point set over which the integral is taken.

By hypothesis there exists for every ε an integer $m(\varepsilon)$, such that

$$I_A(f_m, f_m) < \varepsilon \ , \qquad A = b - a \ , \qquad m, n > m(\varepsilon) \ .$$

Now assume that

$$|f_m - f_m| > \eta \ .$$

We see that the point set over which this inequality holds cannot exceed $\alpha = \varepsilon/\eta^2$, or if $\varepsilon = \eta^3$, it cannot exceed $\alpha = \eta$. Now let $\eta_1 > \eta_2 > \eta_3 > \cdots > \eta_n > \cdots$ be a set of arbitrarily small quantities for which the sum $\eta_1 + \eta_2 + \cdots$ converges, and let $m_1(\varepsilon)$, $m_2(\varepsilon)$, $\cdots$ be a set of integers such that

$$|f_m - f_n| < \eta_p \ , \qquad m, n > m_p(\varepsilon) \ , \qquad p = k \ , \ k+1, \cdots \ .$$

It will be evident that these inequalities hold simultaneously over a set of points of measure $A_k = A - \alpha_k$, where $\alpha_k = \eta_k + \eta_{k+1} + \cdots$. Evidently the set A_k is a subset of A_{k+1}.

From this we conclude that the sequence $\{f_m\}$, $m > m_k(\varepsilon)$, converges uniformly in A_k and converges at every point in the limit set A_∞. This limit set is obviously equal to A, with the possible exception of a set α_∞ of measure zero. Hence we obtain a limit function

$$f(x) = \lim_{k=\infty} f_k(x) \ ,$$

when x is in A_∞, which we define to be zero over α_∞.

In order to show that $f(x)$ is a function of integrable square, consider

$$I_{A_k}(f_m, f_n) \leqq I_A(f_m, f_n) < \varepsilon \ , \qquad m, n > m(\varepsilon) \ .$$

Since the convergence of f_n upon A_n is uniform, we can allow n to become infinite and hence derive

$$I_{A_k}(f_m, f) < \varepsilon \ , \qquad m > m(\varepsilon) \ .$$

Since, moreover, A_k is included in the set A_{k+1}, we can let k become infinite and thus obtain

$$I_{A\infty}(f_m, f) < \varepsilon \ , \qquad m > m(\varepsilon) \ .$$

From this we derive the inequality

$$\int_{A\infty} f^2 \, dx \leqq 2 I_{A\infty}(f_m, f) + 2 \int_{A\infty} f_m^2 \, dx < 2\varepsilon + 2M \ ,$$

which proves that $f(x)$ is a function of integrable square.

From the above argument we see that $f(x)$ is uniquely determined to within a function defined over a set of measure zero, that is to say, a function $g(x)$, not identically zero, which satisfies the equation

$$\int_a^b g^2(x) \, dx = 0 \ .$$

Such a function we shall call a *null function*. Two functions which differ at most by a null function will be said to be equivalent *almost everywhere*.

A definition similar in type to that proposed by Wiener has been made the basis of the study of G. Giorgi and F. Sbrana. These writers, however, interchange the rôle of the operator and the function

and thus define

$$F(z) \to f(x) = \int_{-\infty}^{\infty} f(t)\, G(x-t)\, dt = \int_{-\infty}^{\infty} f(x-t)\, G(t)\, dt \ , \tag{3.2}$$

where we abbreviate

$$G(t) = (1/2\pi\, i) \int_{-i\,\infty}^{i\,\infty} F(w)\, e^{wt} dw \ . \tag{3.3}$$

Their mode of derivation is as follows:

It is obvious that we can write the function $f(x)$ in the form

$$f(x) = \tfrac{1}{2}z \to \left\{ \int_{-\infty}^{x} f(t)\, dt - \int_{x}^{\infty} f(t)\, dt \right\} , \tag{3.4}$$

provided proper restrictions are imposed upon $f(t)$ as, for example, $f(t) = O(t^{-h})$, $h > 1$.

Recalling the identity

$$I(t) = (1/\pi) \int_{-\infty}^{\infty} (\sin \lambda\, t/\lambda)\, d\,\lambda = \begin{cases} 1, & t > 0 \ , \\ -1, & t < 0 \ . \end{cases}$$

we can write (3.4) in the form

$$f(x) = \tfrac{1}{2}z \to \int_{-\infty}^{\infty} f(t) I(x-t)\, dt \ .$$

If we now operate with $F(z)$, this becomes

$$F(z) \to f(x) = \int_{-\infty}^{\infty} f(t)\, G(x-t)\, dt \ ,$$

where we use the abbreviation $G(x-t) = \tfrac{1}{2}\, z\, F(z) \to I(x-t)$.

Since we can also write

$$I(x) = (1/2\pi i) \int_{-\infty}^{\infty} \{ (e^{i\lambda x} - e^{-i\lambda x}) / \lambda \}\, d\lambda$$

$$= (1/2\pi i) \int_{-i\infty}^{i\infty} \{ (e^{wx} - e^{-wx}) / w \}\, dw \ ,$$

it is clear that $G(x)$ can be expressed in the form

$$G(x) = \tfrac{1}{2}\, z\, F(z) \to I(x) = (1/4\pi\, i) \int_{-i\,\infty}^{i\,\infty} [F(w)\, e^{wx} + F(-w)\, e^{-wx}]\, dw$$

$$= (1/2\pi i) \int_{-i\,\infty}^{i\,\infty} F(w)\, e^{wx}\, dw \ ,$$

provided the proper convergence properties are possessed by $F(w)$.

A more extended account of the Fourier operator will be given in section 5 of chapter 6 in connection with its application to the inversion problem of differential equations of infinite order with constant coefficients.

PROBLEM

Given the Fredholm transform

$$T(u) = \int_a^b K(x-t)\, u(t)\, dt \ ,$$

and a function $f(x)$ such that

(a) $f(x)$ is integrable in (a,∞), $f(x) \equiv 0$, $x < a$, $0 \leq a$;

(b) $K(x)$ is finite, continuous, and integrable in $(0,\infty)$;

(c) $h(z) = \int_0^\infty K(t) \cos \pi z t\, dt \ , \qquad k(z) = \int_0^\infty K(t) \sin \pi z t\, dt \ ,$

do not vanish simultaneously for any value of z in $(0,\infty)$.

Show that if $T(u) = f(x)$, we have the inversion

$$u(t) = \text{Real part of} \int_0^\infty dz \int_0^\infty \frac{e^{i\pi z(x-t)}}{h(z) + ik(z)} f(x)\, dx, \; t > a \ ,$$

$$u(t) \equiv 0, \; t < a \ .$$

(Levi-Civita: *loc. cit.*).

Hint: Write

$$u(t) = \int_0^\infty dz \int_0^\infty \cos \pi z(y-t)\, u(y)\, dy \ ;$$

then show that when

$$F(x,y,z) = \text{Real part of } e^{i\pi z(x-t)}/[h(z) + i\,k(z)] \ ,$$

$$\cos \pi z(y-t) = \int_t^\infty K(x-t)\, F(x,y,z)\, dx \ .$$

4. The Operational Symbol of von Neumann and Stone. A still more comprehensive definition has been given by J. von Neumann and M. H. Stone. In the work of the latter the operator is represented by a Lebesgue-Stieltjes integral of the form

$$\int_{-\infty}^\infty F(\lambda)\, dQ(E_\lambda f, g) \ ,$$

where f and g are elements in abstract Hilbert space, E_λ is a family of special operators defined for all real values of λ, $Q(f,g)$ a numerically valued function, and $F(\lambda)$ a function which represents the operational interpretation of the symbol. If we set

$$F(\lambda) = F(x, \lambda i)$$

and

$$E_\lambda f = (1/2\pi) \int_{-\infty}^{\lambda} e^{ivx}\, dv \int_{-\infty}^{\infty} e^{-ix\mu} f(\mu)\, d\mu \ ,$$

the operator of Stone will reduce to (1.2).

The operator of von Neumann and Stone was developed in connection with their generalization of the spectral theory of integral equations and infinite quadratic forms initiated by the researches of D. Hilbert. An account of the spectral theory is given in chapter 12. From the formal results which are developed there the reader can determine the specializations of the general operator which apply in particular cases. The full generality of the operator in the form given above can only be appreciated by a consultation of the original sources.

In view of the importance which the Stieltjes integral has assumed in modern generalizations of the linear operator a brief résumé of its special features will be given here. The integral was first defined by T. J. Stieltjes in connection with his classical researches on continued fractions (see section 7, chapter 1). Extensive accounts of the integral will be found in E. W. Hobson's *Theory of Functions of a Real Variable*, vol. I (3rd ed.), Cambridge (1927), pp. 538-546, in H. Lebesgue's *Leçons sur l'intégration*, (2nd ed.), Paris (1928), pp. 252-313, and in C. J. de la Vallée Poussin's *Integrales de Lebesgue*, Paris (1916) pp. 1-27. Formulations of the integral most useful in the theory of linear operators have been given by M. H. Stone [See *Bibliography*: Stone (2), pp. 158-165; 198-221] and A. Wintner [See *Bibliography*: Wintner (5), pp. 74-105]. The original definition of Stieltjes, valid in the domain of Riemann integration, was extended by J. Radon: Theorie und Anwendungen der absolut additiven Mengenfunktionen, *Sitzungsberichte der Akademie der Wissenschaften*, Wien, (1913), vol. 122, pp. 1295-1438, in particular, pp. 1342-1351. This extension enlarged the domain of functions integrable in the Riemann-Stieltjes sense to include those integrable in the Lebesgue-Stieltjes sense. The Lebesgue-Stieltjes integral thus formulated is often referred to as the Radon-Stieltjes integral. The literature of the subject is now very extensive.*

By a *Stieltjes integral* of a continuous function, $f(x)$, with respect to a function of limited variation, $v(x)$, we mean an integral of the form

$$I = \int_a^b f(x)\, dv \ .$$

which is defined as follows:

Consider the sum

$$I_n = f(t_1)[v(x_1) - v(a)] + f(t_2)[v(x_2) - v(x_1)] + \cdots + f(t_n)[v(b) - v(x_{n-1})] \ ,$$

*In addition to the sources already cited the reader will find the following instructive references: E. B. Van Vleck: Haskin's Momental Problem and its Connection with Stieltjes' Problem of Moments, *Trans. Amer. Math. Soc.*, vol. 18 (1917), pp. 326-330; G. A. Bliss: Integrals of Lebesgue. *Bull. of the Amer. Math. Soc.*, vol. 24 (1917-18), pp. 1-47; T. H. Hildebrandt: On Integrals Related to and Extensions of the Lebesgue Integrals. *Bull. of the Amer. Math. Soc.*, vol. 24 (1917-18), pp. 113-144; 177-202; H. E. Bray: Elementary Properties of the Stieltjes Integral. *Annals of Math.*, vol. 20 (2nd ser.) (1918-19), pp. 177-186.

where $x_1, x_2, \cdots, x_{n-1}$ are a set of points in the interval (a, b) such that

$$a < x_1 < x_2 < \cdots < x_{n-1} < b \ ,$$

and such that $(x_i - x_{i-1}) \to 0$ as $n \to \infty$. The points t_i belong to the intervals $x_i - x_{i-1}$.

We then define as the Stieltjes integral the following limit:

$$I = \lim_{n=\infty} I_n \ .$$

The proof that this limit exists, under the assumptions made concerning $f(x)$ and $v(x)$, can be established easily.*

If $v(x)$ is constant except for a denumerable set of discontinuities of positive saltus $\{S_i\}$ at the points $t_1, t_2, \cdots, t_m, \cdots$, then I reduces to the series

$$I = S_1 f(t_1) + S_2 f(t_2) + \cdots + S_m f(t_m) + \ldots$$

More generally, the function $v(x)$ may be regarded as a density function defined over measurable sets of points of the continuum between a and b. Designating by $\Delta_i v$ the value of this density function over an interval $\Delta_i x$, and t_i a point interior to $\Delta_i x$, we define the integral as the limit

$$I = \lim \sum_{i=1}^{\infty} f(t_i)\, \Delta_i v \ , \qquad \Delta_i x \to 0 \ ,$$

provided the limit exists.

5. *The Operator as a Laplace Transformation.* It will be convenient later to express an operator as a Laplace transformation. For this purpose let us define $f(x)$ as follows:

$$f(x) = \int_L e^{xt}\, v(t)\, dt \ ,$$

where $v(t)$ is a function which generates $f(x)$ when L is a properly defined path in the complex plane.

The operational equivalent of $F(z) \to f(x)$ is clearly contained in a formal manner in the integral representation

$$F(z) \to f(x) = \int_L e^{xt}\, F(t)\, v(t)\, dt \ . \qquad (5.1)$$

If the path L in particular is the real interval $(-\infty, a)$ and if

$$\lim_{t=-\infty} e^{xt}\{F(t)\, v(t)\}^{(n)} = 0$$

for all values of n, then (5.1) can be expanded in the following series by means of continued integration by parts:

$$F(z) \to f(x) = e^{xa} F(a)\, v(a)/x - \int_{-\infty}^{a} e^{xt}\{F(t)\, v(t)\}'\, dt/x$$

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*See, for example, Lebesgue: *loc. cit.*, p. 253.

$$= e^{xa}\{F(a)\,v(a)/x - (F\,v)'/x^2 + (F\,v)''/x^3 - \cdots$$
$$+ (-1)^{n-1}(F\,v)^{(n-1)}/x^n\} +$$
$$(-1)^n \int_{-\infty}^{a} e^{xt}\{F(t)\,v(t)\}^{(n)}\,dt/x^n$$
$$= e^{xa}\{F(a)\,v(a)/x - (F\,v)'/x^2 + (F\,v)''/x^3 - \cdots\}\,. \qquad (5.2)$$

A similar series in terms of inverse factorials is obtained prc vided we assume that, for all positive integral values of n,

$$\lim_{t=-\infty} e^{(x+n)t}\,w_n(t) = 0\ ,$$

where we employ the abbreviation

$$w_0(t) = F(t)\,v(t)\ \ ,\ w_n(t) = \{\frac{d}{dt}\,w_{n-1}(t)\}\,e^{-t}\ .$$

Under this assumption we integrate by parts and thus obtain

$$F(z) \to f(x) = e^{xa}\,w_0(a)/x - \int_{-\infty}^{a} e^{(x+1)}\,w_1(t)\,dt/x$$
$$= e^{xa}\{w_0(a)/x - w_1(a)\,e^a/x(x+1) + w^2(a)\,e^{2a}/x(x+1)(x+2)$$
$$- w_3(a)\,e^{3a}/x(x+1)(x+2)(x+3) + \cdots\}\,. \qquad (5.3)$$

6. *Polar Operators.* By the term *polar operator* we shall mea an operator expansible as a power series in $1/z$,

$$B(x, 1/z) = b_1(x)/z + b_2(x)/z^2 + b_3(x)/z^3 + \cdots\ , \qquad (6.1)$$

where the $b_i(x)$ are functions sharing a common region of definitio and where, by definition,

$$1/z \to u(x) = \int_c^x u(t)\,dt\ . \qquad (6.2)$$

We should first notice the significant fact that the polar operato $1/z$ can also be represented as a non-singular operator of infinite or der. This is accomplished in the following manner:

Let us refer to the transformation

$$u(t) = u(x) + (t-x)\,u'(x) + (t-x)^2\,u''(x)/2! + \cdots$$
$$= e^{(t-x)z} \to u(x) \qquad (6.3)$$

as the *Taylor transform* of $u(x)$.

If we apply this transformation to $u(t)$ in (6.2), we get

$$1/z \to u(x) = \int_c^x e^{(t-x)z}\,dt \to u(x)$$

$$1/z \to u(x) = \{1 - e^{(c-x)z}\}/z \to u(x) \ .$$

Hence we see the essential equivalence for all finite values of c of the two operators $1/z$ and $\{1 - e^{(c-x)z}\}/z$.

The equivalence is seen to have limitations, however, if we observe on the one hand the great generality that can be assumed for $u(x)$ when (6.2) is a Lebesgue integral and on the other hand the restriction that the function operated upon by $\{1 - e^{(c-x)z}\}/z$ must have derivatives of all orders. The equivalence can be partly restored, however, if we make use of (1.2) and replace $F(x, i\lambda)$ by

$$\{1 - e^{(c-x)\lambda i}\}/\lambda i \ .$$

We shall now prove that the function

$$1/z^n \to u(x) = \int_c^x ds \int_c^s ds \cdots \int_c^s u(s)\, ds$$

can be reduced to the single integral

$$1/z^n \to u(x) = \int_c^x (x-s)^{n-1} u(s)\, ds/(n-1)! \ . \qquad (6.4)$$

Let us assume that the formula is true for $n = k$. We shall then have

$$1/z \to \{1/z^k \to u(x)\} = \int_c^x dx \int_c^x (x-s)^{k-1} u(s)\, ds/(k-1)! \ .$$

Applying to this integral the Dirichlet formula for integration over the triangle $s = c$, $s = x$, $x = x$, we find

$$(1/z^{k+1}) \to u(x) = \int_c^x u(s)\, ds \int_s^x (x-s)^{k-1} dx/(k-1)!$$

$$= \int_c^x (x-s)^k u(s)\, ds/k! \ .$$

Since the formula is obviously true for $n = 1$, we now establish its general validity by induction.

Because of the frequent application of *Dirichlet's formula* in the theory of operators we adjoin a brief discussion of it here.

Consider a convex closed curve which lies within a rectangle formed by the lines $x = a$, $x = b$, $y = c$, $y = d$, and for which the points PR and QS are the extreme values of x and y as shown in the figure. (Figure A). Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be continuous functions representing the two arcs PQR and PSR respectively and let $x_1 = x_1(y)$ and $x_2 = x_2(y)$ be, similarly, the equations of the two arcs SPQ and SRQ. Now consider the double integral

$$I = \int_a^b dx \int_{y_1}^{y_2} K(x,y)\, dy$$

where $K(x,y)$ is a function integrable within the contour. It is clear from the definition of a double integral that I may be evaluated also by integrating first with respect to x and this leads to the new form

$$I = \int_c^d dy \int_{x_1}^{x_2} K(x,y)\, dx \; .$$

Equating these values we obtain the general formula

$$\int_a^b dx \int_{y_1}^{y_2} K(x,y)\, dy = \int_c^d dy \int_{x_1}^{x_2} K(x,y)\, dx \; .$$

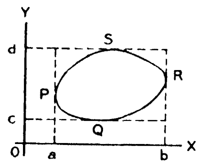

FIGURE A.

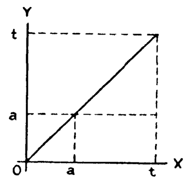

FIGURE B.

Dirichlet's formula follows as a special case by applying the general formula to the triangular region bounded by the lines $x = a$, $x = t$, $x = y$, for which we have (Figure B)

$$\int_a^t dx \int_a^x K(x,y)\, dy = \int_a^t dy \int_y^t K(x,y)\, dx \; .$$

This formula was first given by P. G. L. Dirichlet (1805-1859) in 1837: Sur les séries dont le term général depend de deux angles, et qui servent à exprimer des fonctions arbitraires entre des limites données, *Journal für Math.*, vol. 17 (1837), pp. 35-36, in particular, p. 45.

A useful generalization of the formula was made by W. A. Hurwitz: Note on Certain Iterated and Multiple Integrals, *Annals of Mathematics*, vol. 9 (1908), pp. 183-192, who established the validity of the interchange of the variables of integration for a kernel of the form

$$K(x,y) = (t-x)^{\lambda-1}\,(y-a)^{\mu-1}\,(x-y)^{\nu-1} f(x,y) \;\; , \; \lambda, \mu, \nu, > 0 \; ,$$

where $f(x,y)$ is a continuous function.*

Let us now without loss of generality set $c = 0$.* Applying the Taylor transform to $u(s)$ we find the formal equivalence of the following operations:

$$1/z \to f(x) = (1 - e^{-xz})/z \to f(x) \;\; ,$$

$$1/z^2 \to f(x) = \{1 - (1 + xz)\, e^{-xz}\}/z^2 \to f(x) \;\; ,$$

. . . .

$$\begin{aligned} 1/z^n \to f(x) &= \{1 - [1 + xz + x^2z^2/2! + \cdots \\ &\qquad + x^{n-1} z^{n-1}/(n-1)!]\, e^{-xz}\}/z^n \to f(x) \\ &= \{x^n/n - x^{n+1} z/(n+1) + x^{n+2} z^2/2!\,(n+2) \\ &\qquad - \cdots\}/(n-1)! \to f(x) \; . \end{aligned}$$

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*For a more extended account of the conditions under which this interchange of limits may be effected see J. Pierpont: *The Theory of Functions of Real Variables,* vol. 1 (Boston) 1905, pp. 395-398, and E. W. Hobson: *The Theory of Functions of a Real Variable,* vol. 1 (2nd edition), Cambridge (1921), pp. 479 et seq.

The functions

$$Q_n(x,z) = \{1 - [1 + xz + x^2z^2/2! + \cdots$$
$$+ x^{n+1} z^{n-1}/(n-1)!]\, e^{-xz}\}/z^n \ , \quad n = 1, 2, 3, \cdots ,$$

may be conveniently referred to as *generatrix functions.*

They satisfy the differential equation*

$$L(Q_n) \equiv z\,\partial^2 Q_n/\partial z^2 + (1 + xz + n)\ \partial Q_n/\partial z + n\,x\,Q_n = 0 \ , \tag{6.5}$$

as may be proved by direct substitution of the integral

$$Q_n(x, z) = \int_0^x (x-t)^{n-1} e^{(t-x)z}\, dt/(n-1)!$$
$$= \int_0^x s^{n-1} e^{-sz}\, ds/(n-1)!$$

and its first two z-derivatives in the left member of (6.5).

We thus get

$$L(Q_n) \equiv \int_0^x \{z\, s^{n+1} - (1 + xz + n)\ s^n + nx\, s^{n-1}\} e^{-sz}\, ds/(n-1)! \ .$$

Integrating the second term by parts once and the third term twice, we obtain

$$L(Q_n) \equiv \int_0^x \{z\, s^{n+1} - (1 + xz + n)\ z\, s^{n+1}/(n+1)$$
$$+ x\, z^2 s^{n+1}/(n+1)\}\, e^{-sz}\, ds/(n-1)!$$
$$+ \{-(1 + xz + n)/(n+1) + 1 + xz/(n+1)\}\, x^{n+1} e^{-xz}/(n-1)! \ ,$$

which is seen to reduce to zero.

Since there thus exists a formal equivalence between the polar operators $1/z$, $1/z^2, \cdots, 1/z^n$ and the generatrix functions which possess convergent Taylor's expansions throughout the entire z-plane for all finite values of x, the analytic theory of operators will obviously exhibit fundamental points of difference with the analytic theory of functions. The only exception to this is the case where $c = -\infty$ for which the generatrix function, $Q_n(x, z)$, reduces to $1/z^n$ and the polar operator has a genuine singularity.

It will be important also to notice that the polar operator (6.1) may be replaced by an integral of Volterra kind as follows:

Making use of (6.4) we write

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*This equation is due to L. F. Robertson.

$$B(x, 1/z) \to u(x)$$
$$= \int_0^x \{b_1(x) + b_2(x)(x-t) + b_3(x)(x-t)^2/2! \cdots\} u(t)\, dt$$
$$= \int_0^x K(x,t)\, u(t)\, dt \ ,$$

where we abbreviate

$$b_{n+1}(x) = (-1)^n \partial^n K(x,t)/\partial t^n \,\big|_{t=x} \ .$$

A variant form of considerable use is found for (6.1) by replacing $1/z^n$ by the integral

$$\int_0^\infty \{e^{-tz} t^{n-1}/(n-1)!\}\, dt \ .$$

We thus obtain for (6.1) the expression

$$B(x, 1/z) = \int_0^\infty e^{-tz} \{\sum_{n=1}^{\infty} t^{n-1}\, b_n(x)/(n-1)!\}\, \mathrm{dt} =$$
$$= \int_0^\infty e^{-tz} K(x, x-t)\, dt \ . \tag{6.6}$$

7. *Branch Point Operators.* One finds it interesting next to inquire whether an interpretation can be made of operators with branch points at the origin, as, for example, z^μ, $z^{-\mu}$ (μ a positive fraction), $\log z$, etc.

A long history (see section 5, chapter 1) is attached to the problem of assigning meaning to the operators z^μ and $z^{-\mu}$, but the interpretation of these symbols may now be said to have attained a logical and satisfactory form.

Slightly generalizing a definition stated independently by both N. H. Abel and G. Riemann, we shall mean by the *fractional integration* of a function $u(x)$ the operation

$$z^{-\nu} \to u(x) = {}_cD_x^{-\nu}\, u(x) = \int_c^x \{(x-t)^{\nu-1}/\Gamma(\nu)\}\, u(t)\, dt \ , \quad \nu > 0 \ , \tag{7.1}$$

and by *fractional differentiation*,

$$z^{m+\nu} \to u(x) = {}_cD_x^{m+\nu}\, u(x)$$
$$= \frac{d^{m+1}}{dx^{m+1}} \int_c^x \{(x-t)^{-\nu}/\Gamma(1-\nu)\}\, u(t)\, dt \ , \tag{7.2}$$
$$m = 0, 1, 2, \cdots ; \ 0 < \nu < 1 \ .$$

The constant c is a vital part of the operational symbol and is indispensable in certain problems. Abel let $c = 0$ in his original definition and J. Liouville, as we shall see later, stated a definition

which was equivalent to setting $c = -\infty$. The latter choice served to remove some ambiguities that can arise in application.

The real significance of the definitions just given attaches to the fact that the symbols obey the index law:

$$ {}_cD_x^{-\mu}\, {}_cD_x^{-\nu}\, u(x) = {}_cD_x^{-(\mu+\nu)}\, u(x) \; . \tag{7.3}$$

This can be proved as follows:
By definition we have

$$ {}_cD_x^{-\mu}\, {}_cD_x^{-\nu}\, u(x) = \int_c^x \{ (x-s)^{\mu-1}/\Gamma(\mu) \} \, ds $$
$$ \times \int_c^s \{ (s-t)^{\nu-1}/\Gamma(\nu) \} \, u(t) \, dt \; . $$

Applying to this integral Dirichlet's formula for the interchange of variables, we get

$$ {}_cD_x^{-\mu}\, {}_cD_x^{-\nu}\, u(x) = $$
$$ \int_c^x u(t)\, dt \int_t^x \{ (x-s)^{\mu-1} (s-t)^{\nu-1}/\Gamma(\mu)\Gamma(\nu) \} \, ds \; . $$

Making the transformation $y = (s-t)/(x-t)$ and recalling that

$$ \int_0^1 y^{\nu-1} (1-y)^{\mu-1}\, dy = \Gamma(\mu)\Gamma(\nu)/\Gamma(\mu+\nu) \; , $$

we shall have

$$ {}_cD_x^{-\mu}\, {}_cD_x^{-\nu}\, u(x) = \int_c^x (x-t)^{\mu+\nu-1}\, u(t)\, dt $$
$$ \times \int_0^1 y^{\nu-1}(1-y)^{\mu-1}\, dy/\Gamma(\mu)\, \Gamma(\nu) $$
$$ = \int_c^x (x-t)^{\mu+\nu-1}\, u(t)\, dt/\Gamma(\mu+\nu) $$
$$ = {}_cD_x^{-(\mu+\nu)}\, u(x) \; . $$

Formula (7.1) is seen to be an immediate extension of the polar operator (6.4) where n is replaced by ν and $(n-1)!$ by $\Gamma(\nu)$. This obvious generalization was the one used by Abel in solving the tautochrone problem.*

Liouville without specific reference to Abel formulated a second definition. Since is is clear that

*Solution de quelques problèmes a l'àide d'intégrales définies. *Oeuvres*, vol. 1, pp. 11-27; in particular, p. 17.

$$\frac{d^n}{dx^n} e^{ax} = a^n e^{ax} \quad \text{and} \quad \int_{-\infty}^{x} e^{at} t^{n-1} dt = a^{-n} e^{at} \; , \qquad R(a) > 0 \; ,$$

we may assume as a definition that

$$z^\nu \rightarrow e^{ax} = a^\nu e^{ax} \; , \quad R(a) > 0 \; , \quad \nu \text{ positive or negative.}$$

Hence any function expressible in the series

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}, \; R(a_n) > 0 \; ,$$

possesses the formal derivative

$$z^\nu \rightarrow f(x) = \sum_{n=0}^{\infty} c_n a_n^\nu e^{a_n x} \; .$$

But this definition is seen to be in agreement with the definition of (7.1) provided c is set equal to $-\infty$ and proper uniformity conditions are fulfilled. Thus we have

$$_{-\infty}D_x^\nu f(x) = \frac{d}{dx} \int_{-\infty}^{x} (x-s)^{-\nu} \sum_{n=0}^{\infty} c_n e^{a_n s} \, ds / \Gamma(1-\nu) \; .$$

Making the transformation $s = x - t$, we get

$$_{-\infty}D_x^\nu f(x) = \frac{d}{dx} \int_{0}^{\infty} t^{-\nu} \sum_{n=0}^{\infty} c_n e^{a_n(x-t)} \, dt / \Gamma(1-\nu)$$

$$= \sum_{n=0}^{\infty} c_n a_n^\nu e^{a_n x} \; .$$

An elegant method of attaining the same result is found in a generalization of Cauchy's well-known integral formula,

$$D_x^n f(x) = (n!/2\pi i) \int_c \{f(t)/(t-x)^{n+1}\} \, dt \; ,$$

where the path of integration is taken as a closed curve in the complex plane about the point $t = x$.*

Let us generalize this to fractional values by considering the function

$$_{-\infty}D_x^\nu f(x) = \frac{1}{(e^{-2\pi i\nu} - 1)(-1)^\nu \Gamma(-\nu)} \int_c \{f(t)/(t-x)^{\nu+1}\} \, dt \; ,$$

where the path of integration is now taken about the point $t = x$ and has one branch to infinity so chosen that the integral converges.

From the relation

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*This derivation is found in Laurent. (See *Bibliography*).

$$\Gamma(-\nu) = -\pi/\sin \pi\nu\,\Gamma(\nu+1) \quad ,$$

we see that

$$(e^{-2\pi i\nu} - 1)(-1)^{\nu}\,\Gamma(-\nu)$$
$$= (-1)^{\nu+1}\pi(\cos 2\pi\nu - i\sin 2\pi\nu - 1)/[\sin \pi\nu\,\Gamma(\nu+1)] \quad .$$

If in this formula we set $\nu = n$, where n is a positive integer, it reduces to the proper limiting value, $2\pi i/n!$.

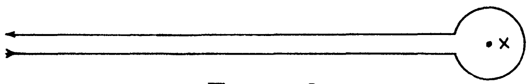

FIGURE 2.

Let us assume that ν is a negative number and let us evaluate this integral over the path indicated in the figure. We thus get

$${}_{-\infty}D_x^{\nu} f(x) = \int_{-\infty}^{x} \{f(t)/(x-t)^{\nu+1}\}\,dt/\Gamma(-\nu) \quad ,$$

which is identical with the Liouville definition. The assumption that ν is a negative number is of course removed in the obvious manner by differentiating the integral m times. It will be convenient for us later (see sections 4 and 7, chapter 8) to refer to the open circuit of figure 2 as a *Laurent circuit* in contrast to the closed *Cauchy circuit* employed in Cauchy's integral formula.

The integral ${}_{-\infty}D_x^{-\nu}\,x^{-n}$ is easily computed as follows:

$${}_{-\infty}D_x^{-\nu}\,x^{-n} = \int_{-\infty}^{x} \{t^{-n}/\Gamma(\nu)\,(x-t)^{1-\nu}\}\,dt \quad .$$

Making the substitution $x - t = xs/(s-1)$, we reduce this to

$${}_{-\infty}D_x^{-\nu}\,x^{-n} = x^{\nu-n}(-1)^{\nu}\int_{0}^{1} s^{\nu-1}(1-s)^{n-\nu-1}\,ds/\Gamma(\nu)$$
$$= x^{\nu-n}(-1)^{\nu}\Gamma(n-\nu)/\Gamma(n) \quad .$$

If we take the first derivative of this function and replace $1 - \nu$ by μ, we obtain Liouville's result,*

$${}_{-\infty}D_x^{\mu}\,x^{-n} = (-1)^{\mu}\,\Gamma(n+\mu)/[\Gamma(n)\,x^{n+\mu}] \quad . \qquad (7.4)$$

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*As a matter of fact, Liouville derived his formula in a somewhat different manner by means of the special Laplace transformation [see (5.1)],

$$\Gamma(n)/x^{n} = \int_{0}^{\infty} t^{n-1}\,e^{-xt}\,dt \quad .$$

Since the Liouville definition assumed that $z^{\mu} \to e^{ax} = a^{\mu}\,e^{ax}$, we get

$$z^{\mu} \to x^{-n} = \int_{0}^{\infty} t^{n+\mu-1}(-1)^{\mu}\,e^{-xt}dt/\Gamma(n) = (-1)^{\mu}\,x^{-n-\mu}\Gamma(n+\mu)/\Gamma(n) \quad .$$

It is thus clear that any function of the form

$$f(x) = \sum_{n=1}^{\infty} a_n x^{-n}$$

possesses a derivative (or integral) in the Liouville sense,

$$_{-\infty}D_x^{\mu} f(x) = (-1)^{\mu} x^{-\mu} \sum_{n=1}^{\infty} a_n \Gamma(n+\mu)/[\Gamma(n)\, x^n] \ . \qquad (7.5)$$

If on the other hand we use the generalized Abel-Riemann definition, we find

$$_cD_x^{-\nu} x^n = \int_c^x (x-t)^{\nu-1} t^n \, dt/\Gamma(\nu) \ .$$

Making the transformation $t = x - (x-c)s$, we can write this as

$$_cD_x^{-\nu} x^n = (x-c)^{\nu} \int_0^1 s^{\nu-1}[(x-c)(1-s)+c]^n \, ds/\Gamma(\nu)$$
$$= (x-c)^{\nu+n} \varphi_n(x,c)\, \Gamma(n+1)/\Gamma(n+1+\nu) \ ,$$

where we abbreviate

$$\varphi_n(x,c) = 1 + (n+\nu)c/(x-c) + (n+\nu)(n+\nu-1)c^2/(x-c)^2 2!$$
$$+ \cdots + (n+\nu)(n+\nu-1)\cdots(\nu+1)\, c^n/(x-c)^n n! \ .$$

From the special value $\varphi(x,0) = 1$ we obtain the familiar formulas

$$_0D_x^{-\nu} x^n = x^{\nu+n} \Gamma(n+1)/\Gamma(n+\nu+1) \ , \qquad n \geqq 0 \ ,$$
$$_0D_x^{\mu} x^n = x^{-\mu+n} \Gamma(n+1)/\Gamma(n-\mu+1) \ . \qquad (7.6)$$

Formulas (7.6) were derived by Peacock from an obvious generalization of the case of integral order. He properly deduces the values

$$z^{\frac{1}{2}} \to a = a\Gamma(1)/\Gamma(\tfrac{1}{2})x^{\frac{1}{2}} = a/(\pi x)^{\frac{1}{2}} \ , \quad z^{-\frac{1}{2}} \to a = 2a(x/\pi)^{\frac{1}{2}} \ ,$$

but is unsuccessful in his attempt to perform the differentiation

$$z^{\frac{1}{2}} \to (1-x^2)^{-\frac{1}{2}} \ .$$

From these results we can immediately construct the general derivative (or integral) for the function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \ .$$

We thus obtain

$$_0D_x^{\mu} f(x) = x^{-\mu} \sum_{n=0}^{\infty} a_n x^n \Gamma(n+1)/\Gamma(n-\mu+1) \ . \qquad (7.7)$$

It is now apparent that there exist essentially two definitions of fractional operators which have different domains of usefulness. A great deal of confusion has arisen over this fact as may be seen from arguments advanced by Kelland in favor of formula (7.5) instead of formula (7.7).

For negative values of n in Louville's definition Kelland devised the new formula

$$_{-\infty}D_x^{\mu}\, x^n = (-1)^{\mu}\,\Gamma(n+1)\sin n\pi\, x^{n-\mu}/[\Gamma(n-\mu+1)\sin(n-\mu)\pi] \ .$$

When n is an integer and μ a fraction, this yields the value 0; when $n - \mu$ is an integer and n a fraction, the value is ∞ ; when n and μ are both integers, $\sin n\pi/\sin(n-\mu)\pi$ is replaced by $1/\cos \mu\pi = (-1)^{\mu}$.

Kelland's argument runs as follows:

Since $\Gamma(p)\,\Gamma(1-p) = \pi/\sin p\pi$, we can write

$$\Gamma(-n) = -\pi/[\sin n\pi\,\Gamma(1+n)] \ ,$$

$$\Gamma(-n+\mu) = \pi/[\sin(-n+\mu)\pi\,\Gamma(1+n-\mu)] \ .$$

Using these values in the Liouville formula (7.5) in which n has been replaced by $-n$, we obtain (7.8). The obvious difficulties which are introduced in the fundamental definition (7.1), $c = -\infty$, due to the behavior of x^n at infinity, substantially nullify this procedure.

This leads to the general observation that the Liouville formula, $_{-\infty}D_x^{-\nu} f(x)$, is applicable in the case where

$$f(-x) = O(x^{-\varepsilon-\nu}) \ , \qquad \varepsilon > 0 \ ,$$

and the Abel-Riemann formula, $_0D_x^{-\nu} f(x)$, when

$$f(1/x) = O(x^{1-\varepsilon}) \ , \qquad \varepsilon > 0 \ ,$$

where the statement $f(x) = O(\varphi)$ has the customary meaning that $\lim_{x\to\infty} |f(x)|/\varphi(x) = \text{a constant.}$*

The definition of fractional operators given by Fourier on pages 561-562 of his *Théorie de la Chaleur* was obtained from his integral representation of $f(x)$,

$$f(x) = (1/2\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos s(x-t)\,dt\,ds \ .$$

Since $d^n \cos s(x-t)/dx^n = s^n \cos\{s(x-t) + n\pi/2\}$ for integral values of n, the generalization

*One should observe that these are sufficient conditions for the existence of the integrals involved provided $f(x)$ is otherwise integrable.

$$z^\mu \to f(x) = (1/2\pi) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, s^\mu \cos\{s(x-t) + \mu\pi/2\}\, dt\, ds$$

is easily made.

One is interested in the question whether this formulates the Liouville or the Abel-Riemann definition. The answer is ambiguous since proper specialization leads to either.*

8. Note on the Complementary Function. A great deal of confusion has been occasioned in the history of fractional operators by the question of the existence of a complementary function for fractional operators. Since the solution of the differential equation

$$d^n \psi(x)/dx^n = 0 \ , \quad n \text{ an integer,}$$

is given by

$$\psi(x) = C_0 + C_1 x + C_2 x^2 + \cdots + C_{n-1}\, x^{n-1} \ ,$$

a function which must therefore always be added to the operator $z^{-n} \to f(x)$, it was argued that a corresponding complementary function should be added to $z^{-\mu} \to f(x)$.

This function would be the solution of the equation

$$z^\mu \to \psi(x) = 0 \ , \quad \mu \text{ a fraction.}$$

Liouville argued that the complementary function was

$$\psi(x) = \sum_{i=0}^{\infty} C_i\, x^i \ , \quad C_i \text{ arbitrary,} \tag{8.1}$$

his reasoning being as follows:

Since

$$z^\mu \to (1/x^n) = (-1)^\mu \Gamma(n+\mu)/\Gamma(n)\, x^{n+\mu} \ ,$$

let us assume that $\psi(x)$ is expansible in a series of the form

$$\psi(x) = \sum_{n=-\infty}^{\infty} A_n/x^n \ .$$

We then obtain

$$z^\mu \to \psi(x) = \sum_{n=-\infty}^{\infty} (-1)^\mu A_n\, \Gamma(n+\mu)/\Gamma(n)\, x^{n+\mu} \ ,$$

which must be identically zero. But this happens only when $n = 0$, -1, -2, $\cdots$, and μ is a fraction. Hence positive values of n are to be excluded and we reach (8.1) as the arbitrary function.

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*In the theory of electrical circuits Oliver Heaviside found frequent use for the operator $p^{\frac{1}{2}}$, where $p = d/dt$. He interpreted $p^{\frac{1}{2}} \to 1$ to mean $1/(\pi t)^{\frac{1}{2}}$. Since $f(t) = 1$ is of the Abel-Riemann type it is obvious that Heaviside's operator must be interpreted by this theory. Specialization of formulas (7.6) shows that this has been done.

As an example Liouville cites the case

$$f(a) = \int_0^\infty \{\cos ax/(1+x^2)\}\, dx = \tfrac{1}{2}\pi e^{-a} ,$$

for which he obtains

$$z^\mu \to f(a) = (-1)^\mu \int_0^\infty \{\cos(ax - \tfrac{1}{2}\mu\pi)\, x^\mu/(1+x^2)\}\, dx$$

$$= (-1)^\mu \tfrac{1}{2}\pi e^{a} + \psi(a) . \qquad (8.2)$$

To compute $\psi(a)$, Liouville makes the transformation $ax = t$ and thus gets

$$z^\mu \to f(a) = (-1)^\mu a^{1-\mu} \int_0^\infty \{\cos(t - \tfrac{1}{2}\mu\pi)\, t^\mu/(a^2+t^2)\}\, dt .$$

If $-1 < \mu < 1$, then $\lim_{a\to\infty} z^\mu \to f(a) = 0$ and $\psi(a) \equiv 0$. If $\mu = -1$, then $\lim_{a\to\infty} z^\mu \to f(a) = A$ (a constant) and $\psi(a) \equiv A$.

Unfortunately for this analysis, if we let $a = 0$ in (8.2) we find

$$A = [-\tfrac{1}{2}\pi + \int_0^\infty \{\cos \tfrac{1}{2}\mu\pi\, x^\mu/(1+x^2)\}\, dx]\, (-1)^\mu$$

$$= \{-\tfrac{1}{2}\pi + \cos \tfrac{1}{2}\mu\pi \cdot \tfrac{1}{2}\pi \csc \tfrac{1}{2}(\mu+1)\pi\}\, (-1)^\mu = 0 .$$

Peacock has a similar difficulty with the Abel-Riemann definition. It is clearer in this case that the equation

$$\int_0^x \{\psi(t)/(x-t)^{1-\nu}\Gamma(\nu)\}\, dt = 0$$

has no solution except the trivial one $\psi(t) = 0$, since we are here concerned with a Volterra integral equation the theory of which is quite complete.

The naïve point of view of Peacock is dominated by an extension of what he calls the *principle of the permanence of equivalent forms*. Although it is stated specifically for algebra he assumes its validity in all symbolic operations. This principle he formulates as follows:

"Direct proposition: *Whatever form is algebraically equivalent to another when expressed in general symbols, must continue to be equivalent whatever the symbols denote.*

"Converse proposition: *Whatever equivalent form is discoverable in arithmetical algebra considered as the science of suggestion, when the symbols are general in their form, though specific in their value, will continue to be an equivalent form when the symbols are general in their nature as well as in their form.*"*

*Peacock's report: (See *Bibliography*) pp. 198-199. Also his *Treatise on Algebra*, vol. 2, Cambridge (1845), pp. 59-60.

Applying this principle Peacock first observes the function

$$z^{-n} \to 0 = x^{n-1}\{C_0/\Gamma(n) + C_1/x\,\Gamma(n-1) + C_2/x^2\Gamma(n-2)$$
$$+ \cdots + C_{n-1}/x^{n-1}\,\Gamma(1)\}\ ,\ n \text{ a positive or negative integer,}$$

and hence he concludes that there exists a complementary function in the general case,

$$z^{-\mu} \to 0 = x^{\mu-1}\{C_0/\Gamma(\mu) + C_1/x\,\Gamma(\mu-1)$$
$$+ C_2/x^2\,\Gamma(\mu-2) + \cdots\}\ .$$

His arguments are ingenious but often misleading and occasionally erroneous, as in the present instance. For example, he considers the identity

$$d^r(ax+b)^n/dx^r = \Gamma(1+n)\ a^r(a+b)^{n-1}/\Gamma(1+n-r)$$
$$+ Cx^{r-1}/\Gamma(-r) + \cdots\ ,$$

where r may have positive or negative integral values.

Generalizing this formula for fractional values of r, he then obtains, for $r = \frac{1}{2}$, $a = b = 1$, the result

$$d^{\frac{1}{2}}(x+1)^2/dx^{\frac{1}{2}} = \Gamma(3)\,(x+1)^{5/2}/\Gamma(5/2) + C_0/x^{\frac{1}{2}}\,\Gamma(-\tfrac{1}{2})$$
$$+ C_1/x^{3/2}\,\Gamma(-3/2) + \cdots$$
$$= 8(x+1)^{3/2}/3\pi^{1/2} + A_0x^{-\frac{1}{2}} + A_1\,x^{-3/2} + \cdots\ .$$

Replacing $(x+1)^2$ by $x^2 + 2x + 1$, we get by means of (7.6) the unambiguous expansion

$$d^{\frac{1}{2}}(x+1)^2/dx^{\frac{1}{2}} = 8x^{3/2}/3\pi^{1/2} + 4x^{\frac{1}{2}}/\pi^{1/2} + x^{-\frac{1}{2}}/\pi^{1/2}\ .$$

If we expand $(x+1)^{3/2} = x^{3/2}(1+1/x)^{3/2}$ by the binomial theorem and compare the terms of this series with the result just written down, we see that the two fractional derivatives are the same, provided we choose

$$A_0 = 0,\ A_n = (-1)^{n+1}\,1\cdot 3\cdot 5\cdots(2n-1)\pi^{-\frac{1}{2}}/2^{n-1}(n+2)!\ ,$$
$$n > 0\ .$$

9. Riemann's Theory. In a paper developed during his student days but posthumously published, Riemann essayed a theory of fractional operators by seeking a generalization of the Taylor expansion

$$u(x+h) = \sum_{n=0}^{\infty} h^n(d^nu/dx^n)/n!\ .$$

For this purpose he assumed the existence of an expansion of the form

$$u(x+h) = \sum_{\nu=-\infty}^{\infty} k_\nu h^\nu \, (d^\nu u/dx^\nu) \ , \tag{9.2}$$

where the ν's form a set of numbers which differ from one another by integers, and the k_ν are constants to be determined.

If now in the Taylor transform

$$u(y) = u(t) + (y-t)\, u'(t) + (y-t)^2\, u''(t)/2! + \cdots \ , \tag{9.3}$$

where we set $y = x + h$ and $t = k$, we can write (9.3) in the form

$$u(x+h) = u(k) + (x-k+h)u'(k) + (x-k+h)^2 u''(k)/2! + \cdots \ . \tag{9.4}$$

Expanding $(x-k+h)^r$ as a series in h, we next obtain

$$(x-k+h)^n = h^n[1 + (x-k)/h]^n$$
$$= h^n \sum_{m=0}^{\infty} \Gamma(n+1)\,(x-k)^m\, h^{-m}/[\Gamma(n-m+1)\Gamma(m+1)] \ .$$

Since, however, we have $1/\Gamma(m+1) = 0$ for $m = -1, -2, \cdots$, the above expansion may be written

$$(x-k+h)^n = \sum_{m=-\infty}^{\infty} \Gamma(n+1)\,(x-k)^m\, h^{n-m}/[\Gamma(n-m+1)\Gamma(m+1)]$$

$$= \sum_{r=-\infty}^{\infty} \Gamma(n+1)\,(x-k)^{n-r}\, h^r/[\Gamma(r+1)\Gamma(n-r+1)] \ .$$

Substituting this value in (9.4), we obtain the expansion

$$u(x+h) = \sum_{r=-\infty}^{\infty} \{h^r/\Gamma(r+1)\}\{u(k)\,(x-k)^{-r}/\Gamma(1-r)$$

$$+ u'(k)\,(x-k)^{1-r}/\Gamma(2-r) + u''(k)\,(x-k)^{2-r}/\Gamma(3-r) + \cdots\} \ .$$

Since the derivative with respect to x of the coefficient of $h^{r-1}/\Gamma(r)$ is the coefficient of $h^r/\Gamma(r+1)$, we may assume, referring to (9.2), that the expression in braces is the desired derivative. If we abbreviate this by $u^{(r)}(x)$ we then obtain, by means of a differentiation with respect to k,

$$d\,u^{(r)}(x)/dk = -u(k)\,(x-k)^{-r-1}/\Gamma(-r) \ ,$$

and hence we derive

$$u^{(r)}(x) = -\int_x^c (x-k)^{-r-1}\, u(k)\, dk/\Gamma(-r)$$

$$u^{(r)}(x) = \int_c^x (x-t)^{-r-1} u(t)\, dt/\Gamma(-r) \ .$$

Riemann then saw fit to add to this definition a complementary function of the form

$$\psi(x) = \sum_{n=1}^{\infty} K_n\, x^{-r-n}/\Gamma(1-n-r) \ ,$$

where the K_n are arbitrary constants, because of the ambiguity in the limit c of the integral.

The essential nature of this complementary function can be easily ascertained. Let us consider the difference

$$C(x) = {}_{c'}D_x^{-\nu}\ {}_cD_x^{-\mu}\, u(x) - {}_cD_x^{-\nu-\mu}\, u(x) \ .$$

When $c' = c$ we have already proved in establishing (7.3) that $C(x) \equiv 0$, but when $c' \neq c$ this will not in general be true. In other words, $C(x)$, the complementary function, measures the deviation of our integration symbol from the index law, this deviation depending clearly upon the choice of c'.

In order to obtain a complementary function which depends upon as many arbitrary parameters as we choose, it is only necessary to proceed as follows:

Let the number n be resolved into p positive fractions $n_1, n_2 \cdots, n_p$, the sum of which equals n. Then we shall have

$${}_cD_x^{-n}\, u = {}_{c_1}D_x^{-n_1}\ {}_cD_x^{-n+n_1}\, u + C_1(x) \ ,$$

$${}_cD_x^{-n+n_1}\, u = {}_{c_2}D_x^{-n_2}\ {}_cD_x^{-n+n_1+n_2}\, u + C_2(x) \ ,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$${}_cD_x^{-n+\sum_{i=1}^{p-1} n_i}\, u = {}_{c_p}D_x^{-n_p}\ {}_cD_x^{0}\, u + C_p(x) \ .$$

The successive substitution of these symbols in the one preceding gives us finally

$${}_cD_x^{-n}\, u = {}_{c_1}D_x^{-n_1}\ {}_{c_2}D_x^{-n_2} \cdots {}_{c_p}D_x^{-n_p}\, u + C_1(x) + {}_{c_1}D_x^{-n_1}\, C_2(x)$$
$$+ {}_{c_1}D_x^{-n_1}\ {}_{c_2}D_x^{-n_2}\, C_3(x) + \cdots + {}_{c_1}D_x^{-n_1}\ {}_{c_2}D_x^{-n_2} \cdots {}_{c_p}D_x^{-n_p}\, C_p(x) \ ,$$

which, since each $C_i(x)$ depends upon the choice of an arbitrary c_i, is a function that depends upon p arbitrary parameters.

A few examples of formulas obtained by an application of these fractional operations to several of the elementary functions are appended below.

TABLE OF FRACTIONAL OPERATIONS

$${}_cD_x^{-\nu}\, k = k(x-c)^{\nu}/\Gamma(1+\nu) \ .$$

$${}_cD_x^{\nu}\, k = k(x-c)^{-\nu}/\Gamma(1-\nu) \ .$$

$${}_0D_x^{-\nu}\, x^n = \Gamma(n+1)\, x^{n+\nu}/\Gamma(n+\nu+1) \ ,$$

$${}_{-\infty}D_x^{-\nu}\, x^{-n} = x^{\nu-n}(-1)^{\nu}\,\Gamma(n-\nu)/\Gamma(n) \ , \quad n-\nu \neq 0 \ ,$$

$${}_0D_x^{\nu}\, x^n = \Gamma(n+1)\, x^{n-\nu}/\Gamma(n-\nu+1) \ ,$$

$${}_{-\infty}D_x^{\nu}\, x^{-n} = x^{\nu+n}(-1)^{\nu}\,\Gamma(n+\nu)/\Gamma(n) \ ,$$

$${}_0D_x^{-\nu}\log x = x^{\nu}[\log x + \Gamma(\nu+1)\,T(\nu)]/\Gamma(1+\nu) \ ,$$

$${}_0D_x^{\nu}\log x = x^{-\nu}[\log x + 1/(1-\nu) + \Gamma(2-\nu)\,T(1-\nu)]/\Gamma(1-\nu) \ ,$$

where we abbreviate

$$T(\nu) = \int_0^1 (1-t)^{\nu-1}\log t\, dt/\Gamma(\nu) \ .$$

$${}_0D_x^{-\nu}\, e^{ax} = e^{ax}\int_0^x t^{\nu-1}\, e^{at}\, dt/\Gamma(\nu) \ ,$$

$${}_{-\infty}D_x^{\nu}\, e^{ax} = a^{\nu}\, e^{ax} \qquad R(a) > 0 \ ,$$

$${}_0D_x^{\nu}\, e^{ax} = e^{ax}\int_0^x t^{-\nu}\, e^{at}\, dt/\Gamma(1-\nu) + x^{-\nu}/\Gamma(1-\nu) \ ,$$

$${}_0D_x^{-\nu}\sin x = \{x^{\nu}/\Gamma(2+\nu)\}\,\Big\{x - \frac{x^3}{(2+\nu)(3+\nu)} + \frac{x^5}{(2+\nu)(3+\nu)(4+\nu)(5+\nu)} - \cdots\Big\} \ ,$$

$${}_0D_x^{\nu}\sin x = \{x^{-\nu}/\Gamma(2-\nu)\}\,\Big\{x - \frac{x^3}{(2-\nu)(3-\nu)} + \frac{x^5}{(2-\nu)(3-\nu)(4-\nu)(5-\nu)} - \cdots\Big\} \ .$$

$${}_0D_x^{-\nu}\cos x = \{x^{\nu}/\Gamma(1+\nu)\}\,\Big\{1 - \frac{x^2}{(1+\nu)(2+\nu)} + \frac{x^4}{(1+\nu)(2+\nu)(3+\nu)(4+\nu)} - \cdots\Big\} \ ,$$

$${}_0D_x^{\nu}\cos x = \{x^{-\nu}/\Gamma(1-\nu)\}\,\Big\{1 - \frac{x^2}{(1-\nu)(2-\nu)} + \frac{x^4}{(1-\nu)(2-\nu)(3-\nu)(4-\nu)} - \cdots\Big\} \ ,$$

$$ {}_0D_x^{-\nu}\, e^{ax} = \{x^\nu/\Gamma(1+\nu)\}\{1 + \frac{ax}{(1+\nu)} + \frac{a^2x^2}{(1+\nu)(2+\nu)} + \frac{a^3x^3}{(1+\nu)(2+\nu)(3+\nu)} + \cdots\}\ , $$

$$ {}_0D_x^{\nu}\, e^{ax} = \{x^{-\nu}/\Gamma(1-\nu)\}\{1 + \frac{ax}{(1-\nu)} + \frac{a^2x^2}{(1-\nu)(2-\nu)} + \frac{a^3x^3}{(1-\nu)(2-\nu)(3-\nu)} + \cdots\}\ , $$

$$ {}_0D_x^{-\nu}\{\sum_{n=0}^{\infty} a_n x^n\} = \{x^\nu/\Gamma(\nu)\}\{\sum_{n=0}^{\infty} B(n+1,\nu)\, a_n x^n\}\ , $$

where $B(m, n) = \Gamma(m)\Gamma(n)/\Gamma(m+n)$,

$$ {}_0D_x^{\nu}\{\sum_{n=0}^{\infty} a_n x^n\} = \{x^{-\nu}/\Gamma(-\nu)\}\{\sum_{n=0}^{\infty} B(n+1,-\nu)\, a_n x^n\}\ , $$

$$ = \{-x^{-\nu}\sin\pi\nu\,\Gamma(1+\nu)/\pi\}\{\sum_{n=0}^{\infty} B(n+1,-\nu)\, a_n x^n\}\ . $$

PROBLEMS

1. Verify the first five formulas in the table of fractional derivatives.
2. Discuss the function
$$ u(x) = {}_cD_x^{-\nu}\, x^{-1}\ . $$
3. Prove that
$$ {}_0D_x^{\frac12}(1-x)^{-\frac12} = \frac{1}{\sqrt{\pi x}}\left(\frac{1}{1-x}\right). $$
4. Prove the formula
$$ {}_0D_x^{\frac12}(1-x^2)^{-\frac12} = \frac{1}{\sqrt{\pi x}}\left[\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac12)\,\Gamma(2n+1)}{\Gamma(n+1)\,\Gamma(2n+\frac12)}x^{2n}\right]. $$
5. Show that
$$ {}_cD_x^{-\nu}\, u(x) = \frac{(x-c)^\nu}{\Gamma(\nu+1)}\sum_{m=0}^{\infty}(-1)^m\frac{\nu}{\nu+m}\frac{(x-c)^m}{m!}u^{(m)}(x)\ . $$

10. Functions Permutable With Unity. The fractional operators that we have just discussed can be regarded also from the point of view of the permutable functions of Volterra, the theory of which we shall consider more adequately in chapter 4. For our present purpose let us define a *function of composition of first kind* by means of the symbol

$$ \psi(x, y) = \int_x^y f(x,t)\, g(t,y)\, dt\ , $$

which we can denote in the convenient form $\psi(x, y) = f * g$. If it

happens that the two functions f and g are so related that

$$f * g = g * f \ ,$$

then f and g are called *permutable functions.*

The *group of the closed cycle* consists of functions permutable with unity; i.e., the class of functions $\{f(x, y)\}$, which satisfy the equation

$$\psi(x, y) = \int_x^y f(x, t) \cdot 1 \, dt = \int_x^y 1 \cdot f(t, y) \, dt \ .$$

Taking partial derivatives with respect to y and x we see that $\psi(x, y)$ satisfies the equation

$$\partial\psi/\partial y = -\partial\psi/\partial x \ ,$$

from which we deduce that ψ is a function of the form $\psi(y - x)$. Moreover since $f(x, y)$ is related to ψ by the equation $\partial\psi/\partial y = f(x, y)$, it follows at once that $f(x, y)$ is also of the form $f(y - x)$.

Hence if $f(x, y)$ and $g(x, y)$ are functions belonging to the group of the closed cycle we shall have

$$f * g = \int_x^y f(x - t) \ g(t - y) \ dt = \int_0^{y-x} f(s) \ g(s - y + x) \ ds \ .$$

Moreover, since $f(x, y) \equiv 1$ is a member of the group, we can compute its successive powers of composition and thus obtain

$$1 * 1 = \overset{*}{1}{}^2 = \int_0^{y-x} ds = y - x \ ,$$

$$1 * \overset{*}{1}{}^2 = \overset{*}{1}{}^3 = \int_0^{y-x} s \, ds = (y - x)^2/2! \ ,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\overset{*}{1}{}^n = \int_0^{y-x} s^{n-2} \, ds/\Gamma(n - 1) = (y - x)^{n-1}/\Gamma(n) \ .$$

If we employ the further abbreviation $y - x = u$ we see that

$$f * \overset{*}{1}{}^n = \int_0^u \{f(s) \ (s - u)^{n-1}/\Gamma(n)\} \ ds \ ,$$

which is the Abel-Riemann definition of fractional integration when n is allowed to assume all positive values.

We are thus able to bring the theory of permutable functions of the group of the closed cycle into relationship with the theory of fractional operators through the conclusion

$${}_0D_u^{-\nu} f(u) = f * \overset{*}{1}{}^\nu = \overset{*}{1}{}^\nu * f \ , \quad u = y - x \ , \quad \nu > 0 \ .$$

11. Logarithmic Operators. Little exploration has yet been made of the properties of the logarithmic operator

$$\varphi(z)\log z \ ,$$

where $\varphi(z) = \varphi_0 + \varphi_1 z^{-1} + \varphi_2 z^{-2} + \cdots$ is a function analytic about infinity. A consistent definition of such an operator, however, can be given without difficulty following a suggestion of Volterra in the theory of permutable functions.

Taking the derivative with respect to ν of

$$z^{-\nu} \to f(x) = \int_0^x (x-t)^{\nu-1} f(t)\, dt/\Gamma(\nu) \ ,$$

we obtain

$$(d/d\nu) z^{-\nu} \to f(x) = -(z^{-\nu}\log z) \to f(x)$$

$$= \int_0^x (x-t)^{\nu-1}\{\log(x-t) - \Gamma'(\nu)/\Gamma(\nu)\} f(t)\, dt/\Gamma(\nu) \ ;$$

that is to say,

$$z^{-\nu}\log z \to f(x) = \int_0^x \{\Gamma'(\nu)/\Gamma(\nu) - \log s\}\, s^{\nu-1} f(x-s)\, ds/\Gamma(\nu) \ . \tag{11.1}$$

For $\nu = 1$, we thus get

$$z^{-1}\log z \to f(x) = -\int_0^x \{C + \log(x-t)\} f(t)\, dt \ ,$$

where C (Euler's constant) is equal to $0.57721\cdots$.

We may then define $\log z \to f(x)$ as follows:

$$\log z \to f(x) = \lim_{\nu=0} \int_0^x \{\nu\psi(\nu) - \nu\log s\}\, s^{\nu-1} f(x-s)\, ds \ , \tag{11.2}$$

where we use the customary abbreviation $\psi(\nu) = \Gamma'(\nu)/\Gamma(\nu)$.

For $f(x) = 1$, this becomes

$$\log z \to 1 = \lim_{\nu=0} \int_0^x \{\nu\psi(\nu) - \nu\log s\}\, s^{\nu-1}\, ds$$

$$= \lim_{\nu=0} \{x^\nu [\psi(\nu) + 1/\nu] - x^\nu \log x\} \ .$$

From the equation defining the ψ function,

$$\psi(\nu+1) - \psi(\nu) = 1/\nu \ ,$$

and the fact that $\psi(1) = -C$, we easily derive

$$\log z \to 1 = -C - \log x \ .$$

Similarly we find

$$\log z \to x = -C\,x - x\log x \ ,$$

$$\log z \to x^n = \lim_{\nu=0} \int_0^x \{\nu\psi(\nu) - \nu\log s\}\, s^{\nu-1}\,(x-s)^n\,ds$$

$$= \lim_{\nu-0} x^{n+\nu}\,[\psi(\nu) + 1/\nu] - x^{n+\nu}\log x$$

$$= -\,\mathrm{C}\,x^n - x^n\log x \ .$$

Thus for the function $f(x) = f_0 + f_1x + f_2x^2 + \cdots$ we are able to obtain the interpretation

$$\log z \to f(x) = -C\,f(x) - f(x)\,\log x \quad . \tag{11.3}$$

Proceeding to more general considerations we find as an analogue of (7.3) the identity

$$z^{-\mu} \to (z^{-\nu}\log z) \to f(x) = z^{-(\mu+\nu)}\log z \to f(x) \ . \tag{11.4}$$

Proof: Since

$$z^{-\nu}\log z \to f(x) = \int_0^x \{\psi(\nu) - \log\,(x-t)\}\,(x-t)^{\nu-1}\,f(t)\ dt/\Gamma(\nu) \ ,$$

we shall have

$$z^{-\mu} \to [z^{-\nu}\log z] \to f(x) = \int_0^x \{(x-t)^{\mu-1}/\Gamma(\mu)\}\,dt$$

$$\times \int_0^t \{\psi(\nu) - \log(t-s)\}\,(t-s)^{\nu-1} f(s)\,ds/\Gamma(\nu) = \int_0^x f(s)\ ds$$

$$\times \int_s^x \{\psi(\nu) - \log\,(t-s)\}\ (x-t)^{\mu-1}\ (t-s)^{\nu-1}\,dt/\Gamma(\mu)\Gamma(\nu) \ .$$

By means of the transformation $y = (t-s)/(x-s)$, this may be written

$$z^{-\mu} \to [z^{-\nu}\log z] \to f(x) = \int_0^x f(s)\ I(x,s)\ ds/\Gamma(\mu)\Gamma(\nu) \ ,$$

where we have

$$I(x,s) =$$

$$\int_0^1 \{\psi(\nu) - \log\,[(x-s)\ y]\}\,(x-s)^{\mu-1}\,(1-y)^{\mu-1}(x-s)^{\nu}\,y^{\nu-1}\,dy$$

$$= (x-s)^{\mu+\nu-1}\,\{\psi(\nu)\Gamma(\mu)\Gamma(\nu)/\Gamma(\mu+\nu)$$

$$- \int_0^1 \log\,[(x-s)\ y]\,(1-y)^{\mu-1}\,y^{\nu-1}\,dy$$

$$= (x-s)^{\mu+\nu-1} \{\psi(\nu)\Gamma(\mu)\Gamma(\nu)/\Gamma(\mu+\nu)$$
$$-\log(x-s)\Gamma(\mu)\Gamma(\nu)/\Gamma(\mu+\nu)$$
$$+\Gamma(\mu)\Gamma(\nu)[\psi(\mu+\nu)-\psi(\nu)]/\Gamma(\mu+\nu)\}$$
$$= (x-s)^{\mu+\nu-1} \{\psi(\mu+\nu)-\log(x-s)\}\Gamma(\mu)\Gamma(\nu)/\Gamma(\mu+\nu) \;.$$

When this is substituted in the integral we see that we have established (11.4).

Making use of this result we are immediately able to interpret the symbol $\varphi(z) \log z \to f(x)$. We thus get

$$\varphi(z) \log z \to f(x) =$$
$$\{\varphi_0 \log z + \varphi_1 z^{-1} \log z + \varphi_2 z^{-2} \log z + \cdots\} \to f(x) =$$
$$-\varphi_0\{C+\log x\} f(x)$$
$$+\int_0^x \{P(x-t) - \log(x-t)\, Q(x-t)\} f(t)\, dt \;,$$

where we abbreviate

$$P(y) = \sum_{n=1}^{\infty} \psi(n)\, \varphi_n\, y^{n-1}/\Gamma(n) \;, \quad Q(y) = \sum_{n=1}^{\infty} \varphi_n\, y^{n-1}/\Gamma(n) \;.$$

It will also be useful for us to have the identity

$$\log z \to \{z^{-\mu} \to f(x)\} = z^{-\mu} \to \{\log z \to f(x)\} \;, \tag{11.5}$$

which assures the commutative property to the operator $\log z$.

Proof: Giving our attention to the left member, we have

$$\log z \to \{z^{-\mu} \to f(x)\} = \log z \to \int_0^x (x-t)^{\mu-1} f(t)\, dt/\Gamma(\mu)$$

$$= \lim_{\nu=0} \int_0^x (x-t)^{\nu-1}\{\nu\,\psi(\nu) - \nu \log(x-t)\}$$
$$\times \int_0^t (t-s)^{\mu-1} f(s)\, ds/\Gamma(\mu)$$

$$= \lim_{\nu=0} \int_0^x f(s)\, ds \int_s^x (x-t)^{\nu-1}(t-s)^{\mu-1}\{\nu\,\psi(\nu)$$
$$-\nu \log(x-t)\}\, dt/\Gamma(\mu)$$

$$= \lim_{\nu=0} \int_0^x f(s)\, ds \int_0^1 \{\nu\,\psi(\nu) - \nu \log[(x-s)(1-y)]\}$$
$$\times (x-s)^{\nu+\mu-1}(1-y)^{\nu-1} y^{\mu-1}\, dy/\Gamma(\mu)$$

$$= \lim_{\nu=0} \int_0^x f(s)\nu[\psi(\nu+\mu) - \log(x-s)]\Gamma(\nu)\, ds/\Gamma(\mu+\nu)$$

$$= \int_0^x f(s)[\psi(\mu) - \log(x-s)]\, ds/\Gamma(\mu) \ .$$

But the last expression is seen to be equivalent to

$$z^{-\mu} \to \{\log z \to f(x)\} \ ,$$

which was the object of the proof.

As in the theory of fractional operators there exists a second definition for $\log z$ which corresponds to the choice of $-\infty$ for the lower limit of integration. We shall derive this formula following a method used by Sbrana.

Referring to formula (3.2) we let $F(z) = (\log z)/z$ and, noting the identity

$$\log z = \int_0^\infty \{(e^{-x} - e^{-xz})/x\}\, dx \ ,^*$$

we obtain for $G(t)$, (3.3), the expression

$$G(t) = (1/2\pi i) \int_{-i\infty}^{i\infty} \{e^{wt}/w\}\, dw \int_0^\infty \{(e^{-x} - e^{-wx})/x\}\, dx$$

$$= \int_0^\infty dx/x \, \{e^{-x}\, (1/2\pi i) \int_{-i\infty}^{i\infty} e^{wt}\, dw/w$$

$$- (1/2\pi i) \int_{-i\infty}^{i\infty} e^{w(t-x)}\, dw/w\} \ .$$

Making use of the identities

$$(1/2\pi i) \int_{-i\infty}^{i\infty} e^{wt}\, dw/w = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

$$(1/2\pi i) \int_{-i\infty}^{i\infty} e^{w(t-x)}\, dw/w = \begin{cases} 0 & \text{for } t < x \\ 1 & \text{for } t > x \ , \end{cases}$$

we then obtain

$$G(t) = \int_0^t (e^{-x} - 1)\, dx/x + \int_t^\infty e^{-x}\, dx/x$$

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*To derive this, we note $1/y = \int_0^\infty e^{-yx}\, dx$ and hence

$$\int_1^z dy/y = \log z = \int_0^\infty -(e^{-yx}/x) \Big|_1^z dx = \int_0^\infty \{(e^{-x} - e^{-xz})/x\}\, dx \ .$$

See Riemann-Weber: *Die Partiellen Differential-gleichungen der Mathematischen Physik*, vol. 1 (1919), § 25 (4); (1900), § 23 (4).

$$G(t) = -\log t + \int_0^\infty e^{-x} \log x \, dx$$
$$= -\log t - C \text{ (Euler's constant)}.$$

When this value is substituted in (3.2) we get the desired formula,

$$z^{-1} \log z \to f(x) = \int_{-\infty}^{x} f(t) \{C + 2 n \pi i - \log (x-t)\} \, dt ,$$

or by one differentiation,

$$\log z \to f(x) = (C + 2 n \pi i) \, f(x)$$
$$+ z \to \int_{-\infty}^{x} f(t) \{\log 1/(x-t)\} \, dt .$$

The operator inverse to $z^{-\nu} \log z$ can be attained by means of the solution of an integral equation due to Volterra.

Since we have by definition

$$z^{-\nu} \log z \to f(x) =$$
$$- \int_0^x (x-t)^{\nu-1} \{\log (x-t) \quad - \psi(\nu)\} \, f(t) dt/\Gamma(\nu) ,$$

the operator inverse to this will naturally be the one which furnishes a solution of the integral equation

$$z^{-\nu} \log z \to f(x) = g(x) .$$

Making use of (11.4) we first reduce the problem to the simpler equation

$$z^{-1} \log z \to f(x) = z^{\nu-1} \to g(x) = h(x) .$$

Defining a new function

$$\lambda(u) = \int_0^\infty \{u^s/\Gamma(1+s)\} \, ds ,$$

we shall prove that the desired inverse is given by the equation

$$f(x) = -z^2 \to \int_0^x \lambda(x-t) \, h(t) \, dt . \qquad (11.6)$$

To show this let us note that

$$z^{-1} \log z \to f(x) = - \int_0^x [C + \log (x-t)] \, f(t) \, dt$$
$$= - \lim_{\nu=1} (d/d\nu) \, \{z^{-\nu} \to f(x)\} .$$

We then form the following identities:

$$\int_0^t \lambda(t-s)\ {}_0D_s^{-\nu} f(s)\ ds =$$

$$\int_0^t \lambda(t-s)\ ds \int_0^s f(r)\,(s-r)^{\nu-1}\,dr/\Gamma(\nu) =$$

$$\int_0^t f(r)\ dr \int_r^t \lambda(t-s)\,(s-r)^{\nu-1}\,ds/\Gamma(\nu) =$$

$$\int_0^t f(r)\ dr \int_0^\infty d\eta \int_r^t (t-s)^\eta (s-r)^{\nu-1}\,ds/\Gamma(1+\eta)\Gamma(\nu) =$$

$$\int_0^t f(r)\ dr \int_0^\infty \{(t-r)^{\eta+\nu}/\Gamma(1+\eta+\nu)\}\ d\eta =$$

$$\int_0^t f(r)\ dr \int_\nu^\infty \{(t-r)^\eta/\Gamma(1+\eta)\}\ d\eta\ .$$

Taking the derivative with respect to ν of the first and last members of this sequence, we get

$$\int_0^t \lambda(t-s)\ (d/d\nu)\ {}_0D_s^{-\nu} f(s)\ ds$$

$$= (d/d\nu) \int_0^t f(r)\ dr \int_\nu^\infty \{(t-r)^\eta/\Gamma(1+\eta)\}\ d\eta$$

$$= - \int_0^t (t-r)^\nu f(r)\ dr/\Gamma(1+\nu)\ .$$

Letting $\nu = 1$, we finally obtain

$$\int_0^t \lambda(t-s)\{z^{-1}\log z \to f(s)\}\ ds = -\,z^{-2} \to f(t)\ ,$$

and hence, since $z^{-1}\log z \to f(s) = h(s)$, we reach the desired result,

$$f(t) = -z^2 \to \int_0^t \lambda(t-s)h(s)\ ds\ ,$$

or in terms of $g(x)$,

$$\begin{aligned} f(t) &= -z^{\nu+1} \to \int_0^t \lambda(t-s)\ g(s)\ ds\ , \\ &= -\,z^{\nu+1} \to \int_0^t \lambda(s)\ g(t-s)\ ds\ . \end{aligned} \tag{11.7}$$

The ideas which we have set forth above are capable of an extensive generalization, which we may describe as follows:

Let us denote by p the derivative operator $p = d/d\nu$, and by $\varphi(p)$ a power series in p. We may then write*

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*It will be noticed that we have replaced $f(t)$ in the integral by $e^{(t-x)z}$, since $f(t) = f(x) + (t-x)f'(x) + (t-x)^2 f''(x)/2! + \cdots = e^{(t-x)z} \to f(x)$. By this device we are able to discuss the pure operational symbol itself.

$$\varphi(p) \to z^{-\nu} = \varphi(p) \to \int_0^x \{(x-t)^{\nu-1}/\Gamma(\nu)\} e^{(t-x)z}\, dt \ ,$$

from which we derive

$$z^{-\nu}\varphi(-\log z) = \int_0^x \{\varphi(p) \to (x-t)^{\nu-1}/\Gamma(\nu)\} e^{(t-x)z}\, dt \ . \tag{11.8}$$

For example, if $\varphi(p) = p^2$, we obtain from (11.8) the formula

$$\begin{aligned} z^{-\nu}\log^2 z &= \int_0^x s^{\nu-1} e^{-sz} \{\log^2 s - 2\psi(\nu)\log s + \psi^2(\nu) \\ &\qquad - \psi'(\nu)\}\, ds/\Gamma(\nu) \ , \\ &= \psi(\nu)\, z^{-\nu}\log z + \int_0^x s^{\nu-1} e^{-sz}\{\log^2 s - \psi'(\nu)\}\, ds/\Gamma(\nu) \ , \end{aligned} \tag{11.9}$$

We shall find it convenient in another place to consider the case where $\varphi(p) = p^n$, from which we obtain,

$$z^{-\nu}\log^n z = (-1)^n \int_0^x \frac{\partial^n}{\partial \nu^n}\{(x-t)^{\nu-1}/\Gamma(\nu)\} e^{(t-x)z}\, dt \ . \tag{11.10}$$

The problem of inversion is similarly generalized. Let us, for example, multiply $z^{-\mu+\nu}$ by $\vartheta(\mu)$ and integrate from 0 to ∞. We thus obtain

$$z^{\nu+1} \int_0^\infty z^{-\mu-1}\vartheta(\mu)\, d\mu = z^{\nu+1}\int_0^x e^{(t-x)z} I(x-t)\, dt \ , \tag{11.11}$$

where we abbreviate

$$I(s) = \int_0^\infty \{\vartheta(\mu)\, s^\mu/\Gamma(\mu+1)\}\, d\mu \ . \tag{11.12}$$

Let us assume that the desired inverse is the function $F(z)$. The function $\vartheta(\mu)$ is then to be determined so that the left member of (11.11) shall equal $F(z)$. That is to say, if we write z in the form $e^{\log z}$, we are to determine $\vartheta(\mu)$ as the solution of the integral equation,

$$\int_0^\infty e^{-\mu \log z}\vartheta(\mu)\, d\mu = z^{-\nu} F(z) \ .$$

This, we observe, is an integral equation of the Laplace type, the inversion of which has been extensively studied. An account of the methods available for its solution has been given in section 7 of chapter 1.

As an example let us determine the inverse of the operator $z^{-\nu}(a - \log z)$.

We first consider the equation,

$$\int_0^\infty e^{-\mu \log z}\,\vartheta(\mu)\,d\mu = 1/(a - \log z) \ , \qquad (11.13)$$

which we note has the solution $\vartheta(\mu) = e^{a\mu}$.

The desired inverse is then derived from (11.11),

$$z^{\nu+1}\int_0^\infty z^{-\mu-1}\,\vartheta(\mu)\,d\mu = z^\nu \int_0^\infty e^{(a-\log z)\mu}\,d\mu = z^\nu/(a - \log z)$$

$$= z^{\nu+1}\int_0^x e^{(t-x)z}\,I(x-t)\,dt \ , \qquad (11.14)$$

where we write $I(s) = \int_0^\infty \{e^{a\mu}\,s^\mu/\Gamma(\mu+1)\}\,d\mu$.

If we set $a=0$, this result is essentially equivalent to (11.7), which we derived by a longer and more difficult argument.

12. Special Operators. In this section we shall list a number of special operators which will be important to us in later chapters.

(a) Of particular importance in the theory of the difference calculus is the operator

$$e^{az} \to f(x) = f(x+a) \ . \qquad (12.1)$$

This follows at once from the Taylor transform of $f(x)$ as we see when we replace $t - x$ by a and u by f in (6.3).

It is obvious that the difference

$$\Delta_x f = f(x+1) - f(x)$$

can be written

$$\Delta_x f = (e^z - 1) \to f(x) \ ,$$

and in general the nth difference, defined by

$$\Delta_x^n f = \Delta[\Delta_x^{n-1} f] = f(x+n) - nf(x+n-1)$$
$$+ n(n-1)\,f(x+n-2)/2! - \cdots \pm f(x) \ ,$$

will appear symbolically

$$\Delta_x^n f = (e^z - 1)^n \to f(x) \ . \qquad (12.2)$$

(b) It is also useful to consider another operator intimately associated with the preceding one.

If, in the series

$$y = u(x) + u(x+1) + u(x+2) + \cdots + u(x+n) + \cdots \ , \qquad (12.3)$$

we replace $u(x+n)$ by $e^{nz} \to u(x)$, we can write (12.3) in the form

$$y = (1 + e^z + e^{2z} + \cdots e^{nz} + \cdots) \to u(x)$$
$$= 1/(1 - e^z) \to u(x) \ .$$

Making use of the well-known expansion of $1/(1-e^z)$*, we can write this

$$y = -(1/z - \tfrac{1}{2} + B_1 z/2! - B_2 z^3/4! + B_3 z^5/6! - \cdots) \to u(x) \ , \tag{12.4}$$

where the B_i are the Bernoulli numbers

$B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66$, $B_6 = 691/2730$, $B_7 = 7/6$, $B_8 = 3617/510$, $B_9 = 43867/798$, $B_{10} = 1222277/2310$.†

In order to explore the convergence of this series we make use of the formula‡

$$B_p = 2(2p)! \ (1 + 1/2^{2p} + 1/3^{2p} + \cdots)/(2\pi)^{2p} \ ,$$

from which we have, in the limit,

$$\lim_{p=\infty} \{B_{p+1}/(2p+2)!\} \times \{(2p)!/B_p\} z^2 = z^2/4\pi^2 \ .$$

*See Whittaker and Watson: *Modern Analysis*, 3rd ed., Cambridge (1920), p. 127.

†Adams in the report of the British Association for 1877 gives tables of the first 62 Bernoulli numbers. These have been extended to 90 by S. Z. Serebrennikoff. See H. T. Davis; *Tables of the Higher Mathematical Functions*, vol. 2 (1935).

‡To achieve this formula we notice that

$$\tfrac{1}{2} i \cot \tfrac{1}{2} ix = \tfrac{1}{2} i \cos \tfrac{1}{2} ix/\sin \tfrac{1}{2} ix = \tfrac{1}{2} (e^{\frac{1}{2}x} + e^{-\frac{1}{2}x})/(e^{\frac{1}{2}x} - e^{-\frac{1}{2}x})$$
$$= \tfrac{1}{2}(1 + e^{-x})/(1 - e^{-x}) = 1/(1 - e^{-x}) - \tfrac{1}{2} \ .$$

But we also know that $\pi \cot \pi z = 1/z + \sum_{n=1}^{\infty} 2z/(z^2 - n^2)$ and hence $1/(1 - e^{-x}) - \tfrac{1}{2} = \tfrac{1}{2} i \cot \tfrac{1}{2} ix = 1/x + \sum_{n=1}^{\infty} 2x/(x^2 + 4\pi^2 n^2) = 1/x + 2x \sum_{n=1}^{\infty} \{1/4\pi^2 n^2 - x^2/(4\pi^2 n^2)^2 + x^4/(4\pi^2 n^2)^3 \cdot \ \}$. Comparing this with the expansion (12.4) we at once obtain

$$2 \sum_{n=1}^{\infty} 1/(4\pi^2 n^2)^p = (1/2^{2p-1}\pi^{2p}) \sum_{n=1}^{\infty} 1/n^{2p} = B_p/(2p)! \ ,$$

from which we derive the desired formula

$$B_p = 2(2p)! \ (1 + 1/2^{2p} + 1/3^{2p} + \cdots)/(2\pi)^{2p} \ .$$

The Bernoulli numbers were first given by James Bernoulli in *Ars Conjectandi* (Basel, 1713, p. 97). He exhibited their usefulness in the summation of the powers of numbers and boasted: "*Huius laterculi beneficio intra semi-quadrantem horae reperi, quod potestates decimae sive quadrato-sursolida mille primorum numerorum ab unitate in summun collecta efficiunt*

91,409,924,241,424,243,424,241,924,242,500 ."

We thus see that the series converges for values of z in the interval $0 < z < 2\pi$, a conclusion that we might have reached directly from the fact that the singularities of $1/(1-e^z)$ are $z = 0$, $2\pi i$, $4\pi i$, $\cdots$.

Equation (12.4) because of its central rôle in the integral calculus of finite differences has been the subject of much study, but most treatments of it introduce an arbitrary constant because of the presence of the polar operator $1/z$. It will be shown later that this indeterminacy can be removed by assuming the integration from $-\infty$. We shall then replace (12.4) by the more comprehensive formula

$$y = \int_{-\infty}^{x} (-1 + \tfrac{1}{2}z - B_1 z^2/2! + B_2 z^4/4! \\ - B_3 z^6/6! + \cdots) \to u(t)\, dt \ . \qquad (12.5)$$

As an example let us consider the summation for the case where $u(x) = 1/x^2$. We should then have

$$y = \int_{-\infty}^{x} [-1/x^2 - 2!/2x^3 - B_1 3!/2!x^4 + B_2 5!/4!x^6 \\ - B_3 7!/6!x^8 + \cdots]\, dx$$

$$= 1/x + 1!/2x^2 + B_1 2!/2!x^3 - B_2 4!/4!x^5 \\ + B_3 6!/6!x^7 - \cdots \ .$$

Although divergent, this series is summable by the method of Borel (see chapter 5) ; applying to it the integral

$$\int_0^\infty e^{-xt}\, t^n\, dt = n!/x^{n+1} ,$$

we get

$$y = \int_0^\infty e^{-xt}\, (1 + t/2 + B_1 t^2/2! - B_2 t^4/4! + B_3 t^6/6! - \cdots)\, dt$$

$$= \int_0^\infty e^{-xt}\{t/(e^t - 1) + t\}\, dt = \int_0^\infty \{te^{-(x-1)t}/(e^t - 1)\}\, dt \ .$$

For $x = 1$ this reduces to the well-known sum

$$y = \int_0^\infty \{t/(e^t - 1)\}\, dt = \pi^2/6 \ .$$

More generally let us write $u(x) = 1/x^m$, $m > 1$. We shall then have

$$y = \int_{-\infty}^{x} \{-1/x^m - m/2x^{m+1} - B_1 m(m+1)/2!x^{m+2} \\ + B_2 m(m+1)(m+2)/4!x^{m+3} - \cdots\}\, dx$$

$$= 1/(m-1)x^{m-1} + 1/2x^m + B_1 m/2!x^{m+1} \\ - B_2 m(m+1)/4!x^{m+2} + \cdots$$

$$y = \int_0^\infty e^{-xt}\,(t^{m-2} + t^{m-1}/2 + B_1 t^m/2! - B_2 t^{m+1}/4! + \cdots)\,dt/\Gamma(m)$$
$$= \int_0^\infty e^{-xt}\,t^{m-2}\{t/(e^t - 1) + t\}\,dt/\Gamma(m)$$
$$= \int_0^\infty e^{-xt}\{t^{m-1}/(1 - e^{-t})\}\,dt/\Gamma(m) \ .$$

For $x = 1$, $m = 2n$, this formula yields the well-known result

$$y = \pi^{2n} B_n\, 2^{2n-1}/(2n)! \ .$$

Hence if $u(x)$ is a function of the form

$$u(x) = u_r/x^r + u_{r+1}/x^{r+1} + u_{r+2}/x^{r+2} \cdots ,$$

where r is a number greater than 1, we shall have

$$y(x) = \int_0^\infty \{e^{-xt}/(1 - e^{-t})\}\{\sum_{m=0}^\infty u_{r+m} t^{r-1+m}/\Gamma(m+r)\}\,dt \ . \tag{12.6}$$

If we adopt the customary notation of the zeta function of Riemann,

$$\zeta(m, x) = 1/x^m + 1/(x+1)^m + 1/(x+2)^m + \cdots ,$$

and note that $y(x)$ is the series

$$y(x) = u_r\,\zeta(r, x) + u_{r+1}\,\zeta(r+1, x) + \cdots ,$$

we obtain from (12.6) the well-known formula

$$\zeta(m, x) = \int_0^\infty \{e^{-xt}\,t^{m-1}/(1 - e^{-t})\}\,dt/\Gamma(m) \ .\text{*}$$

If series (12.3) be truncated at the nth term, so that

$$y = u(x) + u(x+1) + u(x+2) + \cdots + u(x+n-1) \ ,$$

then the finite sum may be expressed symbolically in the form

$$y = (1 - e^{nz})/(1 - e^z) \to u(x) \ . \tag{12.7}$$

This function is readily reduced to derivatives by means of the identity

$$(1 - e^{nz})/(1 - e^z) = \sum_{p=1}^\infty \varphi_p(n)\, z^{p-1}/p! \ , \tag{12.8}$$

where $\varphi_p(n) = n^p - \tfrac{1}{2}pn^{p-1} + {}_pC_2\, B_1\, n^{p-2} - {}_pC_4\, B_2\, n^{p-4} + {}_pC_6\, B_3\, n^{p-6} - \cdots$ is the *Bernoullian polynomial of order p*.

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*See Whittaker and Watson: *Modern Analysis, 3rd ed.*, Cambridge (1920), chapter 13; in particular, p. 266.

PROBLEMS

The following problems contain many of the formal results basic to the discipline generally referred to as the *calculus of finite differences.* The following abbreviations are employed:

$$E_x = 1 + \Delta_x \ ; \quad E_x^p \rightarrow u(x) = u(x+p) \ ;$$

$$\sum_{x=1}^{n} u(x) = \Delta^{-1} u(x) \Big|_{1}^{n+1} = \lim_{x=n+1} \Delta^{-1} u(x) - \lim_{x=1} \Delta^{-1} u(x) \ ;$$

$$x^{(n)} = x(x-1)(x-2) \cdots (x-n+1) = \Gamma(x+1)/\Gamma(x-n+1) \ ,$$

$$x^{(-n)} = 1/[x(x+1)(x+2) \cdots (x+n-1)] = \Gamma(x)/\Gamma(x+n) \ ;$$

$$\Delta^r 0^n = \lim_{x=0} \Delta^r x^n \qquad D^r 0^{(n)} = \lim_{x=0} \frac{d^r}{dx^r} x^{(n)}$$

These last two symbols are called respectively the *differences* and *differential coefficients of zero.* Numerical values for them will be found in Davis: *Tables of the Higher Mathematical Functions,* vol. 2, pp. 212 and 215.

1. Expand $E_x^p \rightarrow u(x) = (1+\Delta_x)^p \rightarrow u(x)$, and obtain Newton's interpolation formula:

$$u(x+p) = u(x) + \Delta u(x) + \frac{1}{2!}\Delta^2 u(x) + \frac{1}{3!}\Delta^3 u(x) + \cdots .$$

2. Prove that

$$\Delta^r 0^r = r(\Delta^r 0^{n-1} + \Delta^{r-1} 0^{n-1}) \ , \quad \Delta^r 0^r = r! \ , \quad \Delta 0^n = 1 \ .$$

3. Establish the following formula:

$$(\Delta - \Delta^2/2 + \Delta^3/2^2 - \Delta^4/2^3 + \cdots)\, 0^{2n-1}$$
$$= (-1)^{n-1}\, 2(2^{2n}-1)\, B_n/n \ ,$$

where B_n is the nth Bernoulli number. (Herschel).

4. Prove that if $P(x)$ is a polynomial with constant coefficients, then

$$P(e^t) = P(1) + t P(E)\, 0 + \frac{t^2}{2!} P(E)\, 0^2 + \frac{t^3}{3!} P(E) 0^3 + \cdots .$$

This is known as *Herschel's theorem,* being found in Sir J. F. W. Herschel's: *Examples of the Calculus of Finite Differences,* (1820).

5. Prove that

$$D^r 0^{(n)} = -n D^r 0^{(n)} + r D^{r-1} 0^{(n)} \ , \quad D^r 0^{(r)} = r! \ , \quad D\, 0^{(n)} = (-1)^{n+1}(n-1)!$$

6. Establish the following expansions:

$$x^n = \sum_{r=0}^{n} x^{(r)} \Delta^r 0^n / r! \ ,$$

$$x^{(n)} = \sum_{r=0}^{n} x^r D^r 0^{(n)} / r! \ .$$

7. Verify the following table of differences:

(*a*) $\Delta x^{(n)} = n x^{(n-1)}$,

(*b*) $\Delta x^{(-n)} = -n x^{(-n-1)}$,

(*c*) $\Delta^n x^n = n!$,

(*d*) $\Delta(ax+b)^{(n)} = an(ax+n)^{(n-1)}$,

$$(e) \quad \Delta(ax+b)^{(-n)} = -an(ax+b)^{(-n-1)} ,$$

$$(f) \quad \Delta u_x v_x = v_{x+1} \Delta u_x + u_x \Delta v_x ,$$

$$(g) \quad \Delta \frac{u_x}{v_x} = \frac{v_x \Delta u_x - u_x \Delta v_x}{v_x v_{x+1}} ,$$

$$(h) \quad \Delta a^{mx+b} = (a^m - 1) a^{mx+b} ,$$

$$(i) \quad \Delta \sin(ax+b) = 2 \sin \tfrac{1}{2}a \cos(ax+b+\tfrac{1}{2}a) ,$$

$$(j) \quad \Delta \cos(ax+b) = -2 \sin \tfrac{1}{2}a \sin(ax+b+\tfrac{1}{2}a) ,$$

$$(k) \quad \Delta^n \log x = \log \frac{(x+n)^p (x+n-2)^q (x+n-4)^r \cdots}{(x+n-1)^P (x+n-3)^Q (x+n-5)^R \cdots} ,$$

where $p, q, r, \cdots$ are the binomial coefficients ${}_nC_0, {}_nC_2, {}_nC_4, \cdots$, and $P, Q, R, \cdots$ the binomial coefficients, ${}_nC_1, {}_nC_2, {}_nC_3, \cdots$.

$$(l) \quad \Delta^n u_x = z^n \rightarrow u_x + \frac{\Delta^n 0^{n+1}}{(n+1)!} z^{n+1} \rightarrow u_x + \frac{\Delta^n 0^{n+2}}{(n+2)!} z^{n+2} \rightarrow u_x + \cdots .$$

8. Verify the following table of inverse differences:

$$(A) \quad \Delta^{-1} x^{(n)} = x^{(n+1)}/(n+1) ,$$

$$(B) \quad \Delta^{-1} x^{(-n)} = x^{(-n+1)}/(-n+1) ,$$

$$(C) \quad \Delta^{-1}(ax+b)^{(n)} = (ax+b)^{(n+1)}/[a(n+1)] ,$$

$$(D) \quad \Delta^{-1}(ax+b)^{(-n)} = (ax+b)^{(-n+1)}/[a(-n+1)] ,$$

$$(E) \quad \Delta^{-1} x \cdot x! = x! ,$$

$$(F) \quad \Delta^{-1}(v_{x+1} \Delta u_x) = u_x v_x - \Delta^{-1}(u_x \Delta v_x) ,$$

$$(G) \quad \Delta^{-1} k a^{mx+b} = k a^{mx+b}/(a^m - 1) ,$$

$$(H) \quad \Delta^{-1} x a^x = \frac{a^x}{a-1} [x - a/(a-1)] ,$$

$$(I) \quad \Delta^{-1} \sin(ax+b) = -\cos(ax + b - \tfrac{1}{2}a)/(2 \sin \tfrac{1}{2}a) ,$$

$$(J) \quad \Delta^{-1} \cos(ax+b) = \sin(ab + b - \tfrac{1}{2}a)/(2 \sin \tfrac{1}{2}a) ,$$

$$(K) \quad \Delta^{-1} a^x u_x = \frac{a^x}{a-1} \left\{ u_x - \frac{a}{a-1} \Delta u_x + \frac{a^2}{(a-1)^2} \Delta^2 u_x + \cdots \right\} .$$

9. Find the value of the sum $1^2 2 + 2^2 2^2 + 3^2 2^3 + \cdots + n^2 2^n$.
Solution: Making use of problem 6 and (K) of problem 8, we have

$$\Delta^{-1} x^2 2^x = \Delta^{-1}(x^{(2)} + x) 2^x = 2^x(x^2 - 4x + 6) .$$

Hence, since $\sum_{x=1}^{n} x^2 2^x = \Delta^{-1} x^2 2^x \Big|_1^{n+1}$

we get

$$\sum_{x=1}^{n} x^2 2^x = 2^x(x^2 - 4x + 6) \Big|_1^{n+1} = 2^{n+1}(n^2 - 2n + 3) - 6 .$$

10. Show that

$$\frac{1!}{x+1} + \frac{2!}{(x+1)(x+2)} + \cdots + \frac{n!}{(x+1)(x+2)\cdots(x+n)}$$

$$= \frac{1}{x-1} \left\{ 1 - \frac{(n+1)!}{(x+1)(x+2)\cdots(x+n)} \right\} .$$

11. If we abbreviate,

$$S_n(p) = 1^n + 2^n + 3^n + \cdots + p^n ,$$

compute $S_2(p), S_3(p), S_4(p), S_5(p)$.

12. Using the notation of problem 11, prove that

$$S_5(p) + S_7(p) = 2[S_3(p)]^2 .$$

13. Prove that

$$1 - \frac{m}{n+1} + \frac{m(m-1)}{(n+1)(n+2)} - \cdots = \frac{n}{m+n}$$

14. Making use of (I), problem 8, show that

$$\sin x + \sin 2x + \sin 3x + \cdots + \sin nx = \frac{\sin \tfrac{1}{2}nx \sin \tfrac{1}{2}(n+1)x}{\sin \tfrac{1}{2}x}$$

15. Making use of (J), problem 8, show that

$$\cos x + \cos 2x + \cos 3x + \cdots + \cos nx = \frac{\cos \tfrac{1}{2}nx \sin \tfrac{1}{2}(n+1)x}{\sin \tfrac{1}{2}x}$$

(c) A third operator useful in many places is the following:*

$$(z+k)^n \rightarrow f(x) = e^{-kx}\{z^n \rightarrow [e^{kx} f(x)]\} . \qquad (12.9)$$

That this operator is valid for fractional as well as integral values of n can be proved as follows:

We first extend the well-known formula of Leibnitz for the differentiation of a product to the case of fractional operators. (See section 2, chapter 4).

For this purpose we write

$$ {}_cD_x^{-\nu}\{U(x)\ V(x)\} = \int_c^x \{U(t)\ V(t)/\Gamma(\nu)\}(x-t)^{\nu-1}\,dt$$

$$= U(x) \int_c^x \{V(t)/\Gamma(\nu)\,(x-t)^{1-\nu}\}\,dt$$

$$- U'(x) \int_c^x \{V(t)/\Gamma(\nu)\,(x-t)^{1-\nu}\}\,dt$$

$$+ \{U''(x)/2!\} \int_c^x \{V(t)/\Gamma(\nu)\,(x-t)^{1-\nu}\}\,dt - \cdots .$$

from which we at once obtain the desired formula,

$$ {}_cD_x^{-\nu}\{U(x)\ V(x)\} = U(x)\ V^{(-\nu)}(x) - \nu\, U'(x)\ V^{(-\nu-1)}(x)$$
$$+ \nu\,(\nu+1)\ U''(x)\ V^{(-\nu-2)}(x)/2! - \cdots . \qquad (12.10)$$

Returning to the main problem, we then write

$$e^{-kx}\, z^n \rightarrow \{e^{kx} f(x)\} = e^{-kx}\, z^{m-\mu} \rightarrow \{e^{kx} f(x)\} ,$$

*See J. Edwards: *Differential Calculus*, 3rd ed. London (1904), p. 70.

where m is an integer and μ is a fraction between 0 and 1. Then in (12.10), setting $U(x)$ equal to e^{kx} and $V(x)$ equal to $f(x)$, we get

$$e^{-kx} z^n \to \{e^{kx} (f(x)\} = e^{-kx} z^m \to \{e^{kx} [f^{(-\mu)}(x)$$

$$-\mu k f^{(-\mu-1)} + \mu(\mu+1) k^2 f^{(-\mu-2)}/2! \cdots]\}$$

$$= e^{-kx} z^m \to [e^{kx}\{1 - \mu k z^{-1} + \mu(\mu+1) k^2 z^{-2}/2! - \cdots\} \to f^{(-\mu)}(x)] \ .$$

This expression reduces to

$$e^{-kx} z^m \to [e^{kx} (1 + k/z)^{-\mu} \to f^{(-\mu)}(x)] =$$

$$(z+k)^m \to [(1+k/z)^{-\mu} \to f^{(-\mu)}(x)] = (z+k)^{m-\mu} \to f(x) \ ,$$

which was to be proved.

(d) (Hadamard's Operator). In the third part of his classical paper on analytic functions Hadamard considered an operator which is closely related to the operator for fractional differentiation and integration. Designating this operator by the symbol $\Omega_x^{-\nu} f(x)$, we may define it as follows:

$$\Omega_x^{-\nu} f(x) = \int_0^x (\log x - \log t)^{\nu-1} f(t)\, d(\log t)/\Gamma(\nu) \ . \tag{12.11}$$

By means of the transformation $t = sx$, this operator may be written in the form

$$\Omega_x^{-\nu} f(x) = \int_0^1 (\log 1/s)^{\nu-1} \{f(sx)/s\}\, ds/\Gamma(\nu) \ .$$

The most significant property of $\Omega_x^{-\nu} f(x)$ is found in the fact that the index law holds for it; that is,

$$\Omega_x^{-\nu}\, \Omega_x^{-\mu} f(x) = \Omega_x^{-(\nu+\mu)} f(x) \ . \tag{12.12}$$

To show this we write

$$\Omega_x^{-\nu}\{\Omega_x^{-\mu} f(x)\} = \int_0^x \{(\log x/t)^{\nu-1}/t\,\Gamma(\nu)\}\, dt$$

$$\times \int_0^t \{(\log t/s)^{\mu-1} f(s)/s\,\Gamma(\mu)\}\, ds$$

$$= \int_0^x \{f(s)/s\}\, ds \int_s^x \{(\log x/t)^{\nu-1} (\log t/s)^{\mu-1}/t\,\Gamma(\nu)\,\Gamma(\mu)\}\, dt \ .$$

If we abbreviate the second integral by $I(x, s)$ and apply to it the transformation $\log t = y$, we then obtain

$$I(x, s) = \int_{\log s}^{\log x} (-y + \log x)^{\nu-1} (y - \log s)^{\mu-1} dy/\Gamma(\nu)\,\Gamma(\mu) \;.$$

Making the further transformation $r = (-y + \log x)/(\log x - \log s)$, this integral becomes

$$I(x, s) = (\log x/s)^{\nu+\mu-1} \int_0^1 s^{\nu-1} (1 - s)^{\mu-1} ds/\Gamma(\nu)\Gamma(\mu)$$

$$= (\log x/s)^{\nu+\mu-1} /\Gamma(\mu + \nu) \;,$$

which, when substituted above, is seen to establish the identity.

If $f(x)$ is a function analytic about the origin and vanishing there,

$$f(x) = f_1x + f_2x^2 + f_3x^3 + \cdots \;,$$

we see from the well-known formula

$$\int_0^1 (\log 1/t)^{\nu-1} t^{m-1} dt = \Gamma(\nu)/m^\nu$$

that the Hadamard operator transforms $f(x)$ as follows:

$$\Omega_x^{-\nu} f(x) = \sum_{m=1}^{\infty} f_m x^m m^{-\nu} \;. \tag{12.13}$$

(*e*) (*The generatrix operators of Laplace*). The basis of the calculus of generatrix functions which P. S. Laplace adopted as the analytical method of his *Théorie des Probabilités* is created from two operators G and D defined thus:

If $f(x)$ is a function defined by the series

$$f(x) = \sum_{n=-\infty}^{\infty} a(n)\, x^n \;,$$

where $a(n)$ is a function of n, we shall then have, by definition,

$$Ga(n) = f(x) \quad \text{and} \quad Df(x) = a(n) \;. \tag{12.14}$$

The function $f(x)$ is called the *generatrix* of $a(n)$ and $a(n)$ is called the *déterminante* of $f(x)$.

The use of these symbols and their connection with other operators may be illustrated in the following way:

Let us assume that $f(x)$ is expansible in the series $f(x) = \sum_{n=0}^{\infty} a(n) x^n$. Then since $f(x)/x^m = \sum_{n=0}^{\infty} a(n)\, x^{n-m}$, we obtain from (12.14),

$$Ga(n + m) = f(x)/x^m \;.$$

Also, since

$$(1/x - 1)\, f(x) = a(0)/x + [a(1) - a(0)] + \cdots$$
$$[a(n+1) - a(n)]x^n + \cdots ,$$

we have

$$G[\Delta a(n)] = (1/x - 1)\, f(x) ,$$

where $\Delta\, a(n) = a(n+1) - a(n)$.

More generally, but in similar fashion, we prove that

$$G\,\Delta^m\, a(n) = (1/x - 1)^m\, f(x) .$$

In order to apply this formula to the calculus of finite differences we write

$$f(x)/x^m = (1 + 1/x - 1)^m\, f(x) = [1 + m(1/x - 1)$$
$$+ m(m-1)\,(1/x - 1)^2/2! + \cdots]\, f(x) .$$

Operating upon both members with D we then get

$$Df(x)/x^m = a(n+m) = a(n) + m\,\Delta\, a(n)$$
$$+ m(m-1)\,\Delta^2\, a(n)/2! + \cdots , \qquad (12.15)$$

which we recognize as Newton's formula for interpolation.

For a more complete account of the significance of this operator the reader is referred to section 7, chapter 1.

(*f*) (The "*Calcul de Généralisation*" of Oltramare). An operator resembling in some ways that of Laplace was made the basis of a calculus by G. Oltramare which was developed and applied by C. Cailler and D. Mirimanoff in numerous ways.

The calculus is based upon the operational symbol

$$Ge^{au} = \varphi(x+a) ,$$

or more generally,

$$Ge^{au+bv+cw+\cdots} = \varphi(x+a,\, y+b,\, z+c, \cdots) .$$

From the expansion

$$Ge^{au} = G[1 + au + (au)^2/2! + (au)^3/3! + \cdots]$$
$$= \varphi(x) + a\varphi'(x) + a^2\varphi''(x)/2! + \cdots$$

we derive the fundamental operator

$$Gu^n = \varphi^{(n)}(x) . \qquad (12.16)$$

In illustration of the application of his calculus, Oltramare applied it to the solution of the functional equation

$$f(x, y) - af(x-1, y+1) = 0 . \qquad (12.17)$$

Since we have by definition

$$Ge^{xu+yv} = f(x, y)$$

and

$$Ge^{(x-1)u+(y+1)v} = f(x-1, y+1) \ ,$$

equation (12.17) can be written symbolically

$$Ge^{xu+yv}\ (1 - ae^{-u+v}) = 0 \ ,$$

from which we get

$$1 - ae^{-u+v} = 0 \ .$$

Making use of this relationship between u and v we write

$$e^{xu+yv} = a^x e^{(x+y)v} = a^{-y}\ e^{(x+y)u} \ .$$

Operating upon these functions and recalling definitions, we then obtain

$$Ge^{xu+yv} = f(x, y) = a^x \varphi(x+y) = a^{-y} \vartheta(x+y) \ ,$$

where $\varphi(z)$ and $\vartheta(z)$ are arbitrary functions. Since the last term may be written $a^x[a^{-(x+y)}\ \vartheta(x+y)]$, it is clear that these solutions are essentially equivalent.

The calculus is also extended to functions represented by definite integrals

$$Gf(u) = \int_a^\beta Gg(u, t)\ dt \ ;$$

for example

$$1/(u+a)^n = \int_0^\infty e^{-at}\ e^{-tu}\ t^{n-1}\ dt/\Gamma(n) \ ,$$

$$G\,1/(u+a)^n = \int_0^\infty e^{-at}\ t^{n-1}\ \varphi(x-t)\ dt/\Gamma(n) \ .$$

The Liouville definition of fractional differentiation is obtained as a special case of this formula. Making the specialization $a = 0$, and $n = -\mu + m$ and recalling (12.16), we then get

$$Gu^{\mu-m} = d^{\mu-m}\varphi(x)/dx^{\mu-m} = \int_0^\infty t^{m-\mu-1}\ \varphi(x-t)\ dt/\Gamma(m-\mu) \ ,$$

which we see is identical with the definition of Liouville discussed in section 7.

It will be seen from this description that the *calcul de généralisation* is not the development of a principal of functional transformation but is essentially a table of symbol equivalents which is indispensable for its application. The table given below includes some of the most common elements.

TABLE OF G– TRANSFORMATIONS

(1) $Gu^n = \phi^{(n)}(x)$,

(2) $G(u+a)^{-n} = \int_0^\infty e^{-at}\, t^{n-1}\, \phi(x-t)\, dt/\Gamma(n)$.

(3) $Gu^{\mu-n} = \phi^{(\mu-n)}(x) = \int_0^\infty t^{n-\mu-1}\, \phi(x-t)\, dt/\Gamma(n-m)$.

(4) $Ge^{au} = \phi(x+a)$. (5) $Ge^{\pi u} = \phi(x)$.

(6) $Ge^{au^2} = (1/\pi^{\frac{1}{2}}) \int_{-\infty}^{\infty} e^{-y^2}\, \phi[x + 2/(ay)^{\frac{1}{2}}]\, dy$.

(7) $G\, 1/(e^{au} + e^{-au}) = \frac{1}{2} \int_{-\infty}^{\infty} \{\phi(x+ayi)/(e^{\frac{1}{2}\pi y} + e^{-\frac{1}{2}\pi y})\}\, dy$.

(8) $G\, 1/\cos au = \int_{-\infty}^{\infty} \{\phi(x+at)/(e^{\frac{1}{2}\pi t} + e^{-\frac{1}{2}\pi t})\}\, dt$.

(9) $G\, 1/(a^2 + b^2u^2) = (1/\pi) \int_{\infty}^{\infty} \phi(x+yi) + \phi(x-yi)\, dy$

$$\times \int_0^\infty \{\cos yv/(a^2+b^2v^2)\}\, dv .$$

(10) $G \sin au = (1/2i)\{\phi(x+ai) - \phi(x-ai)\}$.

(11) $G \cos au = \frac{1}{2}\{\phi(x+ai) + \phi(x-ai)\}$.

(12) $G\, e^{-u^2/4q^2} \cos au$

$$= (q/\pi^{\frac{1}{2}}) \int_{-\infty}^{\infty} e^{-q^2t^2}\, \{\phi[x+(t+a)i] + \phi[x+(t-a)i]\}\, dt .$$

(13) $G\, e^{-y^2/4u}/u^{\frac{1}{2}} = (1/\pi^{\frac{1}{2}}) \int_0^\infty \phi(x-v^2) \cos yv\, dv$.

(14) $G\, 1/(a+bu+cu^2) = \{1/(b^2-4ac)^{\frac{1}{2}}\} \int_0^\infty e^{-bt/2c}\{e^{t(b^2-4ac)^{1/2}/2c}$

$$- e^{t(b^2-4ac)^{1/2}/2c}\}\, \phi(x-t)\, dt , \qquad b^2 > 4ac .$$

$$= \{2/(4ac-b^2)^{\frac{1}{2}}\} \int_0^\infty e^{-bt/2c} \sin\{(4ac-b^2)^{\frac{1}{2}}\, t/2c\}\, \phi(x-t)\, dt ,$$

$$b^2 < 4ac .$$

(15) $G\, 1/(a^4+b^4u^4) = (1/\pi) \int_0^\infty \{\phi(x+yi) + \phi(x-yi)\}\, dy$

$$\times \int_0^\infty \{\cos yv/(a^4+b^4v^4)\}\, dv .$$

(16) $$G\, e^{-a/u}/u^n = \pi i \int_0^\infty \int_0^\infty e^{-ay/t}\, y^{n-1} \{\sin t/t^{n+1}\}\{\phi(x+yi) - \phi(x-yi)\}\, dy\, dt\ .$$

(17) $$G\, e^{-au^{1/2}} = (1/\pi^{\frac{1}{2}}) \int_{-\infty}^{\infty} e^{-t^2} \phi[x - a^2/4t^2]\, dt\ .$$

(18) $$G\, e^{-au^{1/4}} = (1/\pi) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-s^2-t^2} \phi(x - a/64s^4t^2)\, ds\, dt\ .$$

(19) $$G\, 1/(1-e^{au}) = -\tfrac{1}{2}\phi(x) - \tfrac{1}{2} i \int_{-\infty}^{\infty} \{(e^{\pi y} + e^{-\pi y})\ \phi(x+ayi)/(e^{\pi y} - e^{-\pi y})\}\, dy\ .$$

(20) $$G\, 1/(1+e^{au}) = \tfrac{1}{2}\phi(x) + i \int_{-\infty}^{\infty} \{\phi(x+ayi)/(e^{\pi y} - e^{-\pi y})\}\, dy\ .$$

(21) $$G\, e^{bu}/(1-e^{au}) = -\tfrac{1}{2}\phi(x+b) - \tfrac{1}{2} i \int_{-\infty}^{\infty} \{(e^{\pi y} + e^{-\pi y})\ \phi(x+b+ayi)/(e^{\pi y} - e^{-\pi y})\}\, dy\ .$$

(22) $$G\, 1/(p+qu) = \int_0^\infty e^{-pt}\, \phi(x-qt)\, dt\ .$$

The efficacy of this table in the solution of certain types of integral equations will be evident from an example.

Let us consider

$$\int_0^\infty e^{-pt}\, \varphi(x-qt)\, dt = f(x)\ ,$$

which from (5) and (22) can be written

$$G\, 1/(p+qu) = G\, e^{xu}\ .$$

From this we at once derive

$$G\, 1 = \varphi(x) = G\, e^{xu}\, (p+qu) = p\, f(x) + q\, f'(x)\ .$$

It will be clear, however, that this process is entirely formal and the operator gives no information regarding the validity of the solution thus attained.

The method within its formal scope, however, is general and yields the solution of the integral equation

$$\int_\alpha^\beta \varphi(x-T_1, y-T_2, \cdots)\, T\, dt = f(x, y, \cdots)\ ,$$

where T, T_1, T_2, $\cdots$ are functions of t, in the symbolic form

$$\varphi(x, y, \cdots) = G\{e^{xu+yv+}\cdots / \int_a^\beta e^{-(T_1u + T_2v + \ldots)}\, T\, dt\} \,.$$

PROBLEMS

1. Given $F(z) = e^{az}$, compute $F(z) \to e^{bx}/(1+x)$.
2. Prove that

$$e^{(az)^2} \to e^{-x^2} = \phi(a)\, e^{-x^2/(1+4a^2)} \,,$$

where we abbreviate

$$\phi(a) = 1 - 2!\, a^2 + \frac{4!}{(2!)^2} a^4 - \frac{6!}{(3!)^2} a^6 + \cdots \,.$$

See C. W. Oseen: Sur une application des méthodes sommatoires de M. M. Borel et Mittag-Leffler. *Arkiv for Mat., Astro. och Fysik,* vol. 12 (1917), No. 16, pp. 1-13.

3. Defining $\theta = xz$, prove that

$$e^{a\theta} \to u(x) = u(x\, e^a) \,, \quad e^{a\,r\theta} \to u(x) = u[x/(1-ax)] \,.$$

4. If $\theta = xz$, show that

$$\theta \to \theta^{(n)} = \theta \cdot \theta^{(n)} + \theta \frac{\partial}{\partial \theta} \theta^{(n)} \,.$$

Hence show that

$$\begin{aligned}
\theta^{(2)} &= \theta + \theta^2 \,, \\
\theta^{(3)} &= \theta + 3\,\theta^2 + \theta^3 \,, \\
\theta^{(4)} &= \theta + 7\,\theta^2 + 6\,\theta^3 + \theta^4 \,, \\
\theta^{(5)} &= \theta + 15\,\theta^2 + 25\,\theta^3 + 10\,\theta^4 + \theta^5 \,, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\theta^{(n)} &= S_1^n\,\theta + S_2^n\,\theta^2 + S_3^n\,\theta^3 + \cdots + S_n^n\,\theta^n \,,
\end{aligned}$$

where the coefficients are defined in terms of *differences of zero,*

$$S_r^n = [\Delta^r\, 0^n]/r! \,.$$

(For numerical values see Davis: *Tables of the Higher Mathematical Functions,* vol. 2, p. 212).

5. If $\theta = xz$, show that

$$\begin{aligned}
x^2 z^2 &= \theta(\theta-1) = \theta^{(2)} - \theta \,, \\
x^3 z^3 &= \theta(\theta-1)(\theta-2) = \theta^{(3)} - 3\,\theta^{(2)} + 2\,\theta \,, \\
x^4 z^4 &= \theta(\theta-1)(\theta-2)(\theta-3) = \theta^{(4)} - 6\,\theta^{(3)} + 11\theta^{(2)} - 6\,\theta \,, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
x^n z^n &= T_n^n\,\theta^{(n)} + \cdots + T_3^n\,\theta^{(3)} + T_2^n\,\theta^{(2)} + T_1^n\,\theta \,,
\end{aligned}$$

where the values T_r^n are defined in terms of the *differential coefficients of zero,*

$$T_r^n = [\Delta^r 0^{(n)}]/r! \,.$$

(For numerical values see Davis: *Tables of the Higher Mathematical Functions,* vol. 2, p. 215).

6. Prove that

$$\theta^{(n)} \to \sum_{m=0}^{n-1} A_m (\log x)^m = 0 \,.$$

7. Prove that

$$\theta^{(n)} \to (x^m u) = x^m (\theta + m)^{(n)} \to u(x) \,,$$

and hence show that

$$F(\theta) \to (x^m u) = x^m[F(\theta + m) \to u(x)] \,.$$

13. The General Analytic Operator. Having thus surveyed what we might call the elementary operators, we turn to a more comprehensive theory which will include the preceding as special cases. This theory centers about the properties of what we shall refer to as the *general analytic linear operator,* which possesses the expansion.

$$F(x,z) = z^{\mu}\varphi(\log z)\{A_1(x,z) + B_1(x,1/z)\} + z^{\nu}\{A_2(x,z) + B_2(x,1/z)\} \ . \tag{13.1}$$

The functions $A_i(x,y)$, $B_i(x,y)$ are analytic in y about $y=0$, the latter vanishing there;

$$\begin{aligned} A_i(x,y) &= a_{i0}(x) + a_{i1}(x)y + a_{i2}(x)y^2 + \cdots \ , \\ B_i(x,y) &= b_{i1}(x)y + b_{i2}(x)y^2 + b_{i3}(x)y^3 + \cdots \end{aligned} \tag{13.2}$$

The functions $a_{ij}(x)$, $b_{ij}(x)$ possess a common domain of existence (R), such that if x is restricted to this domain, the functions $A_i(x,y)$ and $B_i(x,y)$ exist for values of y within a second domain S.*

Although it is our purpose to explore in the ensuing pages the many specializations of which the operator $F(x,z)$ is capable and to claim for it an important rôle in the general theory of linear functional equations, no attempt is made or even contemplated to endow it with the title of *the general linear operator.* The intriguing generality of linear operation in the abstract has been made the basis of the school of mathematical philosophy inaugurated by the *general analysis* of E. H. Moore and is the object of the more recent investigations of N. Wiener, J. von Neumann, H. Weyl, and M. H. Stone through the medium of the Fourier integral and the Stieltjes-Lebesgue integral.

We should also observe that the operators $B_i(x,1/z)$ are in many cases essentially included in the operators $A_i(x,z)$ since $1/z^n$ can be replaced by the generatrix function $Q_n(x,z) = [1 - \{1 + xz + (xz)^2/2! + \cdots + (xz)^{n-1}/(n-1)!\}\, e^{-xz}]/z^n$, provided the constant limit of the integral is zero (or more generally any finite constant). In this manner the operators $B_i(x,1/z)$ are transformed formally into operators without singularities. When, however, the constant limit of the integral is infinite, this is no longer the case since the generatrix function then becomes

$$\lim_{c=-\infty} Q_n(x-c,z) = 1/z^n \ ,$$

and the polar character of the operator cannot be removed.

One of the most interesting features of the operational calculus from the present point of view is the variety of approaches to basic formulas. Emil L. Post (see *Bibliography*) has employed the following definitions in an effective manner.

*While this statement is true in general, exceptions will appear in subsequent developments.

$$\left\{ f(D) \right\}_{X_0}^{X} \phi(x) = \lim_{\Delta x \to 0} \left\{ f(\Delta/\Delta x) \right\}_{X_0}^{X} \phi(x) = \lim_{\Delta x \to 0} \left\{ f(1 - e^{-\Delta x})/\Delta x \right\}_{X_0}^{X} \phi(x)$$

$$= \lim_{\substack{\Delta x \to 0 \\ p \cdot \{(X - X_0)/\Delta x\}}} [f(1/\Delta x)\, \phi(X) - f'(1/\Delta x)\, \phi(X - \Delta x)/1!\, \Delta x + \cdots$$

$$+ (-1)^p f^{(p)}(1/\Delta x)\, \phi(X - p\Delta x)/p!\, \Delta x^p] \;,$$

$$\left\{ f(D) \right\}_{-\infty}^{X} \phi(x) = \lim_{\Delta x \to 0} [f(1/\Delta x)\, \phi(X) - f'(1/\Delta x)\, \phi(X - \Delta x)/1!\, \Delta x + \cdots] \;.$$

As an example we set $f(D) = \log D$, $\phi(x) \equiv 1$, X_0 finite. We then have by the first formula

$$\left\{ \log\, (\Delta/\Delta\, x) \right\}_{X_0}^{X} 1 = \log(1/\Delta\, x) - 1 - 1/2 - \cdots - 1/p$$

$$= -\, (1 + 1/2 + \cdots + 1/p - \log p) - \log(p\, \Delta\, x) \;.$$

As $\Delta\, x \to 0$, p increases indefinitely and $p\, \Delta\, x$ approaches $X - X_0$. Furthermore, as $p \to \infty$, the bracket has the limit C (Euler's constant) and we thus get the value

$$\left\{ \log(D) \right\}_{X_0}^{X} 1 = -\, C - \log(X - X_0) \;.$$

14. The Differential Operator of Infinite Order. Limiting our attention for the present to the operator

$$A(x, z) = a_0(x) + a_1(x) z + a_2(x) z^2/2! + \cdots \;, \qquad (14.1)$$

we shall first consider the bounds to be imposed upon its generality. The content of this discussion is based upon the work of C Bourlet and S. Pincherle, the development of the former being closely followed.

Let us denote by S a linear operator and by $S(u)$ the value obtained when S operates upon some member, u, of a class of functions which we may denote by A.*

The following definitions will be useful:

An operator will be called *continuous* when

$$\lim_{h \to h'} S[u(x, h)] = S[\lim_{h \to h'} u(x, h)] \;,$$

and *regular* when the result of operating upon every function analytic in a certain domain D is a function analytic in a second domain D'.

An operator will be called *uniform* if it sets up a one-to-one correspondence between analytic functions. Otherwise it will be *multiform*. For example, the definite integral is uniform but the indefinite integral is multiform.

The following theorem shows the formal scope of the operator (14.1):

---

*Bourlet uses the word "*transmutation*" for S and "*transmuée*" for $S(u)$. He says of the word operator: "Opération était un terme trop vague et qui se prêtait mal à la formation de mots dérivés."

Theorem 1. Every uniform, continuous, regular, linear operator can be expressed in the form

$$S(u) = A(x, z) \rightarrow u(x) \ ,$$

where $A(x, z)$ *is defined by* (14.1) *and the coefficients are*

$$a_m(x) = S(x^m) - {}_mC_1\, x\, S(x^{m-1}) + {}_mC_2\, x^2\, S(x^{m-2}) - \cdots \pm x^m S(1) \ ,$$

in which ${}_mC_1$, ${}_mC_2$, $\cdots$, *are binomial coefficients.*

Proof: For the proof of this we let $S(u)$ be a uniform, continuous, regular, linear operator and we determine a function such that the difference

$$S(u) - a_0 u$$

will be identically zero when $u = 1$. We thus find that $a_0 = S(1)$.

Similarly a second function, $a_1(x)$, can be determined so that it will satisfy the condition

$$S(u) - a_0 u - a_1 u' = 0$$

when u is set equal to x. It will be readily seen that

$$a_1(x) = S(x) - xS(1) \ .$$

In similar fashion $a_2(x)$ can be so constructed that

$$S(u) - a_0 u - a_1 u' - a_2 u''/2! = 0 \ ,$$

when $u = x^2$; thus we get

$$a_2(x) = S(x^2) - 2xS(x) + x^2 S(1) \ .$$

Continuing this process we obtain without difficulty the general formula

$$a_m(x) = S(x^m) - {}_mC_1\, x\, S(x^{m-1}) + {}_mC_2\, x^2\, S(x^{m-2}) - \cdots \pm x^m S(1) \ , \tag{14.2}$$

which can be established for the general index by induction.

Now consider the series

$$T(u) = A(x, z) \rightarrow u(x)$$

which for every analytic function $u(x)$ that renders this series convergent will define a uniform, continuous, regular linear operator.

Then the difference

$$U \equiv S - T$$

is a uniform, regular, continuous linear operator for which, moreover, the transformation of 1, x, x^2, $\cdots$, namely, of all the integral powers of x, is equal to zero.

Now suppose that $u(x)$ is a function regular in the neighborhood of x_0,

$$u(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)^2 + \cdots + A_m(x-x_0)^m + \cdots ,$$

and consider the polynomial

$$u_m(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)^2 + \cdots + A_m(x-x_0)^m$$

We shall then have

$$S(u_m) - T(u_m) = U(u_m) = 0$$

for all values of m.

But since the operator was assumed to be continuous, it follows that

$$U(u) = U(\lim_{m\to\infty} u_m) = \lim_{m\to\infty} U(u_m) = 0 .$$

We thus establish the desired identity

$$S(u) \equiv T(u) .$$

It is possible to extend the application of theorem 1 to special cases even when $T(u)$ does not converge for analytic functions regular about x. Thus it is clear that $T(u_m)$ will always exist and it may happen that $\lim T(u_m)$ will exist without $T(\lim u_m)$ existing also. In this case we may assign to $S(u)$ the value $\lim T(u_m)$ much in the same way as values are assigned to divergent series.

If we indicate by means of a subscript the variable to which the operator applies, theorem 1 can be put in a form that is often useful in application.

Thus replacing $u(t)$ in the operator $S_t(u)$ by its Taylor transform (6.3) and making use of the linear character of the operator, we shall obtain

$$S_t(u) = J_0\, u(x) + J_1\, u'(x) + J_2\, u''(x)/2! + \cdots$$
$$+ J_n\, u^{(n)}(x)/n! + \cdots , \qquad (14.3)$$

where we have used the abbreviation

$$J_n = S_t[(t-x)^n] .$$

PROBLEMS

1. Derive Taylor's expansion from theorem 1 by assuming $S(u) = u(a)$.
2. Assume that

$${}_cD_x^{-\nu} \to u(x) = [\sum_{m=0}^{\infty} a_m(x) z^m] \to u(x) ,$$

and compute the coefficients $a_m(x)$ by means of (14.2) and the table of fractional derivatives given in section 9. Now compare with the expansion given in problem 5, section 9, and hence establish the identity

$$\frac{\nu}{\nu+m}=1-\frac{m}{\nu+1}+\frac{m(m-1)}{(\nu+1)(\nu+2)}-\cdots .$$

3. Define $S(u)=\sum_{n=1}^{x} u(n)$, and compute the coefficients $a_m(x)$ by means of (14.2). Then compare the resulting expansion with formula (12.8) and derive a relationship between the Bernoulli polynomial of degree p and the sums

$$S_n(x)=\sum_{m=1}^{x} m^n , \qquad n=0,1,2,\cdots,p .$$

15. Differential Operators as a Cauchy Integral. In the last section we saw how a differential operator of infinite order could be expressed in a form analogous to Taylor's series. We shall now express $S(u)$ as a contour integral in the complex plane.

Thus let us consider the operator

$$S(u)=A(x,z)\to u(x) , \tag{15.1}$$

where $A(x,z)$ is defined by (14.1) in which $a_0(x)$, $a_1(x),\cdots$ are functions analytic in a domain R. Then employing Cauchy's formula,

$$u^{(m)}(x)=(m!/2\pi i)\int_c [u(z)/(z-x)^{m+1}]\, dz ,$$

we have upon substitution in (15.1) the formula

$$S(u)=\lim_{m\to\infty}(1/2\pi i)\int_c [u(z)\,\varphi_m(x,z)/(z-x)]\, dz ,$$

where we abbreviate

$$\varphi_m(x,z)=a_0(x)+a_1(x)/(z-x)+\cdots+a_m(x)/(z-x)^m .$$

If, as m increases indefinitely, φ_m converges to a function $\varphi(x,z)$ we attain the formula

$$S(u)=(1/2\pi i)\int_c u(z)\,[\varphi(x,z)/(z-x)]\, dz . \tag{15.2}$$

It is of special interest to note the case in which an operator of form (15.1) furnishes a transformation for every function analytic in a given domain. This is supplied by the following theorem:

Theorem 2. A necessary and sufficient condition that the operator defined by (15.1) furnishes a transformation for every function $u(x)$ regular in a domain of radius ϱ around the point x_0, is that the series

$$\varphi(x,z)=a_0(x)+a_1(x)/(z-x)+a_2(x)/(z-x)^2+\cdots$$

shall be convergent for every value of z which satisfies the condition

$$|z-x_0|=\varrho .$$

That the condition is necessary is proved as follows: If $S(u)$ existed for every function analytic in a region R of radius ϱ around x_0, it would exist, in particular, for the function $u(x) = 1/(z—x)$.

Since for this function we have $u(x) = 1/(z—x)$, $u'(x) = 1/(z—x)^2, \cdots, u^{(n)}(x) = n!/(z—x)^{n+1}, \cdots$, by substitution in $S(u)$ we get

$$S[1/(z—x)] = \psi(x, z)/(z—x) \quad ,$$

where $\psi(x, z)$ is defined as above.

But this series must be convergent for every value of z for which $|z - x_0| = \varrho$.

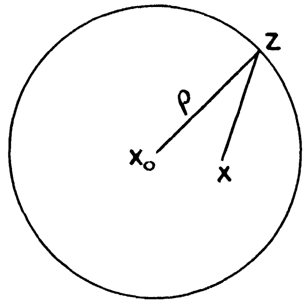

FIGURE 3

That the condition is sufficient follows at once from the fact that $\psi(x, z)$ converges on the boundary of a circle of radius ϱ described around x as center, for if $u(z)$ is a function analytic within and on the circle of circumference C, then

$$T(u) = (1/2\pi i) \int_C [u(z)/(z—x)] \, \psi(x, z) \, dz$$

exists and defines a transformation for ever function analytic in the region. Such an operator, $S(u)$, will be called *complete* in the domain R.

16. The Generatrix of Differential Operators. The function $A(x, z)$ defined explicitly by (14.1) we shall refer to as the *generatrix* of the differential operator.*

The central theorem relating to the generatrix of a uniform, continuous, and regular operator is the following:

Theorem 3. The generatrix of a uniform, continuous, and regular operator, which is also complete in a certain domain around the point x_0, is, for this value of x_0, a transcendental entire function in z of genus 1 or 0.

*This term is due to Lalesco. See *Bibliography*: Lalesco (1), p. 193. Bourlet uses the term "*fonction operative.*"

Proof: From the definition of completeness the series

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

must be convergent for $|z| = 1/\varrho$, where ϱ is the radius of the domain of regularity around x. Hence $A(x, z)$ is convergent for every value of z and, consequently, $A(x, z)$ is an entire function of this variable.

Moreover, if we designate by ε a positive number arbitrarily small, we shall have, for a sufficiently large m, the inequality

$$|a_m|/(\varrho + \varepsilon)^m < 1 \text{ , and hence } |a_m|/m! < (\varrho + \varepsilon)^m/m! \ .$$

Also, if η is any positive number, we have for a sufficiently large value of m, the inequality $(\varrho + \varepsilon)^m < (m!)^\eta$, so that

$$|a_m|/m! < 1/(m)^{1-\eta} \ . \tag{16.1}$$

We now make use of a lemma due to J. Hadamard.*

Lemma. If, in an entire function in z, the coefficient of z^m remains less than $1/(m!)^{1/\lambda}$, the function is of genus less than λ where λ is not an integer.

Referring now to (16.1) we see that, since the coefficient is smaller than $1/(m!)^{1-\eta}$ in absolute value, we shall have $\lambda = 1/(1-\eta)$, and since η is a number as small as we wish the function must be of class 1 or 0.

17. Five Operators of Analysis. The object of the present volume is to exhibit the formal unification attained in the solution of linear functional equations by means of an exploration of the properties of the general analytic operator of section 13. This is most clearly seen if we translate five important operators of anaylsis into the form given.

The first operator is the polynomial operator of section 2,

$$F_p(x, z) = A_0(x) + A_1(x)z + A_2(x)z^2 + \cdots A_p(x)z^p \ , \tag{17.1}$$

where the $A_i(x)$ are functions of x which share a common region of definition.

The second operator is the operator of the Fredholm transformation,

$$A\,u(x) + \int_a^b K(x, t)\,u(t)\,dt \ , \tag{17.2}$$

where $K(x, t)$ is subject to certain limitations which will not be specified here. If we apply to $u(t)$ the Taylor transformation (6.3) we can write (17.2) in the symbolic form

*Etude sur les properiétés des fonctions entières. *Journal de mathématiques*, 4th ser., vol. 9 (1893), p. 172. See also section 2, chapter 5.

$$\{A + \int_a^b K(x, t)\, e^{(t-x)z}\, dt\} \to u(x) \quad .$$

We shall refer to the function

$$F_f(x, z) = A + \int_a^b K(x, t)\, e^{(t-x)z}\, dt \qquad (17.3)$$

as the *Fredholm generatrix of first kind* when $A = 0$ and the *Fredholm generatrix of second kind* when $A = 1$.

The third operator is derived from the Volterra transformation of a function,

$$A\, u(x) + \int_c^x K(x, t)\, u(t)\, dt \ , \qquad (17.4)$$

which can be written by means of the Taylor transformation of $u(t)$ as

$$\{A + \int_c^x K(x, t)\, e^{(t-x)z}\} \to u(x) \quad ;$$

or by applying the Taylor transformation to $K(x, t)$, as

$$\{A + K(x, x)/z - K_t'(x, x)/z^2 + K_t''(x, x)/z^3$$
$$- K_t'''(x, x)/z^4 + \cdots\} \to u(x)$$
$$= \{A + \int_0^\infty e^{-tz} K(x, x-t)\, dt\} \to u(x) \quad .$$

If $A = 0$, we shall refer to the function

$$F_v(x, z) = A + \int_c^x K(x, t)\, e^{(t-x)z}\, dt \qquad (17.5)$$

or its symbolic equivalent,

$$F_v(x, z) = A + \int_0^\infty e^{-tz} K(x, x-t)\, dt \ , \qquad (17.6)$$

as the *Volterra generatrix of first kind*, and if $A = 1$, as the *Volterra generatrix of second kind*.

A fourth type of operator to which mathematicians have devoted much attention is the operator of the difference transformation,

$$\varphi_0(x)\, u(x) + \varphi_1(x)\, u(x+\omega) + \varphi_2(x)\, u(x+2\omega)$$
$$+ \cdots + \varphi_n(x)\, u(x+n\omega) \ ,$$

where the functions $\varphi_i(x)$ have a common region of definition.

The generatrix corresponding to this transformation is easily

obtained by replacing $u(x + r\omega)$ by $e^{r\omega z} \to u(x)$. We thus get for the generatrix

$$F_d(x, z) = \varphi_0(x) + \varphi_1(x)\, e^{\omega z} + \varphi_2(x)\, e^{2\omega z} + \cdots + \varphi_n(x)\, e^{n\omega z} \;. \tag{17.7}$$

Closely associated with the last function is the generatrix of the q-difference operator,

$$\varphi_0(x)\, u(x) + \varphi_1(x)\, u(q_1 x) + \varphi_2(x)\, u(q_2 x) + \cdots + \varphi_n(x)\, u(q_n x) \;.$$

Replacing $u(q_i x)$ by its Taylor transformation $e^{\omega_i x z} \to u(x)$, where $\omega_i = q_i - 1$, we obtain the desired generatrix,

$$F_q(x, z) = \varphi_0(x) + \varphi_1(x)\, e^{\omega_1 x z} + \varphi_2(x)\, e^{\omega_2 x z} + \cdots + \varphi_n(x)\, e^{\omega_n x z} \;. \tag{17.8}$$

The equations corresponding to these operators, i. e.,

$$\begin{aligned} F_p(x, z) &\to u(x) = f(x) \;, \\ F_f(x, z) &\to u(x) = f(x) \;, \\ F_v(x, z) &\to u(x) = f(x) \;, \\ F_d(x, z) &\to u(x) = f(x) \;, \\ F_q(x, z) &\to u(x) = f(x) \;, \end{aligned}$$

are referred to as *differential equations, integral equations of Fredholm type, integral equations of Volterra type, difference equations,* and *q-difference equations* respectively. Variants of these types are, of course, common in mathematical literature, but without exception they can be included with suitable restrictions under the general theory of the analytic operator.

For example, the mixed integral equation the development of which is largely due to A. Kneser, i. e.,*

$$u(x) + \sum_{i=1}^{n} \varphi_i(x)\, u(\zeta_i) + \int_a^b K(x, t)\, u(t)\, dt = f(x) \;,$$

where the $\{\zeta_i\}$ form a discrete set of points in the interval (a, b), is included in the general theory under the formulization

$$\{1 + \sum_{i=1}^{n} \varphi_i(x)\, e^{(\zeta_i - x)z} + \int_a^b K(x, t)\, e^{(t-x)z} dt\} \to u(x) = f(x) \;.$$

*Belastete Integralgleichungen. *Rendiconti di Palermo*, vol. 37 (1914), pp. 169-197.

CHAPTER III.

THE THEORY OF LINEAR SYSTEMS OF EQUATIONS.

1. Preliminary Remarks. It will become apparent in the subsequent development of our subject that the theory of operators may be envisaged as an aspect of the theory of linear systems of equations, where the number of variables and the number of equations are infinite. Such a system may be conveniently represented in the following notation:

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n + \cdots = b_1 \ , \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n + \cdots = b_2 \ , \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n + \cdots = b_3 \ , \\
\quad . \quad\quad . \quad\quad . \quad\quad . \quad\quad . \quad\quad . \quad\quad . \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n + \cdots = b_m \ , \\
. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .
\end{array}
\qquad (1.1)
$$

The matrix of the system will be designated by

$$A = ||a_{ij}||$$

and the unit matrix by

$$I = ||\delta_{ij}|| \ ,$$

where δ_{ij} (Kronecker's symbol) is zero for $i \neq j$ and unity for $i = j$.

The law of multiplication for finite matrices will be assumed to hold in the infinite case provided the infinite series which compose the elements of the product matrix converge. Thus if we have

$$A = ||a_{ij}|| \qquad \text{and} \quad B = ||b_{ij}|| \ ,$$

the product may be written

$$A \times B = C = ||c_{ij}|| \ ,$$

where we assume the existence of the sum

$$c_{ij} = \sum_{n=1}^{\infty} a_{in}\, b_{nj} \ .$$

The matrix *conjugate* to A will be designated by the customary symbol A', that is

$$A' = ||a_{ji}|| \ ,$$

and the *reciprocal* or *inverse* matrix by A^{-1}, that is,

$$A^{-1} = \left\| \frac{A_{ji}}{|A|} \right\| , \quad |A| \neq 0 ,$$

where A_{ij} is the cofactor of the element a_{ij} and $|A|$ is the determinant of A.

It is clear, however, that a unique inverse may not exist for A. Definitions and theorems pertaining to this situation will be given in section 9.

In most of the ensuing discussion it will be assumed for convenience that the elements a_{ij} are real, but this is not in general a restrictive condition. Many of the results hold for what is called the *Hermitian matrix,*

$$H = \|a_{ij}\| ,$$

where we assume that $a_{ij} = \bar{a}_{ji}$. The bar over the a designates the complex conjugate of a_{ji}.

2. Types of Matrices. It will be found that there are several types of infinite matrices which are especially useful in the theory of operators and we shall confine our attention particularly to these.

(*A*) The *secular* or *normal* matrix will refer to the limiting form, as n approaches infinity, of the following:

$$S_n = \left\| \begin{array}{ccccccc} 1+a_{11} & a_{12} & a_{13} & & \cdots & & a_{1n} \\ a_{21} & 1+a_{22} & a_{23} & & \cdots & & a_{2n} \\ a_{31} & a_{32} & 1+a_{33} & & \cdots & & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & & \cdots & & 1+a_{nn} \end{array} \right\| \tag{2.1}$$

The determinant of S_n we can write briefly as

$$D(n) = | \delta_{ij} + a_{ij} | ,$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$. It can be proved without difficulty that $D(n)$ has the following development:

$$D(n) = 1 + \frac{1}{1!} \sum_{r_1} a_{r_1 r_1} + \frac{1}{2!} \sum_{r_1, r_2} \begin{vmatrix} a_{r_1 r_1} & a_{r_1 r_2} \\ a_{r_2 r_1} & a_{r_2 r_2} \end{vmatrix}$$

$$+ \cdots + \frac{1}{n!} \sum_{r_i} \begin{vmatrix} a_{r_1 r_1} & a_{r_1 r_2} & \cdots & a_{r_1 r_n} \\ a_{r_2 r_1} & a_{r_2 r_2} & \cdots & a_{r_2 r_n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{r_n r_1} & a_{r_n r_2} & \cdots & a_{r_n r_n} \end{vmatrix} , \tag{2.2}$$

where $r_1, r_2, \cdots, r_n$ run independently over all the values from 1 to n.

*See G. Kowalewski: *Einführung in die Determinantentheorie.* Leipzig (1909), pp. 455-456.

It will be useful in another place also to have an explicit expansion of the cofactors of the elements of $D(n)$. For this purpose we introduce the determinant in the form

$$\begin{vmatrix} 1+a_{rr}, & a_{rs}, & a_{rk_1}, & \cdots, & a_{rk_{n-2}} \\ a_{sr}, & 1+a_{ss}, & a_{sk_1}, & \cdots, & a_{sk_{n-2}} \\ a_{k_1r}, & a_{k_1s}, & 1+a_{k_1k_1}, & \cdots, & a_{k_1k_{n-2}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k_{n-2}r}, & a_{k_{n-2}s}, & a_{k_{n-2}k_1}, & \cdots, & 1+a_{k_{n-2}k_{n-2}} \end{vmatrix} . \tag{2.3}$$

The cofactor of a_{sr}, $r \neq s$, can be expanded thus:

$$-D_{rs} = a_{rs} + \frac{1}{1!}\sum_{r_1}\begin{vmatrix} a_{rs}, & a_{rr_1} \\ a_{r_1s}, & a_{r_1r_1} \end{vmatrix} + \frac{1}{2!}\sum_{r_1r_2}\begin{vmatrix} a_{rs}, & a_{rr_1}, & a_{rr_2} \\ a_{r_1s}, & a_{r_1r_1}, & a_{r_1r_2} \\ a_{r_2s}, & a_{r_2r_1}, & a_{r_2r_2} \end{vmatrix}$$

$$+\cdots+\frac{1}{(n-2)!}\sum_{r_1r_2\cdots r_{n-2}}\begin{vmatrix} a_{rs}, & a_{rr_1}, & \cdots, & a_{rr_{n-2}} \\ a_{r_1s}, & a_{r_1r_1}, & \cdots, & a_{r_1r_{n-2}} \\ \cdot & \cdot & \cdot & \cdot \\ a_{r_{n-2}s}, & a_{r_{n-2}r_1}, & \cdots, & a_{r_{n-2}r_{n-2}} \end{vmatrix}, \tag{2.4}$$

where $r_1, r_2, \cdots, r_{n-2}$ range independently over all the values $k_1, k_2, \cdots, k_{n-2}$.*

(B) Another array that will be useful to us is the *triangular matrix,* which is the limiting form of the following:

$$\begin{Vmatrix} a_{11}, & a_{12}, & a_{13}, & \cdots, & a_{1n} \\ 0, & a_{22}, & a_{23}, & \cdots, & a_{2n} \\ 0, & 0, & a_{33}, & \cdots, & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \cdots, & a_{nn} \end{Vmatrix} \tag{2.5}$$

If we replace the elements $\{a_{ij}\}$ of the matrix by the special set $\{a_{j-i}\}$, $a_{j-i} = 0$, $i > j$, $j = 1, 2, \cdots, n+1$, and if we abbreviate the determinant formed by omitting the first column and the last row by the symbol

$$D_n(a_0, a_1, a_2, \cdots, a_n) \;;$$

*Kowalewski, *loc. cit.*, p. 468.

that is,

$$D_n(a_0, a_1, a_2, \cdots, a_n) = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_1 \end{vmatrix} , \qquad (2.6)$$

we can establish an important connection with the inversion of power series. This connection we find in the fact that the series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots, \ a_0 \neq 0 , \qquad (2.7)$$

has for its reciprocal the expansion

$$1/f(x) = (1/a_0)\{1 - D_1(x/a_0) + D_2(x/a_0)^2 - D_3(x/a_0)^3 + \cdots\} . \qquad (2.8)$$

Some interesting conclusions follow from this simple fact. For example, since $1/\{1/f(x)\} = f(x)$, we derive at once the following theorem in determinants:

$$D_n\{1, -D_1/a_0, D_2/a_0^2, -D_3/a_0^3, \cdots, (-1)^n D_n/a_0^n\} = (-1)^n a_n/a_0 .$$

It is well known in the theory of infinite series that if

$$\lim_{n\to\infty} | D_{n+1}/D_n | \text{ and } \lim_{n\to\infty} | D_n |^{1/n}$$

exist, they are equal to each other. Moreover, if either limit exists, it is equal to a_0/ϱ, where ϱ is the radius of convergence of series (2.8). Understanding then that the existence of the limits is implied, we can write

$$\lim_{n=\infty} | D_{n+1}/D_n | = \lim_{n\to\infty} | D_n |^{1/n} = a_0/\varrho . \qquad (2.9)$$

For example, from the expansion $\cos x^{\frac{1}{2}} = 1 - x/2! + x^2/4! - \cdots$ and the fact that $1/(\cos x^{\frac{1}{2}})$ has its nearest singularity at $x = \pi^2/4$, we are able to infer that

$$\pi^2/4 = \lim_{n=\infty} | \delta_{n-1}/\delta_n | = \lim_{n=\infty} | \delta_n |^{-1/n} ,$$

where we abbreviate

$$\delta_n = D_n[1, -1/2!, 1/4!, -1/6!, \cdots, 1/(2n)!] .$$

Similarly, since the Bernoulli numbers B_n are defined by the series

$$x/(e^x - 1) = 1 - x/2 + B_1 x^2/2! - B_2 x^4/4! + B_3 x^6/6! - \cdots \qquad (2.10)$$

and since

$$(e^x - 1)/x = 1 + x/2! + x^2/3! + \cdots ,$$

we see that

$$B_n = (-1)^{n+1}(2n)!\,D_{2n}[1, 1/2!, 1/3!, 1/4!, \cdots, 1/(2n+1)!] .$$

(C) The third kind of array with which we shall be concerned is the *Laurent matrix,* defined as the extension to infinity of the following:

$$\left\| \begin{matrix} \cdots & a_{-3-3} & a_{-3-2} & a_{-3-1} & a_{-30} & a_{-31} & a_{-32} & a_{-33} & \cdots \\ \cdots & a_{-2-3} & a_{-2-2} & a_{-2-1} & a_{-20} & a_{-21} & a_{-22} & a_{-23} & \cdots \\ \cdots & a_{-1-3} & a_{-1-2} & a_{-1-1} & a_{-10} & a_{-11} & a_{-12} & a_{-13} & \cdots \\ \cdots & a_{0-3} & a_{0-2} & a_{0-1} & \boxed{a_{00}} & a_{01} & a_{02} & a_{03} & \cdots \\ \cdots & a_{1-3} & a_{1-2} & a_{1-1} & a_{10} & a_{11} & a_{12} & a_{13} & \cdots \\ \cdots & a_{2-3} & a_{2-2} & a_{2-1} & a_{20} & a_{21} & a_{22} & a_{23} & \cdots \\ \cdots & a_{3-3} & a_{3-2} & a_{3-1} & a_{30} & a_{31} & a_{32} & a_{33} & \cdots \end{matrix} \right\| . \qquad (2.11)$$

This matrix is closely related to the theory of Laurent series in the theory of functions of a complex variable as will be later pointed out, and its name is due to this fact.* Historically it was one of the first matrices to be employed in the solution of infinite systems of equations, a special matrix of this type being used by G. W. Hill (1838-1914) in his theory of the motion of the moon.†

If we make the particular specialization

$$a_{pq} = c_{q-p} ,$$

then we have what is called the matrix of an *L-form,* that is to say, the bilinear form

$$L(x,y) = \sum_{p,q=1}^{\infty} c_{q-p}\, x_p\, y_q . \qquad (2.12)$$

The elements of the matrix are obviously the coefficients of the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n . \qquad (2.13)$$

*See O. Toeplitz: Analytische Theorie der L-Formen. *Mathematische Annalen,* vol. 70 (1911), pp. 351-376.

†On the Part of the Motion of the Lunar Perigee which is a Function of the Mean Motions of the Sun and Moon. *Acta Mathematica,* vol. 8 (1886), pp. 1-36. First published at Cambridge, Mass. in 1877. See sections 7 and 8, chapter 9.

(D) A fourth array associated with a number of interesting applications is what is called the *Jacobi matrix*, namely,

$$\left\|\begin{array}{cccccc} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right\| . \tag{2.14}$$

This matrix is the matrix of the *Jacobi bilinear form*,

$$J(x,y) = \sum_{p=1}^{\infty} a_{pp}\, x_p\, y_p + \sum_{p=1}^{\infty} (a_{p,p+1}\, x_p\, y_{p+1} + a_{p+1,p}\, x_{p+1}\, y_p) \ , \tag{2.15}$$

which has been especially studied by O. Toeplitz.*

An interesting connection is established between this matrix and certain expansions in continued fractions as follows:

Let us specialize the elements by writing $a_{pp} = a_p$, $a_{pq} = a_{qp} = b_p$. Then consider the following associated system of linear equations:

$$\begin{aligned} & a_1x_1 + b_1x_2 = 1 \ , \\ & b_1x_1 + a_2x_2 + b_2x_3 = 0 \ , \\ & \qquad b_2x_2 + a_3x_3 + b_3x_4 = 0 \ , \\ & \qquad\qquad b_3x_3 + a_4x_4 + b_4x_5 = 0 \ , \\ & \qquad\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{2.16}$$

We then easily derive the following equations:

$$x_1 = \frac{1}{a_1 + b_1(x_2/x_1)} \ , \quad (x_n/x_{n-1}) = \frac{-b_{n-1}}{a_{n-1} + b_n(x_{n+1}/x_n)} \, , \quad n > 1 \ .$$

Employing these ratios successively, we obtain the following continued fraction as a formal evaluation of x_1:

$$x_1 = \cfrac{1}{a_1 - \cfrac{b_1^2}{a_2 - \cfrac{b_2^2}{a_3 - \cfrac{b_3^2}{a_4 - \cdots}}}} \tag{2.17}$$

*Zur Theorie der quadratischen Formen von unendlichvielen Veränderlichen, *Göttinger Nachrichten* (1910), pp. 489-506.

3. The Convergence of an Infinite Determinant. The theory of infinite determinants, initiated by the researches of G. W. Hill mentioned in section 2, was essentially founded in 1886 by H. Poincaré, who succeeded in giving an analytical justification to the methods which Hill had so successfully applied. It is to H. von Koch (1870-1924), however, that the theory owes most of its development. In a long series of papers, the first published in 1892 and the last in 1922, the significance of infinite determinants and their varied applications to the problems of modern analysis were thoroughly explored. These papers are notable both for the clarity of their style and for their ingenuity of attack upon the complexities of the problem. A bibliography of the contributions of von Koch will be found in *Acta Mathematica*, vol. 45 (1925), pp. 345-347.

The theorem which is the subject of this section, was first proved by Poincaré;* we follow here, however, the exposition of von Koch.†

Theorem 1. The infinite determinant $D = \lim_{n \to \infty} D(n)$ *will exist provided both the product of the diagonal elements and the sum of the non-diagonal elements converge absolutely.*‡

Proof: Since the hypothesis that $\sum_{ij=1}^{\infty} | a_{ij} |$ converges carries with it the convergence of the infinite product

$$Q = \prod_{i=1}^{\infty} (1 + \sum_{j=1}^{\infty} | a_{ij} |) \ ,$$

we conclude that the product

$$P = \prod_{i=1}^{\infty} (1 + \sum_{j=1}^{\infty} a_{ij})$$

also converges.

Now consider the two products

$$P_n = \prod_{i=1}^{n} (1 + \sum_{j=1}^{n} a_{ij}) \ , \ Q_n = \prod_{i=1}^{n} (1 + \sum_{j=1}^{n} | a_{ij} |) \ .$$

If in P_n we set certain values of a_{ij} equal to zero and alter the signs of others it is clear that we shall have the expansion of $D(n)$. Hence to each term in the development of $D(n)$ there occurs a term

*Sur les déterminants d'ordre infini. *Bulletin de la Soc. Mathématique de France*, vol. 14 (1886), pp. 77-90.

†Sur les déterminants infinis et les équations différentielles linéaires. *Acta Mathematica*, vol. 16 (1892), pp. 217-295; in particular, pp. 219-221.

‡A useful sufficiency condition for this convergence is the existence of the double integral

$$\int_0^{\infty} \int_0^{\infty} | a_{ij} | \, di \, dj \ .$$

in the development of Q_n, but the correspondence is such that no term in $D(n)$ is superior in value to its corresponding term in Q_n. We thus see that $|D(n)| < Q_n$. Furthermore, we notice that $D(n+p) - D(n)$ represents the sum of those terms in the development of $D(n+p)$ which vanish when the terms $a_{ik}[i,k = (n+1) \cdots (n+p)]$ are replaced by zero. But to each of these terms there corresponds in the development of $Q_{n+p} - Q_n$, a term of equal or greater absolute value.

Hence we attain the inequality

$$|D(n+p) - D(n)| \leqq |Q_{n+p} - Q_n| .$$

We are thus able to conclude, since Q_n converges, that there exists for each positive ε a positive integer n' such that $|D(n+p) - D(n)| < \varepsilon$ when $n > n'$ and p is any positive integer. The theorem is thus established.

Determinants which satisfy the conditions of the theorem are said to be of *normal form.*

The following corollary is of great importance in the applications of infinite determinants.

Corollary. A determinant of normal form remains convergent if the elements of any row (or column) are replaced by a series of quantities which are all smaller in absolute value than a given positive number.

Proof: For simplicity of exposition let us replace the elements of the first row

$$1 + a_{11}, a_{12}, a_{13}, \cdots, a_{1n}, \cdots$$

by the elements $m_1, m_2, m_3, \cdots, m_n, \cdots$, which satisfy the inequality

$$|m_r| < m .$$

Let the new values of $D(n)$ and D be denoted by $D'(n)$ and D' respectively. Moreover, denote by P'_n and Q'_n the products obtained by suppressing in P_n and Q_n the factors which correspond to the index one. We see that no term of $D'(n)$ can have a greater modulus than that of the corresponding term in the expansion of $m\,Q'_n$. Hence, reasoning as before, we have

$$|D'(n+p) - D'(n)| < m\,Q'_{n+p} - m\,Q'_n .$$

This establishes the corollary.

We also note the following propositions, useful in application, which are easily established on the basis of the preceding arguments:

(1) *Theorem 1 and the corollary apply with equal validity to the determinant of Laurent type,*

$$D = |a_{ij}| , \ i, j = -\infty \text{ to } +\infty .$$

(2) *If in a determinant of normal form two rows or two columns are interchanged, then the determinant changes sign; the determinant is zero if two rows or two columns are identical.*

(3) *A determinant of normal form can be developed according to the elements of any row or of any column.*

PROBLEMS

1. Compute the first five Bernoulli numbers from the determinant

$$B_n = (-1)^{n+1}\,(2n)!\,D_{2n}[1, 1/2!, 1/3!, 1/4!, \cdot\,\cdot, 1/(2n+1)!] \ .$$

2. Given

$$D_n = D_n[1, 1/2!, 1/4!, \cdots, 1/(2n)!] \ ,$$

show that

$$\lim_{n\to\infty} | D_{n-1}/D_n | = \lim_{n=\infty} | D_n |^{-1/n} = \tfrac{1}{4}\,\pi^2 \ .$$

3. Prove that if the determinant

$$\begin{vmatrix} 1 & a_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ b_1 & 1 & a_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & 1 & a_3 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & b_m & 1 \end{vmatrix}$$

is to converge absolutely as $m \to \infty$, then it is both necessary and sufficient that the series

$$\sum_{i=1}^{\infty} a_i\, b_i$$

converge absolutely. (von Koch).

4. Show that the following determinant is convergent and compute its numerical value approximately. (L. L. Smail: *Theory of Infinite Processes*).

$$\begin{vmatrix} 1 & 1/2^2 & 1/2^4 & 1/2^8 & \cdots \\ 1/2 & 1+1/2! & 1/2^7 & 1/2^{11} & \cdots \\ 1/2^3 & 1/2^6 & 1+1/3! & 1/2^{16} & \cdots \\ 1/2^5 & 1/2^{10} & 1/2^{15} & 1+1/4! & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

4. The Upper Bound of a Determinant (*Hadamard's Theorem*). Many proofs have been given of the celebrated theorem first proved by J. Hadamard which gives an upper bound to the value of a determinant. Because of the fundamentally different ideas involved we shall reproduce four of these proofs.

The theorem of Hadamard is one of the most thoroughly proved theorems in mathematics, its truth having been established in many ingenious ways. The first general proof of the theorem for complex elements was given by J. Hadamard: Résolution d'une question relative aux déterminants. *Bull. des Sciences Math.*, vol. 17 (2nd series) (1893), pp. 240-246. See also: *Comptes Rendus*, vol. 116 (1893), pp. 1500-1501. It appears, however, that as early as 1867 J. J. Sylvester, in connection with inverse orthogonal matrices, had constructed matrices whose determinants yielded the maximum value case of Hadamard's theorem. Thus

see: Thoughts on Inverse Orthogonal Matrices, *Phil. Mag.*, vol. 34 (series 4), (1867), pp. 461-475. Also Hadamard's theorem was known apparently to Lord Kelvin in 1885 and a proof was communicated to him in 1886 by T. Muir. Proposed as a problem in the *Educational Times*, the theorem was proved by E. J. Nanson in 1901: A Determinant Inequality, *Messenger of Mathematics*, vol. 31 (1901), pp. 48-50. In his *Geometrie der Zahlen*, Leipzig (1896), p. 183, H. Minkowski found the theorem as a consequence of the Jacobi transformation of quadratic forms. The literature of this theorem, proofs other than those already mentioned, discussions of the maximum value, etc., include the following items:

L. Amoroso: Sul valore massimo di speciali determinanti. *Giornale di Mat.*, vol. 48, (1910), pp. 305-315.

G. Barba: Intorno al teorema di Hadamard sui determinanti a valore massimo. *Giornale di Mat.*, vol. 71 (1933), pp. 70-86.

W. Blaschke: Ein Beweis für den Determinantensatz Hadamards. *Archiv der Math. und Physik*, vol. 20 (3rd series) (1913), pp. 277-279.

T. Boggio: Nouvelle démonstration du théorème de M. Hadamard sur les déterminants. *Bull. des Sciences Math.*, vol. 35 (2nd series) (1911), pp. 113-116.

M.Cipolla: Sul teorema di Hadamard relativo al modulo massimo di un determinante. *Giornale di Mat.*, vol. 50 (1912), pp. 355-359.

A. Colucci: Sui valori massimi dei determinanti ad elementi + 1 *e* —1. *Giornale di Mat.*, vol. 64 (1926), pp. 217-221.

R. Courant and D. Hilbert: *Methoden der mathematischen Physik*, vol. 1, (1924), p. 24.

E. W. Davis: The Maximum Value of a Determinant. *Johns Hopkins Circular*, vol. 2 (1882), pp. 22-23; *Bulletin of the American Math. Soc.*, vol. 14 (1907), pp. 17-18.

A. C. Dixon: On the greatest Value of a Determinant whose Constituents are Limited. *Proc. of the Cambridge Phil. Soc.*, vol. 17 (1913), pp. 242-243.

A. L. Dixon: A Proof of Hadamard's Theorem as to the Maximum Value of the Modulus of a Determinant. *Quarterly Journal of Math.* (*Oxford series*), vol. 3 (1932), pp. 224-225.

E. Fischer: Über den Hadamardschen Determinantensatz. *Archiv der Math. und Physik*, vol. 13 (3rd series) (1908), pp. 32-40.

T. Hayashi: Hadamard's Theorem on the maximum Value of a Determinant. *Proceedings of the Tokyo Math. Soc.*, vol. 5 (2nd series) (1909), pp. 104-109; Démonstration élémentaire du théorème de M. Hadamard sur la valeur maximum du déterminant. *Giornale di Mat.*, vol. 48 (1910), pp. 253-258.

A. Kneser: *Die Integralgleichungen und ihre Anwendungen in der mathematische Physik.* Braunschweig (1911); 2nd ed. (1922); § 55 1st ed., § 61 2nd ed.

G. Kowalewski: *Einführung in die Determinantentheorie.* Leipzig (1909), p. 460; *Die klassischen Probleme der Analysis des Unendlichen.* Leipzig (1910), p. 378 et seq.; *Integralgleichungen.* Berlin and Leipzig (1930), pp. 106-109.

T. Kubota: Hadamard's Theorem on the maximum Value of a Determinant. *Tôhoku Math. Journal*, vol. 2 (1912), pp. 37-38.

A. Molinari: Sul teorema di Hadamard. *Atti dei Lincei*, vol. 22 (5th series) (1913), pp. 11-12.

E. Pascal: *Die Determinanten. Leipzig* (1900), §53, pp. 180-184.

T. Peyovitch: Sur la valeur maxima d'un déterminant. *Bull. de la Société Math.*, vol. 55 (1927), pp. 218-221.

U. Scarpis: Sui determinanti di valore massimo. *Rendiconti Istituto Lombardo*, vol. 31 (2nd series) (1898), pp. 1441-1446.

I. Schur: Über die characteristischen Wurzeln einer linearen Substitution mit einer Andwendung auf die Theorie der Integralgleichungen. *Mathematische Annalen*, vol. 66 (1909), pp. 488-510, in particular, p. 496; Über eine Klasse von Mittelbildungen mit Anwendungen auf die Determinantentheorie. *Sitzungsberichte der Berliner Math. Gesellschaft*, vol. 22 (1923), pp. 9-20.

F. R. Sharpe: The Maximum Value of a Determinant. *Bull. of the American Math. Soc.*, vol. 14 (1907), pp. 121-123.

O. Szász: Ein elementarer Beweis des Hadamardschen Determinantensatz. *Math.-naturw. Berichte aus Ungarn*, vol. 27 (1913), pp. 172-180 (translation of original paper in *Math. és phys. Lapok* (Budapest), vol. 19 (1910), pp. 221-227; Über eine Verallgemeinerung des Hadamardschen Determinantensatzes. *Monatshefte für Math. und Physik*, vol. 28 (1917), pp. 253-257.

L. Tonelli: Sul teorema di Hadamard relativo al valor maggiorante di un determinante. *Giornale di Mat.*, vol. 47 (1909), pp. 212-218.

W. Wirtinger: Sur le théorème de M. Hadamard relatif aux déterminants. *Bull. des Sciences Math.*, vol. 31 (2nd series) (1907), pp. 175-179. A German translation of this article is found in *Monatshefte für Math. und Physik*, vol. 18 (1907), pp. 158-160.

Proofs of the Hadamard theorem will also be found in any standard text on integral equations. Historical references to the theorem are given by T. Muir: An Upper Limit for the Value of a Determinant. *Trans. of the Royal Soc. of South Africa*, vol. 1 (1908), pp. 323-334, and also by E. Hellinger and O. Toeplitz: (see *Bibliography*), pp. 1356-1357. Bibliographies are found in the last reference and also in T. Muir: *The History of Determinants* (1900-1920), London and Glasgow (1930), chap. I(a), pp. 90-101.

Theorem 2. If the elements of the determinant $D = [a_{ij}]$, $ij \leqq n$, satisfy the condition $|a_{ij}| \leqq A$, then the following inequality holds:

$$|D| \leqq A^n \cdot n^{n/2} .$$

First Proof (Hadamard): Employing the symbol $D = [a_{ij}]$ for the determinant in order to avoid confusion with the symbol for the absolute value of the elements, $|a_{ij}|$, we replace each element by its conjugate imaginary value and thus attain the new determinant $\overline{D} = [\overline{a}_{ij}]$.

From the elements of the determinant D, let us now construct a matrix of any p rows (T) and from the corresponding elements of $\overline{D}$, a second matrix $(\overline{T})$. Forming the product of these two matrices we then obtain a determinant P_p, where

$$P_p = (T)(\overline{T}) = [s_{ij}] \quad , \quad i,j = 1, 2, \cdots, p \ ,$$

the elements of which are

$$s_{ij} = a_{i1}\overline{a}_{j1} + a_{i2}\overline{a}_{j2} + \cdots + a_{in}\overline{a}_{jn} \ . \ *$$

But the determinant P_p can be written in the form

$$P_p = s_{pp} P_{p-1} + Q_p \ , \tag{4.1}$$

where

$$Q_p = \begin{vmatrix} s_{11} & \cdots & s_{1,p-1} & s_{1,p} \\ \cdot & \cdot & \cdot & \cdot \\ s_{p-1,1} & \cdots & s_{p-1,p-1} & s_{p-1,p} \\ s_{p,1} & \cdots & s_{p,p-1} & 0 \end{vmatrix} .$$

We now fix our attention upon the adjoint determinant of Q_p, the elements of which we shall designate by S_{ij}. From the defini-

*It will be convenient in this section to adopt the convention that the product of two matrices is a determinant the elements of which are obtained by the combination of row with row rather than row with column as assumed in the definition of section 1.

tion of the adjoint we can immediately write*

$$S_{pp} = P_{p-1} \text{ and } S_{hh} = Q_{p-1}{}^{(h)} \quad , \; h = 1, 2, \cdots, p-1 \quad ,$$

where $Q_{p-1}^{(h)}$ is a determinant of order $p-1$ formed by omitting the hth row and hth column from Q_p.

If we now consider the minor

$$s_2 = \begin{vmatrix} S_{hh} & S_{hp} \\ S_{ph} & S_{pp} \end{vmatrix}$$

and recall the theorem stated in the note, specialized for $m = 2$, we have

$$s_2 = P_{p-2}{}^{(h)} \, Q_p \quad ,$$

where $P_{p-2}{}^{(h)}$ is the complement of s_2 in Q_p. From this it follows that

$$P_{p-2}{}^{(h)} \, Q_p = S_{hh} \, S_{pp} - S_{hp} \, S_{ph} = P_{p-1} Q_{p-1}{}^{(h)} - | \, S_{ph} \, |^2 \quad , \qquad (4.2)$$

where $| \, S_{ph} \, |$ is the modulus of S_{ph}, the last term appearing because the two determinants S_{ph} and S_{hp} are conjugate.

We now proceed by induction to prove that Q_p is essentially a negative quantity or zero. Since $Q_2 = - | \, s_{12} \, |^2$ is negative and P_p is always positive, it follows from (4.2), namely, $Q_3 \, P_1 = P_2 \, Q_2 - | \, s_{3h} \, |^2$, that Q_3 is also negative or zero. By similar reasoning, Q_4, Q_5, $\cdots$, and finally Q_p are also negative or zero.

Returning now to (4.1) we see that for $p = 2$ we shall have

$$P_2 = s_{11} \, s_{22} - | \, s_{12} \, |^2 \quad ,$$

from which we get $P_2 \leqq s_{11} \, s_{22}$.

Similarly, for P_3 we have

$$P_3 = s_{33} \, P_2 + Q_3 \quad ,$$

*The adjoint of $a = [a_{ij}]$ is the determinant $A = [A_{ij}]$, where the A_{ij} are the cofactors or algebraic complements of a_{ij}. The fundamental relation between A and a is given by $A = a^{n-1}$. More generally if we define

$$M = \begin{vmatrix} A_{11} \cdots A_{1m} \\ \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \\ A_{m1} \cdots A_{mm} \end{vmatrix}$$

and if N is the determinant

$$\begin{vmatrix} a_{m+1,m+1} \cdots a_{n,m+1} \\ \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \\ a_{m+1,n} \;\; \cdots a_{n,n} \end{vmatrix} ,$$

we shall then have $M = N \, a^{m-1}$.

$$P_3 \leqq s_{33} P_2 \leqq s_{11} s_{22} s_{33} \ ,$$

and hence in general,

$$P_3 \leqq s_{11} s_{22} s_{33} \cdots s_{nn} \ . \tag{4.3}$$

Making use of this fundamental inequality and assuming the existence of a value A such that $|\, a_{ij} \,| \leqq A$, we have finally

$$D^2 = P_n \leqq A^{2n} n^n \ , \ |\, D \,| \leqq A^n \cdot n^{n/2} \ .$$

It is interesting to note that this theorem gives the best upper bound in particular instances since the equality sign may actually hold. This we observe is true only when $Q_n = 0$ and this is the case only when $s_{ij} = 0$, $i \neq j$. Hadamard cites an example due to J. J. Sylvester:*

$$D_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix} = 16 \ .$$

Similar determinants of maximum value can also be constructed for $n = 12$ and $n = 20$.

Second Proof (Wirtinger): Making use of the symbol

$$s_i = a_{i1} \bar{a}_{i1} + \cdots + a_{in} \bar{a}_{in} \ ,$$

we divide the elements of the ith row of D as well as of the determinant $\bar{D}$ by $(s_i)^{\frac{1}{2}}$. Adopting the convenient abbreviations $a_{ij}/(s_i)^{\frac{1}{2}} = b_{ij}$ and $\bar{a}_{ij}/(s_i)^{\frac{1}{2}} = \bar{b}_{ij}$, so that

$$b_{i1} \bar{b}_{i1} + \cdots + b_{in} \bar{b}_{in} = 1 \ , \tag{4.4}$$

we replace D and $\bar{D}$ by the two new determinants

$$\Delta = [b_{ij}] \text{ and } \bar{\Delta} = [\bar{b}_{ij}] \ ,$$

where

$$\Delta = D/(s_1 s_2 \cdots s_n)^{\frac{1}{2}} \ ; \ \bar{\Delta} = \bar{D}/(s_1 s_2 \cdots s_n)^{\frac{1}{2}} \ . \tag{4.5}$$

Let us now consider the problem of finding the maximum value of the function $\Delta \bar{\Delta}$ of the $2n^2$ variables b_{ij}, $\bar{b}_{ij}$, subject to the conditions (4.4). This problem is solved by the Lagrange rule as follows:†

Phil. Mag., vol. 34 (1867), pp. 461-475. See historical note at the beginning of this section.

†See E. Goursat (Hedrick translation): *Mathematical Analysis*. Boston, (1904), p. 129.

We differentiate partially

$$\Delta\bar{\Delta}-\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i b_{ij}\bar{b}_{ij}$$

with respect to the variables b_{ij}, $\bar{b}_{ij}$ and set each resulting expression equal to zero. We shall then obtain the $2n^2$ equations

$$\bar{\Delta}\cdot A_{ij}=\lambda_i\bar{b}_{ij}\ ,\ \Delta\cdot\bar{A}_{ij}=\lambda_i b_{ij}\ , \tag{4.6}$$

where we have made the abbreviations

$$A_{ij}=\frac{\partial\Delta}{\partial b_{ij}}\qquad \bar{A}_{ij}=\frac{\partial\bar{\Delta}}{\partial\bar{b}_{ij}}\ .$$

These expressions are clearly the cofactors of the b_{ij} and $\bar{b}_{ij}$, elements in Δ and $\bar{\Delta}$ respectively.

By a simple combination we shall obtain

$$\bar{\Delta}\sum_{j=1}^{n}A_{ij}b_{ij}=\lambda_i\sum_{j=1}^{n}b_{ij}\bar{b}_{ij}\ .$$

Then since $\Delta\equiv\sum_{j=1}^{n}A_{ij}\ b_{ij}$, we have, upon recalling (4.4),

$$\Delta\bar{\Delta}=\lambda_i=\lambda\ .$$

If we now form the determinant $[A_{ij}]$ which is equal to Δ^{n-1},* and recall (4.6), it is clear that we shall obtain the relation

$$[A_{ij}\bar{\Delta}]=\bar{\Delta}^n[A_{ij}]=\bar{\Delta}^n\Delta^{n-1}=\lambda^n[\bar{b}_{ij}]=\lambda^n\bar{\Delta}\ ,$$

or

$$(\Delta\bar{\Delta})^{n-1}=\lambda^n\ .$$

Hence, since $\lambda^{n-1}=\lambda^n$ and $\lambda=0$ is obviously impossible, it follows that $\lambda=1$, and the desired inequality is established; that is,

$$\Delta\bar{\Delta}\leqq 1\ .$$

It then follows from (4.5) that

$$D\cdot\bar{D}\leqq s_1\cdot s_2\cdots s_n\ .$$

This inequality is seen to coincide with (4.3) and the theorem follows as before.

Third Proof (Boggio): This proof is based upon the possibility of determining a transformation of the elements D and $\bar{D}$ by means of which they will go into determinants of the same value,

$$D=B=[b_{ij}]\ ,\ \bar{D}=\bar{B}=[\bar{b}_{ij}]\ ,$$

but with elements so related that

$$\sum_{i=1}^{n}b_{ri}\bar{b}_{si}=0\ ,\ r\neq s\ ,\ r,s=1,2,\cdots,n\ . \tag{4.7}$$

Let us write the desired transformation in the form

*M. Bôcher: *Higher Algebra.* (1907), p. 33.

$$a_{ri} = b_{ri} + \sum_{k=1}^{r-1} m_{rk}\, b_{ki} \ , \ r,i = 1, 2, \cdots, n \ . \tag{4.8}$$

For the determination of the m_{rs} we multiply (4.8) by $\overline{b}_{si}$ and sum, obtaining

$$\sum_{i=1}^{n} a_{ri}\, \overline{b}_{si} = \sum_{i=1}^{n} b_{ri}\, \overline{b}_{si} + \sum_{k=1}^{r-1} \sum_{i=1}^{n} m_{rk}\, b_{ki}\, \overline{b}_{si} \ ,$$

which, imposing condition (4.7), gives

$$\sum_{i=1}^{n} a_{ri}\, \overline{b}_{si} = m_{rs} \sum_{i=1}^{n} b_{si}\, \overline{b}_{si} \ , \ s < r \ .$$

We notice from (4.8) that D and B have the same value since each line of B differs from the corresponding line of D by a linear combination of the elements of the preceding line. The same applies to $\overline{D}$ and $\overline{B}$.

Hence by virtue of (4.7) we can write

$$D\overline{D} = \begin{vmatrix} \sum_{i=1}^{n} b_{1i}\, \overline{b}_{1i} & 0 & \cdots & 0 \\ 0 & \sum_{i=1}^{n} b_{2i}\, \overline{b}_{2i} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \sum_{i=1}^{n} b_{ni}\, \overline{b}_{ni} \end{vmatrix}$$

$$= \prod_{r=1}^{n} \left(\sum_{i=1}^{n} b_{ri}\, \overline{b}_{ri} \right) \ . \tag{4.9}$$

But from (4.8) and its companion formula associated with $\overline{B}$, we have

$$b_{ri}\, \overline{b}_{ri} = a_{ri}\, \overline{a}_{ri} - \sum_{s=1}^{r-1} \overline{m}_{rs}\, a_{ri}\, \overline{b}_{si} - \sum_{s=1}^{r-1} m_{rs}\, \overline{a}_{ri}\, b_{si} + \sum_{s=1}^{r-1} \sum_{t=1}^{r-1} m_{rs}\, \overline{m}_{rt}\, b_{si}\, \overline{b}_{ti} \ .$$

Summing and noting formulas (4.7) and (4.8) we get

$$\sum_{i=1}^{n} b_{ri}\, \overline{b}_{ri} = \sum_{i=1}^{n} a_{ri}\, \overline{a}_{ri} - \sum_{i=1}^{n} \sum_{s=1}^{r-1} \overline{m}_{rs}\, \overline{b}_{si} (b_{ri} + \sum_{k=1}^{r-1} m_{rk}\, b_{ki})$$

$$- \sum_{i=1}^{n} \sum_{s=1}^{r-1} m_{rs}\, b_{si} (\overline{b}_{ri} + \sum_{k=1}^{r-1} \overline{m}_{rk}\, \overline{b}_{ki}) + \sum_{i=1}^{n} \sum_{s=1}^{r-1} m_{rs}\, \overline{m}_{rs}\, b_{si}\, \overline{b}_{si}$$

$$= \sum_{i=1}^{n} a_{ri}\, \overline{a}_{ri} - \sum_{i=1}^{n} \sum_{s=1}^{r-1} \overline{m}_{rs}\, m_{rs}\, b_{si}\, \overline{b}_{si} \ . \tag{4.10}$$

Since the second term of the right-hand member is positive or zero, it follows that we shall have

$$\sum_{i=1}^{n} b_{ri}\, \overline{b}_{ri} \leqq \sum_{i=1}^{n} a_{ri}\, \overline{a}_{ri} \ ,$$

and consequently by (4.9),

$$D\overline{D} \leqq \prod_{r=1}^{n} \sum_{i=1}^{n} a_{ri}\, \overline{a}_{ri} \ ,$$

which is once more the inequality (4.3).

Formula (4.10) itself is of interest since it shows that the equal sign holds only when $m_{rs} = 0$; that is to say, when D is an *orthogonal* determinant.*

Fourth Proof (Molinari): The following proof is interesting in that it depends entirely upon geometrical intuitions.

Limiting ourselves first to the case $n = 3$, let us write

$$D_3 = \frac{1}{6} \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

and observe the fact that this is the volume of a tetrahedron with vertices at the points (x_1,y_1,z_1), (x_2,y_2,z_2), (x_3,y_3,z_3), $(0,0,0)$. Since this determinant is invariant with respect to any orthogonal transformation, let us pass the X-axis through the point (x_1,y_1,z_1) and the XY-plane through the point (x_2,y_2,z_2). D_3 then becomes

$$D_3' = \frac{1}{6} \begin{vmatrix} X_1 & X_2 & X_3 \\ 0 & Y_2 & Y_3 \\ 0 & 0 & Z_3 \end{vmatrix} = \frac{1}{6} X_1 Y_2 Z_3 ,$$

which is the volume of a tetrahedron with vertices at the points

$$(X_1,0,0,), (X_2,Y_2,0), (X_3,Y_3,Z_3), (0,0,0) .$$

But these elements cannot have a value greater than the longest edge of the tetrahedron extending from the origin. If these elements do not exceed A in absolute value, this edge is not larger than $A\,3^{\frac{1}{2}}$. Therefore we have

$$| D_3 | \leqq A^3 3^{3/2} .$$

This argument is generalized without difficulty to the determinant $D_n = [a_{ij}]$ by passing the X-axis through the point $(a_{11}, a_{12}, \cdots, a_{1n})$, the XY-plane through $(a_{21}, a_{22}, \cdots, a_{2n})$, and the hyperplane $(x, y, \dots, t)$ through the point $(a_{n-1,1}, a_{n-1,2}, \cdots, a_{n-1,n})$.

The determinant D_n then becomes

$$D_n' = [A_{ij}] \text{ , where } A_{ij} = 0 \text{ for } i < j .$$

If the elements a_{ij} do not exceed A in absolute value then $| A_{ij} | \leqq A n^{\frac{1}{2}}$ and hence we get $| D_n | \leqq A^n n^{n/2}$.

5. *Determinants Which Do Not Vanish.* It will be useful for us in another place to have the following two theorems relating to determinants with positive lower bounds. The proofs given are due to H. von Koch† although the results were known to L. Lévy in 1881, to Desplanques in 1887 and to J. Hadamard in 1903.‡

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*See G. Kowalewski, *Determinantentheorie.* Leipzig (1909).

†*Jahresbericht der Deutschen Mathematiker-Vereinigung,* vol. 22 (1913), pp. 285-291.

‡Lévy, Sur la possibilité de l'équilibre électrique. *Comptes Rendus,* vol. 93 (1881), pp. 706-708; Desplanques: Théorème d'algèbre. *Journal de Math. Spéc.,* vol. 11 (1887), pp. 12-13; J. Hadamard: *Leçons sur la propagation des ondes.* Paris (1903), pp. 13-14. See also T. Muir: *The History of Determinants* (1900-1920). London and Glasgow (1930), pp. 68-69.

Theorem 3. The determinant $A = [a_{ij}]$ is different from zero if for each i the diagonal element is in absolute value larger than the sum of the absolute values of all the non-diagonal elements in the same row; namely,

$$|a_{ii}| > \sum_{k=1}^{i-1} |a_{ik}| + \sum_{k=i+1}^{n} |a_{ik}| \; .$$

Proof: By assumption, since $a_{ii} \neq 0$, we can set $a_{ik}/a_{ii} = -b_{ik}$, $k \neq i$. If we then adopt the abbreviations

$$b_{ii} = 0 \, , D(b) = [\delta_{ik} - b_{ik}] \; ,$$

$$\delta_{ik} = \begin{cases} 1 \; , \; i = k \\ 0 \; , \; i \neq k \end{cases} ,$$

we can write A in the form

$$A = a_{11} \, a_{22} \cdots a_{nn} \, D(b) \; .$$

If we make the further abbreviations

$$b_{ik}^{(2)} = \sum_{m=1}^{n} b_{im} \, b_{mk} \, , \cdots \; , \; b_{ik}^{(p)} = \sum_{m=1}^{n} b_{im}^{(p-1)} \, b_{mk} \, , \cdots \; ,$$

we then have by the ordinary properties of determinants,*

$$\begin{aligned} &D(b) \; D(-b) = D(b^{(2)}) \; , \\ &D(b^{(2)}) D(-b^{(2)}) = D(b^{(4)}) \; , \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &D[b^{(2^{p-1})}] \; D[-b^{(2^{p-1})}] = D[b^{(2^p)}] \; . \end{aligned}$$

Multiplying these equations together and removing the factor common to both sides, we obtain

$$D(b) D(-b) D[-b^{(2)}] \cdots D[-b^{(2^{p-1})}] = D[b^{(2^p)}] \; . \tag{5.1}$$

But by hypothesis $s_i = \sum_{k=1}^{n} |b_{ik}| \leqq \varepsilon < 1$, and hence we get

$$\sum_{k=1}^{n} |b_{ik}^{(2)}| \leqq \varepsilon \sum_{k=1}^{n} |b_{ik}| = \varepsilon \, s_i \; ,$$

or more generally,

$$\sum_{k=1}^{n} |b_{ik}^{(\nu)}| \leqq \varepsilon^{\nu-1} \sum_{k=1}^{n} |b_{ik}| = \varepsilon^{\nu-1} s_i \; . \tag{5.2}$$

We thus reach the result that $\lim_{\nu=\infty} b_{ik}^{(\nu)} = 0$ and from this and the definition of $D(b^{(\nu)})$ we conclude that

$$\lim_{\nu=\infty} D(b^{(\nu)}) = 1 \; .$$

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*See Kowalewski: *Determinantentheorie*, pp. 126-7.

Making use of this fact we see from (5.1) that if p be chosen sufficiently large the right member of (5.1) does not vanish and consequently none of the factors of the left member is zero. From this conclusion the non-vanishing of $D(b)$ in particular is proved and with it the non-vanishing of the original determinant A.*

Theorem 4. Under the assumptions of theorem 3 the following inequality holds:

$$|A| \geqq |a_{11}\, a_{22} \cdots a_{nn}|\, e^S (1-\varepsilon)^{S/\varepsilon} ,$$

where $S = \sum_{i=1}^{n} s_i$, $s_i = \sum_{k=1}^{n} |b_{ik}|$, *and* $\varepsilon < 1$.

Proof: To prove this we start with the expansion

$$\log D(b) = -\sum_{\nu=1}^{\infty} \sum_{i=1}^{n} b_{ii}^{(\nu)}/\nu .$$

The proof of this essential formula *ab initio* would require a considerable amount of algebraic manipulation which would be equivalent to developing the well-known formula of Fredholm for the case where the integrals are replaced by sums:

$$\log D(\lambda) = -\sum_{\nu=1}^{\infty} \lambda^\nu A_\nu/\nu ,$$

where we abbreviate

$$A_n = \int_a^b \cdots \int_a^b K(s_1,s_2)K(s_2,s_3) \cdots K(s_n,s_1)\, ds_1 ds_2 \cdots ds_n ,$$

$$D(\lambda) = 1 - \lambda \int_a^b K(s_1,s_1)\, ds_1 + \cdots$$

$$+ \frac{(-\lambda)^n}{n!} \int_a^b \cdots \int_a^b K\begin{pmatrix} s_1\, s_2 \cdots s_n \\ s_1\, s_2 \cdots s_n \end{pmatrix} ds_1\, ds_2 \cdots ds_n + \cdots ,$$

in which we use the customary notation

$$K\begin{pmatrix} x_1\, x_2 \cdots x_n \\ y_1\, y_2 \cdots y_n \end{pmatrix} = [K(x_i,y_j)] .$$

If we replace this transcendental expansion by the expanded form of the determinant

*Theorem 3 is also true if there exists an index p so that we have for every value of i the inequality

$$|b_{i1}^{(p)}| + |b_{i2}^{(p)}| + \cdots |b_{in}^{(p)}| < 1 .$$

Without essential extension of the argument already given, this follows from the formula

$$D(b)D(qb)D(q^2b) \cdots D(q^{p-1}b) = D(b^{(p)}) ,$$

where $q = e^{2\pi i/p}$.

$$[\delta_{ij} - \lambda K(x_i,y_j)]$$

according to (2.2), we see the essential equivalence between the algebraic series and $D(\lambda)$. In this situation the integrals defining A_ν are replaced by

$$A_1 = \sum_{i=1}^{n} K(x_i,y_i) \quad , A_2 = \sum_{ij=1}^{n} K(x_i,y_j)\, K(x_j,y_i) = \sum_{i=1}^{n} K^{(2)}(x_i,y_i) \quad ,$$

$$A_3 = \sum_{ijk=1}^{n} K(x_i,y_j)\, K(x_j,y_k)\, K(x_k,y_i) = \sum_{ik=1}^{n} K^{(2)}(x_i,y_k)\, K(x_k,y_i)$$

$$= \sum_{i=1}^{n} K^{(3)}(x_i,y_i) \quad .$$

Setting $\lambda = 1$ and replacing $K(x_i,y_j)$ by b_{ij}, we obtain the desired formula.

Making use of (5.2) we reach the inequality

$$\log |\, D(b)\,| \geqq -\left(\varepsilon \sum_{i=1}^{n} s_i/2 + \varepsilon^2 \sum_{i=1}^{n} s_i/3 + \cdots\right)$$

$$= \sum_{i=1}^{n} s_i\{1 + \log(1-\varepsilon)/\varepsilon\}$$

$$= S\{1 + \log(1-\varepsilon)^{1/\varepsilon}\} \quad .$$

Hence we attain the inequality

$$|\, D(b)\,| \geqq e^S(1-\varepsilon)^{S/\varepsilon} \, ,$$

from which the general theorem follows as an immediate consequence.

The application of these results is made to infinite determinants if we assume that both $\prod_{i=1}^{\infty} |\, a_{ii}\,|$ and $\sum_{ij=1}^{\infty} |\, a_{ij}\,|$ converge.

6. *The Method of the Liouville-Neumann Series.* We proceed now to a discussion of methods for solving the set of equations (1.1). Three essentially distinct procedures have been developed for this purpose. The first one is based upon the convergence properties of the *Liouville-Neumann* series; the second, which we shall call the *method of segments* (*méthode des réduites*), considers the limiting form of the solution of a finite set of equations when the number of variables becomes infinite; the third is based on the theory of infinite bilinear forms. Historically the first method was suggested by the solutions achieved by J. Liouville (1809-1882) and C. Neumann (1832-1925) in the theory of integral equations; the second was essentially employed by J. Fourier (1768-1830) in the celebrated problem treated in section 8; the third was inaugurated by D. Hilbert in connection with his treatment of integral equations.

The following theorem is due to H. von Koch and is fundamental in all applications of the Liouville-Neumann series:

Theorem 5. If in system (1.1) *the quantities x_i and b_i satisfy the conditions*

$$| x_i | < X \ , \ | b_i | < B \, | a_{ii} | \ ,$$

where X and B are finite positive magnitudes, and if, further, we have

$$\sum_{j \neq i} | a_{ij} | \leqq \varepsilon \, | a_{ii} | \ , \quad 0 < \varepsilon < 1 \ ,$$

then there exists one and only one solution of system (1.1) *and this solution can be developed in the Liouville-Neumann series,*

$$x_i = b_i' + \sum_j b_{ij} \, b_j' + \sum_j b_{ij}^{(2)} \, b_j' + \cdots \ , \qquad \textbf{(6.1)}$$

where we employ the abbreviation

Proof: In order to prove this we write

$$b_i' = b_i / a_{ii} \ , \ b_{ij} = - a_{ij} / a_{ii} \ , \ b_{ij}^{(p)} = \sum_{k=1}^{\infty} b_{ik}^{(p-1)} \, b_{kj} \ , \ b_{ii} = 0 \ .$$

$$t_i = b_i' + \sum_j b_{ij} \, b_j' + \sum_j b_{ij}^{(2)} \, b_j' + \cdots \ ,$$

$$[t_i] = | b_i' | + \sum_j | b_{ij} \, b_j' | + \sum_j | b_{ij}^{(2)} \, b_j' | + \cdots \ .$$

Employing the inequality (5.2) and the assumption

$$s_i = \sum_{j=1}^{\infty} | b_{ij} | \leqq \varepsilon \ ,$$

we get $| t_i | \leqq B/(1-\varepsilon)$. The convergence of $\sum_j | b_{ij} | [t_j]$ enables us to write

$$\sum_j b_{ij} \, t_j = \sum_j b_{ij} \, b_j' + \sum_j b_{ij}^{(2)} \, b_j' + \cdots = t_i - b_i' \ ;$$

that is to say,

$$\sum_j a_{ij} \, t_j = b_i \ .$$

In order to prove that this solution is unique we designate by $x_1, x_2, x_3, \cdots$ any arbitrary solution of the original equations and then form the sum

$$\varphi^{(p)}(x_i) = x_i + \sum_j b_{ij} \, x_j + \cdots + \sum_j b_{ij}^{(p)} \, x_j \ .$$

Substituting b_i' for x_i we get

$$\begin{aligned} \varphi^{(p)}(b_i') &= \varphi^{(p)}(x_i - \sum_j b_{ij} \, x_j) \\ &= \varphi^{(p)}(x_i) - \varphi^{(p)}(\sum_j b_{ij} \, x_j) = x_i - \sum_j b_{ij}^{(p+1)} \, x_j \ . \end{aligned}$$

Making use of (5.2) we reach the inequality

$$| x_i - \varphi^{(p)}(b_i') | \leqq \varepsilon^p s_i X ,$$

and hence, since $t_i = \lim_{p=\infty} \varphi^{(p)}(b_i')$, we get

$$x_i = t_i .$$

J. L. Walsh has applied the method of successive approximations to the Liouville-Neumann series for the triangular system*

$$\begin{aligned} x_1 + a_{12} x_2 + a_{13} x_3 + \cdots &= b_1 , \\ x_2 + a_{23} x_3 + \cdots &= b_2 , \\ x_3 + \cdots &= b_3 , \\ \cdots\cdots\cdots \end{aligned} \tag{6.2}$$

The resulting theorem cannot be derived as a special case under the conditions stated by von Koch, since limitations imposed upon the constants b_i are essential to the convergence of the series defining x_i. The results of Walsh are stated in the following theorem:†

Theorem 6. If there exist positive constants b, M, and P such that the coefficients of (6.2) *satisfy the conditions*

$$|b_k| \leqq M b^k , \ \sum_{j=k+1}^{\infty} |a_{kj}| \leqq P , \ k = 1, 2, \cdots , \ b < 1/P , \ b \leqq 1 ,$$

then system (6.2) *has one and only one solution for which* $|x_k| \leqq m g^k$, $g < 1/P$, $g \leqq 1$.‡

*It should be observed that the general system (1.1) can be reduced to a diagonal system by means of the following transformation provided the principal (diagonal) minors of $| a_{ij} |$ do not vanish:

$$c_{1k} = a_{1k} , \ d_1 = b_1 ,$$

$$c_{nk} = \begin{vmatrix} a_{11} \, a_{12} \cdots a_{1,n-1} \, a_{1k} \\ \cdots\cdots\cdots \\ a_{n1} \, a_{n2} \cdots a_{n,n-1} \, a_{nk} \end{vmatrix} , \ d_n = \begin{vmatrix} a_{11} \, a_{12} \cdots a_{1,n-1} \, b_1 \\ \cdots\cdots\cdots \\ a_{n1} \, a_{n2} \cdots a_{n,n-1} \, b_n \end{vmatrix} , \ n > 1 .$$

In this manner we obtain the diagonal system

$$\begin{aligned} c_{11} u_1 + c_{12} u_2 + c_{13} u_3 + \cdots + c_{1n} u_n + \cdots &= d_1 , \\ c_{22} u_2 + c_{23} u_3 + \cdots + c_{2n} u_n + \cdots &= d_2 , \\ c_{33} u_3 + \cdots + c_{3n} u_n + \cdots &= d_3 , \\ \cdots\cdots\cdots\cdots \end{aligned}$$

This transformation is due to Th. Kötteritzsch: *Zeitschrift für Math. und Physik*, vol. 15 (1870), pp. 1-15, 229-268. See also, F. Riesz: *Les systèmes d'équations linéaires*, (1913), Paris, pp. 11-12.

†On the Solution of Linear Equations in Infinitely Many Variables by Successive Approximations. *Amer. Journal of Math.*, vol. 42 (1920), pp. 91-96.

‡This theorem may be slightly generalized by replacing the second condition with $\sum_{j=k+1}^{\infty} | a_{kj} | \leqq P$ for every k greater than a fixed K, assuming, however, the convergence of the series for all values of k.

Proof: Let us consider the following set of approximations:

$$x_k^{(1)} = b_k \ , \ k = 1, 2, \cdots ,$$
$$x_k^{(i+1)} = b_k - [a_{k,k+1} x_{k+1}^{(i)} + a_{k,k+2} x_{k+2}^{(i)} + \cdots] \ ,$$
$$i = 1, 2, \cdots .$$

From this set we form the expressions

$$x_k^{(1)} = b_k \ ,$$
$$x_k^{(2)} - x_k^{(1)} = - [a_{k,k+1} x_{k+1}^{(1)} + a_{k,k+2} x_{k+2}^{(1)} + \cdots] \ ,$$
$$x_k^{(3)} - x_k^{(2)} = - [a_{k,k+1} (x_{k+1}^{(2)} - x_{k+1}^{(1)}) + a_{k,k+2} (x_{k+2}^{(2)} - x_{k+2}^{(1)}) + \cdots] \ . \tag{6.3}$$

Making use of the conditions imposed by the theorem we obtain the majorant

$$\begin{aligned} x_k &= x_k^{(1)} + (x_k^{(2)} - x_k^{(1)}) + (x_k^{(3)} - x_k^{(2)}) + \cdots \\ &<< M\, b^k + P\, M\, b^{k+1} + P^2\, M\, b^{k+2} + \cdots \\ &= M\, b^k/(1-P\, b) \ , \end{aligned} \tag{6.4}$$

where the symbol $<<$ has its customary meaning "is term by term less than in absolute value."

The set of values $\{x_k\}$ is easily seen to furnish a solution of the original system since, if we add the equations of (6.3) and sum the absolutely convergent double series by columns, we get

$$x_k = b_k - \sum_{j=k+1}^{\infty} a_{kj}\, x_j \ .$$

The bound for x_k stated in the theorem is obvious from (6.4).

The uniqueness of the solution is obtained from a consideration of the differences $y_k = x_k' - x_k''$, where x_k' and x_k'' are solutions of (6.2) under the conditions of the theorem. Obviously the set $\{y_k\}$ furnishes a solution of the homogeneous system:

$$\begin{aligned} y_1 + a_{12}\, y_2 + a_{13}\, y_3 + \cdots &= 0 \ , \\ y_2 + a_{23}\, y_3 + \cdots &= 0 \ , \\ y_3 + \cdots &= 0 \ , \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \end{aligned} \tag{6.5}$$

We employ successive approximation to solve this system, thus obtaining

$$y_k^{(1)} = y_k \ , \quad k = 1, 2, \cdots ,$$
$$y_k^{(i+1)} = - \sum_{j=k+1}^{\infty} a_{kj}\, y_j^{(i)} \ , \ i = i, 1, 2, \cdots .$$

Noting the inequality $|y_k| \leqq N X^k$, $X < 1/P$, $X \leqq 1$, we conclude that

$$|y_k^{(i+1)}| \leqq N(PX)^i X^k ,$$

which converges to zero as i approaches infinity. But from (6.5) we have $y_k^{(i)} = y_k^{(1)} = y_k$, and hence we see that $y_k = 0$. This result establishes the uniqueness of the solution obtained for (6.2).

The actual solution obtained by this method is the Liouville-Neumann series previously stated in (6.1).

The following generalization of the theorem just proved is useful in application and may be established by arguments essentially similar to those employed above:

Theorem 7. If system (6.2) *is such that* $\sum_{j=k+1}^{\infty} |a_{kj}|^p \leqq P^p$ *for every value of* k, $b_k \leqq M b^k$, $b < 1/[1 + P^{p/(p-1)}]^{(p-1)/p}$, *then system* (6.2) *has a unique solution for which* $|x_k| \leqq M g^k$, $g < 1/[1 + P^{p/(p-1)}]^{(p-1)/p}$.

As in the previous case the actual solution is obtained as a Liouville-Neumann series.

7. *The Method of Segments.* The following method may be described as one which defines regions of validity for

$$\lim_{m=\infty} x_i^{(m)} = x_i , \quad i = 1, 2, 3, \cdots, m ,$$

where the values $\{x_i\}$ are the solutions of a set of m linear equations in m unknowns.

The following theorem was discovered by A. Pellet in 1914* and independently by A. Wintner in 1925,† although the fundamental idea was developed in 1899 by E. Lindelöf‡ in discussing the problem of the existence of implicit functions:

Theorem 8. If in the system $x_i - \psi_i = c_i$, $i = 1, 2, \cdots, \infty$, *where*

$$\psi_i = \sum_{k=1}^{\infty} a_{ik} x_i ,$$

the constants c_i *are bounded and the coefficients* a_{ik} *are subject to the condition*

$$S_i = \sum_{k=1}^{\infty} |a_{ik}| < 1 , \quad i = 1, 2, \cdots, \infty , \tag{7.1}$$

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*Sur la méthode des réduites. *Bulletin de la Société Mathématique*, vol. 42 (1914), pp. 48-53.

†Ein Satz über unendliche Systeme von linearen Gleichungen. *Mathematische Zeitschrift*, vol. 24 (1925-1926), p. 266. See also: Zur Hillschen Theorie der Variation des Mondes. *Ibid.*, pp. 257-265; in particular, p. 265 et. seq.

‡Démonstration élémentaire de l'existence des fonctions implicites, *Bulletin des Sciences Mathématiques*, vol. 23, 2nd ser., (1899), pp. 68-75.

then the x_i exist and are equal to the limiting form as $m \to \infty$ of $x_i^{(m)}$, determined by solving the reduced system

$$x_i^{(m)} - \sum_{k=1}^{m} a_{ik}\, x_k^{(m)} = c_i \ , \quad i = 1, 2, 3, \cdots, m \ . \qquad (7.2)$$

Moreover, if C is the largest of the numbers $|c_i|$, and S the largest of the sums S_i, then the solutions x_i are subject to the limiting condition $|x_i| \leqq C/(1 - S)$.

Proof: The proof of this theorem depends upon the fact that if the system

$$x_i - f_i(x_1, x_2, \cdots, x_n) = 0 \ , \quad i = 1, 2, \cdots, n \ , \qquad (7.3)$$

where the $f_i(x_1, x_2, \cdots, x_n)$ are analytic functions of the n variables, has a majorant system $X_i - F_i(X_1, X_2, \cdots, X_n) = 0$, where the functions $F_i(X_1, X_2, \cdots, X_n)$ are positive dominants of $f_i(x_1, x_2, \cdots, x_n)$, and if there exists a set of positive values $X_1^{(0)}, X_2^{(0)}, \cdots, X_n^{(0)}$ for which $X_i - F_i(X_1, X_2, \cdots, X_n)$ are all positive or zero, then there exists a set of solutions $x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}$ of (7.3) for which

$$X_i^{(0)} \geqq |x_i^{(0)}| \ , \quad i = 1, 2, \cdots, n \ .^*$$

In order to make application to the system $x_i - \psi_i = c_i$, we consider the majorant system,

$$X_i - \Psi_i = C_i \ , \text{ where } \Psi_i = \sum_{k=1}^{\infty} A_{ik} X_k \ , \ A_{ik} \geqq |a_{ik}| \ ,$$

and $C_i \geqq |c_i|$. Now let us assume the restrictive inequality

$$\sum_{k=1}^{\infty} A_{ik} < 1 \ .$$

Then if C' is the largest of the numbers $|C_i|$, and S' the largest of the sums $\sum_{k=1}^{\infty} A_{ik}$, it is clear that the set of values $X_1^{(0)} = X_2^{(0)} = \cdots = X_i^{(0)} = \cdots = C'/(1 - S')$ will satisfy the inequality

$$X_i - \Psi_i - C_i > 0 \ .$$

Hence we are able to infer the existence of a set of solutions of the original system.

It is also clear that the same reasoning applies to the reduced system (7.2) and hence the method of segments may be employed in obtaining the solution of the original set of equations.

The method of segments is closely related to the well known *rule of Cramer*, which applies in the ordinary finite case. This rule is gen-

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*The paper by Lindelöf gives essentially this theorem. It is also given by Pellet: Des équations majorantes. *Bulletin de la Société Mathématique*, vol. 37 (1909), pp. 93-101.

erally employed to obtain the solutions of the successive segments of the infinite system.

8. Applications of the Method of Segments. The efficacy of the method of segments in attaining results in the transcendental case from the algebraic case may be illustrated by the following elegant solution of the problem of the elastic plate. This solution is due to H. W. March.*

The problem of the elastic plate is concerned with the determination of a function $V(x,y)$ representing the deflection of a uniformly loaded rectangular plate of dimensions (a,b). In the theory of elasticity† it is shown that $V(x,y)$ satisfies the differential equation

$$\nabla^4 V(x,y) = A \ , \tag{8.1}$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \ , \quad A = 3p/[2\,E\,h^2/(1-\sigma^2)] \ ,$$

where p is the uniform load, h is half the thickness of the plate, and E and σ are elastic constants determined from the material of the plate.

When the plate is clamped at the edges, $V(x,y)$ is subject to the following conditions:

$$V = 0 \ , \ x = 0 \ , \ x = a \ ; \ y = 0 \ , \ y = b \ ; \tag{8.2,a}$$

$$\frac{\partial V}{\partial x} = 0 \ , \ x = 0 \ , \ x = a \ ; \ \frac{\partial V}{\partial y} = 0 \ , \ y = 0 \ , \ y = b \ . \tag{8.2,b}$$

Let us now consider the function

$$V(x,y) = V_1(x,y) + V_2(x,y) \ ,$$

where we abbreviate

$$V_1(x,y) = P\,x(x-a)\,y(y-b) \ , \ P = A/8 \ ,$$

$$V_2(x,y) = P\{\sum_n [a_n/(\lambda_n b + \sinh \lambda_n b)]\,[(y-b)\sinh \lambda_n y$$

$$+ y \sinh \lambda_n (y-b)] \sin \lambda_n x + \sum_m [b_m/(\mu_m a + \sinh \mu_m a)]$$

$$\times [(x-a) \sinh \mu_m x + x \sinh \mu_m (x-a)] \sin \mu_m y\} \ ,$$

$$\lambda_n = n\pi/a \ , \ \mu_m = m\pi/b \ .$$

It is seen that the function thus defined satisfies both the differential equation and the boundary conditions (8.2,a). The problem, then,

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*The Deflection of a Rectangular Plate Fixed at the Edges. *Trans. of the American Math. Soc.*, vol. 27 (1925), pp. 307-318.

†A. E. H. Love: *Theory of Elasticity*, 3rd ed. (1920), p. 496.

is to determine the coefficients so that the conditions (8.2,b) are also satisfied; we have

$$(\partial V/\partial x)|_{x=0} = P\{-a\,y(y-b) + \sum_n [\lambda_n a_n/(\lambda_n b + \sinh \lambda_n b)]$$
$$\times [(y-b)\sinh \lambda_n y + y \sinh \lambda_n (y-b)]$$
$$- \sum_m b_m \sin \mu_m y\} = 0 \quad ; \qquad (8.3)$$

$$(\partial V/\partial y)|_{y=0} = P\{-b\,x(x-a) - \sum_n a_n \sin \lambda_n x + \sum_m [\mu_m b_m/(\mu_m a$$
$$+ \sinh \lambda_m a)][(x-a)\sinh \mu_m x + x \sinh \mu_m (x-a)]\} = 0. \qquad (8.4)$$

It will be seen that these equations are also those which hold on the edges $x = a$ and $y = b$.

Equation (8.3) will be found to have the solution

$$b_m = (2/b) \int_0^b \{-a\,y(y-b) + \sum_n [\lambda_n a_n/(\lambda_n b + \sinh \lambda_n b)]$$
$$\times [(y-b)\sinh \lambda_n y + y \sinh \lambda_n (y-b)]\} \sin \mu_n y \, dy \; .$$

Considerations of uniform convergence show that it is possible to interchange the order of integration and summation signs in this equation. We thus obtain the system

$$b_m = (8a/b\mu_m^3) - (8\mu_m/b) \sum_n \lambda_n^2 a_n \, R(\lambda_n b)/(\lambda_n^2 + \mu_n^2)^2 \; , \qquad (8.5)$$

where we employ the abbreviation

$$R(\lambda_n b) = (1 + \cosh \lambda_n b)/(\lambda_n b + \sinh \lambda_n b) \; .$$

Similarly, we obtain from (8.4) by symmetry the system

$$a_n = (8b/a\lambda_n^3) - (8\lambda_n/a) \sum_m \mu_m^2 b_m \, R(\mu_m a)/(\lambda_n^2 + \mu_m^2)^2 \; . \qquad (8.6)$$

It we set $b = \eta\, a$, $0 \leqq \eta \leqq 1$, that is to say, if we consider the case of a rectangular plate, equations (8.5) and (8.6) take the following forms:

$$b_n = (8\eta^2 a^3/n^3 \pi^3)$$
$$- (8n\,\eta^2/\pi) \sum_m [m^2 a_m R(m\,\pi\,\eta)/(m^2 \eta^2 + n^2)^2] \; ,$$

$$a_n = (8\eta\, a^3/n^3 \pi^3)$$
$$- (8n\,\eta^2/\pi) \sum_m [m^2 b_m R(m\,\pi/\eta)/(m^2 + n^2 \eta^2)^2] \; .$$

For the square plate, i.e., $\eta = 1$, $b_n = a_n$, the two systems just written down become

$$b_n = K/n^3 - \sum_m A_{nm}\, b_m \;,\quad n = 1, 3, 5, \cdots$$

where we abbreviate $K = 8a^3/\pi^3$, $A_{nm} = (8/\pi)\, nm^2\, R(m\pi) / (n^2 + m^2)^2$.

The first four equations of this system reduce explicitly to the following:

$$\begin{aligned}
1.546b_1 &+ 0.229b_3 + 0.094b_5 + 0.050b_7 + \cdots = K \;, \\
0.065b_1 &+ 1.212b_3 + 0.165b_5 + 0.111b_7 + \cdots = .03705K \;, \\
0.016b_1 &+ 0.099b_3 + 1.128b_5 + 0.114b_7 + \cdots = .008\;K \;, \\
0.0061b_1 &+ 0.048b_3 + 0.081b_5 + 1.091b_7 + \cdots = .00292K \;.
\end{aligned} \tag{8.7}$$

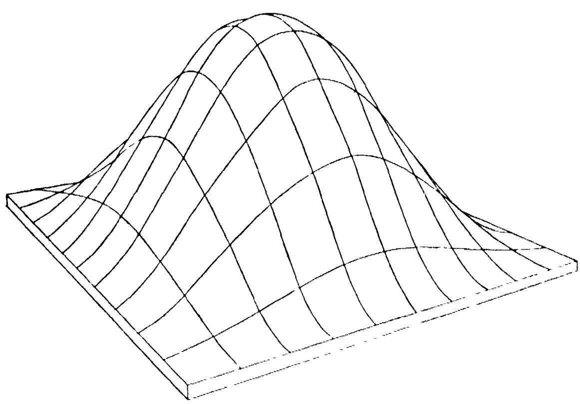

FIGURE 1

From the explicit formula for A_{nm} we have the inequality

$$A_{nm} < [8nm^2/\pi(n^2 + m^2)^2] \;,$$

and hence, summing over the odd integers,

$$\sum_m A_{nm} < \sum_m 8\, n\, m^2/\pi(n^2 + m^2)^2$$

$$< \tfrac{1}{2} \int_0^\infty [8nm^2/\pi(n^2+m^2)^2]\, dm \;.$$

Since the value of the integral is exactly unity, it follows at once that we may apply theorem 8 to obtain a numerical solution of system (8.7). Since the right-hand members are obviously bounded, the solution will also be bounded.

Applying the method of segments we find the values

$b_1 = 0.6475K,\ b_3 = -0.0040K,\ b_5 = -0.0017K,\ b_7 = -0.006K,$

$b_9 = -0.0003K,\ b_{11} = -0.0001K,\ b_{13} = -0.00006K, \cdots$.

When these values are properly substituted and a computation made of the displacements of the elastic surface

$$V(x, y) = V_1(x, y) + V_2(x, y) \ ,$$

for $P = 1,\ a = b = 1$, the following table is obtained:

x	$y = .1$	.2	.3	.4	.5	.6	.7	.8	.9
.1	.0001	.0006	.0010	.0013	.0014	.0013	.0010	.0006	.0001
.2	.0006	.0019	.0032	.0040	.0043	.0040	.0032	.0019	.0006
.3	.0010	.0032	.0052	.0066	.0071	.0066	.0052	.0032	.0010
.4	.0013	.0040	.0066	.0084	.0091	.0084	.0066	.0040	.0013
.5	.0014	.0043	.0071	.0091	.0097	.0091	.0071	.0043	.0014
.6	.0013	.0040	.0066	.0084	.0091	.0084	.0066	.0040	.0013
.7	.0010	.0032.	0052	.0066	.0071	.0066	.0052	.0032	.0010
.8	.0006	.0019	.0032	.0040	.0043	.0040	.0032	.0019	.0006
.9	.0001	.0006	.0010	.0013	.0014	.0013	.0010	.0006	.0001

A graphical representation of these values is found in figure 1.

In the example just discussed, the conditions assuring a convergent solution of the infinite system of equations were fulfilled. It is interesting to observe that solutions can be obtained by the method of segments in certain special cases where the solutions fail to converge. The following example, originally due to J. Fourier (1768-1830), illustrates this point.* We follow a modification due to F. Riesz.†

Let us solve Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

for the case where $V(x, y)$ is subject to the conditions

$$V(0,y) = 1 \ , \ V(x, \pm \tfrac{1}{2}\pi) = 0 \ , \ \lim_{x \to \infty} V(x,y) = 0 \ .$$

Obviously we can write $V(x, y)$ as the series

$$V(x,y) = \sum_{m=1}^{\infty} x_m\, e^{-(2m-1)x} \cos(2m-1)y \ ,$$

the constants x_m to be determined from the conditions

*See his *Théorie analytique de la chaleur.* (1822), arts. 171-178.

†*Les systèmes d'équations linéaires a une infinité d'inconnues* (1913), pp. 2-6.

$$\sum_{m=1}^{\infty} x_m \cos(2m-1)y = 1 \ .$$

We now proceed formally. Differentiating this equation an infinite number of times and setting $y = 0$ in each equation, we obtain the following system:

$$\begin{aligned} \sum x_m &= 1 \ , \\ \sum (2m-1)^2 x_m &= 0 \ , \\ \sum (2m-1)^4 x_m &= 0 \ , \\ \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \end{aligned}$$

Clearly the conditions of theorem 8 are violated. Applying the method of segments, however, we find

$$x_m^{(k)} = \frac{2m-3}{2m-1} \, \frac{m+k-1}{m-k-1} \, x_{m-1}^{(k)} \ ,$$

$$x_1^{(k)} = \frac{3^2 \cdot 5^2 \cdot 7^2 \cdots (2k-1)^2}{8 \cdot 24 \cdots (4k^2-4k)} \sim \frac{4}{\pi} \ .$$

It thus follows that

$$x_2^{(k)} \sim -\frac{1}{3} x_1 = -\frac{1}{3} \, \frac{4}{\pi} \ ,$$

$$x_3^{(k)} \sim -\frac{3}{5} x_2 = \frac{1}{5} \, \frac{4}{\pi} \ ,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x_m^{(k)} \sim -\frac{2m-3}{2m+1} x_m = (-1)^{m-1} \frac{1}{2m-1} \, \frac{4}{\pi} \ ,$$

and we obtain the well-known solution

$$V(x,y) = (4/\pi) \sum_{m=1}^{\infty} (-1)^{m-1} e^{-(2m-1)x} \cos(2m-1)y/(2m-1) \ .$$

If we now substitute the values of x_m in the original system, it is clear that we obtain divergent series in all but the first instance. However, if we employ Borel's method of summation* it is possible to show that all these divergent series are summable to zero.

For example, we shall have

$$\begin{aligned} \sum (2m-1)^2 x_m &= (4/\pi)\,(1 - 3 + 5 - 7 + \cdots) \\ &= (4/\pi) \int_0^{\infty} e^{-t} (t - t^3/2! + t^5/4! - t^7/6! + \cdots)\, dt \end{aligned}$$

*See section 4, Chapter 5.

$$\sum (2m-1)^2 x_m = (4/\pi) \int_0^\infty e^{-t}\, t \cos t \, dt = 0 \quad ;$$

$$\sum (2m-1)^4 x_m = (4/\pi)(1 - 3^3 + 5^3 - 7^3 + \cdots)$$

$$= (4/\pi) \int_0^\infty e^{-t} [(t-t^3)\cos t - 3t^2 \sin t] \, dt = 0 \ .$$

If we admit the validity of Borel summability in the present situation, we see how it is possible to give at least a rationale to the application which we have discussed here.

9. The Hilbert Theory of Linear Equations in an Infinite Number of Variables. The theory sketched in this section was initiated by D. Hilbert in his classical papers on the theory of integral equations published between 1904 and 1910 (See *Bibliography*). These fruitful ideas led immediately to an extensive development by E. Schmidt, O. Toeplitz, E. Hellinger, I. Schur, and numerous others. A short bibliography of some of the important memoirs will be found in chapter 12.

We shall begin by introducing the concept of *Hilbert space*. By such a space is meant an infinite array of numbers $x_1, x_2, x_3, \cdots, x_n, \cdots$, which obey ordinary associative and commutative laws and satisfy the condition

$$\sum_{n=1}^{\infty} |x_n|^2 \leqq M \ ,$$

where M is finite. It is obvious that no essential restriction is imposed by assuming that $M = 1$.

This concept has been broadened by a number of writers prominent among whom are J. von Neumann, M. H. Stone, S. Banach, and T. H. Hildebrandt. A complete account of these recent developments will be found in the treatise of Stone: *Linear Transformations in Hilbert Space* (American Math. Soc. Colloquium Publications, 1932), viii + 622 p. The following postulates are due to von Neumann [See *Bibliography*: von Neumann (1), p. 14 et seq.]

A class, H, of elements $f, g, \cdots$ is called a Hilbert space provided:

I. The elements form a linear space; i. e., there exist commutative and associative operations for the elements and a null element with the properties $f + 0 = f$, $f \cdot 0 = 0$, $0 \cdot f = 0$.

II. There exists a numerically-valued function (f,g) defined for every pair of elements with the properties: (1) $(af,g) = a(f,g)$, where a is a complex number; (2) $(f_1 + f_2, g) = (f_1,g) + (f_2,g)$; (3) $(g,f) = \overline{(f,g)}$; (4) $(f,f) \geqq 0$, the equal sign holding only if $f = 0$.

III. For every value of n there exists a set of n linearly independent elements of H.

IV. H is separable; i. e., there exists a denumerable, infinite set of elements $\{f_i\}$, such that for every g in H and every positive ε there exists an n for which $|f_n - g| < \varepsilon$.

V. H is complete; i. e., if a sequence $\{f_i\}$ of H satisfies the condition $|f_m - f_n| \to 0$, $m,n \to \infty$, then there exists an element f such that $|f - f_n| \to 0$, $n \to \infty$.

The Hilbert theory of the inversion of a system of linear equations takes its departure from the related problem of the definition and inversion of the bilinear form

$$A(x,y) = \sum_{ij=1}^{\infty} a_{ij}\, x_i\, y_j \ .$$

Definition: The bilinear form $A(x,y)$ is called *limited* if there exists a positive number M, independent of n, x, and y, such that if x and y both belong to Hilbert space, we have

$$\left| \sum_{ij=1}^{n} a_{ij}\, x_i\, y_j \right| \leqq M \ .$$

The form

$$A_n(x,y) = \sum_{ij=1}^{n} a_{ij}\, x_i\, y_j$$

is called the nth *segment* (Abschnitt) of $A(x,y)$.

Although for convenience much of the following discussion will assume that the coefficients and variables in $A(x,y)$ are real, most of the theorems will apply with equal validity to the *Hermitian form*

$$H(x,\bar{x}) = \sum_{ij=1}^{\infty} a_{ij}\, x_i\, \bar{x}_j \ ,$$

where we assume $a_{ij} = \bar{a}_{ji}$. The bars over the a and the x denote the complex conjugates of a_{ji} and x_j respectively.

In general we shall designate a bilinear form by the symbol $A(x,y)$, the corresponding quadratic form by $A(x,x)$ and the matrix of the form, namely $\| a_{ij} \|$, by A. In some cases, however, where no ambiguity results, it will be convenient to represent bilinear and quadratic forms by A. The form conjugate to $A(x,y)$, that is to say, with matrix equal to A', will be designated by $A'(x,y)$. The unit form, that is to say, the form with matrix I, will be represented by $I(x,y)$.

The following theorem is fundamental in the theory of limited bilinear forms:

Theorem 9. If $A(x,y) = \sum_{ij=1}^{\infty} a_{ij}\, x_i\, y_j$ *and* $B(x,y) = \sum_{ij=1}^{n} b_{ij}\, x_i\, y_j$ *are limited forms, then the product form*

$$C(x,y) = \sum_{ij=1}^{\infty} c_{ij}\, x_i\, y_j \ ,$$

where $C = A\,B$, *that is, where*

$$\| c_{ij} \| = \| a_{ij} \| \cdot \| b_{ij} \| = \| a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots \| ,$$

is also a limited form.

Proof: In order to show this it is necessary to prove that

$$| \sum_{i\,j=1}^{n} [\sum_{k=1}^{\infty} a_{ik} b_{kj}] x_i y_j | \leqq M ,$$

where M is a number independent of n, x_i, and y_j.

This sum may be written in the form

$$\sum_{k=1}^{\infty} (\sum_{i=1}^{n} a_{ik} x_i) (\sum_{j=1}^{n} b_{kj} y_j) . \tag{9.1}$$

Lemma. (The Schwarz inequality). If $u_1, u_2, \cdots$ and $v_1, v_2, \cdots$ are two sets of real numbers, and if

$$u_1^2 + u_2^2 + \cdots + u_n^2 + \cdots \text{ and } v_1^2 + v_2^2 + \cdots + v_n^2 + \cdots$$

converge, then

$$u_1 v_1 + u_2 v_2 + \cdots + u_n v_n + \cdots$$

also converges and satisfies the inequality

$$| \sum_{p=1}^{\infty} u_p v_p | \leqq \sqrt{\sum_{p=1}^{\infty} u_p^2} \sqrt{\sum_{p=1}^{\infty} v_p^2} .$$

Proof: Since $\displaystyle\sum_{p=1}^{n} (\lambda u_p + \mu v_p)^2 \equiv$

$$\lambda^2 \sum_{p=1}^{n} u_p^2 + 2 \lambda \mu \sum_{p=1}^{n} u_p v_p + \mu^2 \sum_{p=1}^{n} v_p^2 \geqq 0 ,$$

it follows that the discriminant must be either negative or zero; i.e.,

$$(\sum_{p=1}^{n} u_p v_p)^2 - \sum_{p=1}^{n} u_p^2 \sum_{p=1}^{n} v_p^2 \leqq 0 ,$$

from which we derive

$$| \sum_{p=1}^{n} u_p v_p | \leqq \sqrt{\sum_{p=1}^{n} u_p^2} \sqrt{\sum_{p=1}^{n} v_p^2} .$$

From the assumption made as to the convergence of the left-hand member the lemma at once follows.

The Schwarz inequality was given by H. A. Schwarz in his memoir: Über ein die Flächen kleinsten Flächeninhalts betreffen des Problem der Variationsrechnung. *Acta soc. sc. Fennicae*, vol. 15 (1885), pp. 315-362. This inequality for finite summation, however, was given by J. L. Lagrange for three terms: *Nouv. Mem. Acad. Berlin* (*Oeuvres*, vol. 3, p. 662 et seq.) and by A. L. Cauchy: *Cours d'analyse de l'école polytech., Analyse algebrique* (1812), note 2, theorem 16 (*Oeuvres*, vol. 3, 2nd ser., p. 373 et seq.) for the general case. The theorem is thus often referred to as the *Lagrange-Cauchy inequality.*

We note that the Schwarz inequality holds *mutatis mutandis* for integrals. Thus, if $u(t)$ and $v(t)$ are real, continuous functions of t in the interval (ab), then the following inequality holds:

$$\left|\int_a^b u(t)\, v(t)\, dt\right|^2 \leqq \int_a^b |u(t)|^2\, dt \cdot \int_a^b |v(t)|^2\, dt \ .$$

This inequality can be generalized from the criteria that a real quadratic form be positive definite, that is to say, that the quadratic form

$$A = \sum_{ij=1}^{n} a_{ij}\, x_i\, x_j \ , \quad (a_{ij} = a_{ji})$$

shall assume no negative values for any values of the variables $x_i\, x_j$. In order for the quadratic form A to be positive definite, it is both necessary and sufficient that the n principal minors

$$A_0 = 1 \ , \ A_1 = a_{11} \ , \ A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \ , \cdots , \ A_n = |a_{ij}| \ ,$$

shall be positive or zero.

Let us now employ the abbreviation

$$(u\, v) = \int_a^b u(t)\, v(t)\, dt \ .$$

From the obvious inequality

$$\int_a^b \left[\sum_{i=1}^{n} x_i\, u_i(t)\right]^2 dt \geqq 0 \ ,$$

where $u_1(t)$, $u_2(t)$, $\cdots$, $u_n(t)$ form a set of real continuous functions in the interval (ab), we derive the positive definite quadratic form

$$U = \sum_{ij=1}^{n} U_{ij}\, x_i\, x_j \ , \quad (U_{ij} = U_{ji})$$

where we abbreviate

$$U_{ij} = (u_i\, u_j) \ .$$

Hence the sequence of determinants

$$U_0 = 1 \, , \ U_1 = (u_1\, u_1) \ , \ U_2 = \begin{vmatrix} (u_1\, u_1) & (u_1\, u_2) \\ (u_2\, u_1) & (u_2\, u_2) \end{vmatrix} \ , \cdots , U_n = |(u_i\, u_j)| \ ,$$

must be positive or zero. The Schwarz inequality is clearly the special case

$$U_2 \geqq 0 \ .$$

The determinants U_1, U_2, $\cdots$, U_n are called Gram determinants after J. P. Gram who first employed them in his notable paper: Über die Entwickelung reeller Funktionen in Reihen mittelst der Methode der kleinsten Quadrate. *Journal für Math.*, vol. 94 (1883), pp. 41-73.

The Schwarz formula has also been extended in another direction by O. Hölder, who established the following inequality:*

*Über einen Mittelwertsatz. *Göttinger Nachrichten* (1889), pp. 38-47.

$$|\sum_{p=1}^{n} u_p v_p|^m \leqq \{\sum_{p=1}^{n} |u_p|^{m/(m-1)}\}^{m-1} \{\sum_{p=1}^{n} |v_p|^m\}, \quad m > 1 .$$

The Schwarz inequality is the special case $m = 2$.

Closely related to the Hölder inequality is the following:

$$\{\sum_{p=1}^{n} |u_p + v_p|^m\}^{1/m} \leqq \{\sum_{p=1}^{n} |u_p|^m\}^{1/m} + \{\sum_{p=1}^{n} |v_p|^m\}^{1/m}, \quad m > 1 ,$$

which was established by H. Minkowski in 1907* and which is called the *Minkowski inequality.*

For the case $m = 2$ it follows as an immediate consequence of the Schwarz inequality. Thus we have

$$2 |\sum_{p=1}^{n} u_p v_p| \leqq 2 \{\sum_{p=1}^{n} |u_p^2|^2 \sum_{p=1}^{n} |v_p^2|^2\}^{\frac{1}{2}} ,$$

and hence

$$\sum_{p=1}^{n} |u_p^2|^2 + 2 |\sum_{p=1}^{n} u_p v_p| + \sum_{p=1}^{n} |v_p|^2 \leqq \sum_{p=1}^{n} |u_p|^2 + 2 (\sum_{p=1}^{n} |u_p|^2 \sum_{p=1}^{n} |v_p|^2)^{\frac{1}{2}} + \sum_{p=1}^{n} |v_p|^2 ,$$

from which it follows that

$$\sum_{p=1}^{n} |u_p + v_p|^2 \leqq \{ (\sum_{p=1}^{n} |u_p|^2)^{\frac{1}{2}} + (\sum_{p=1}^{n} |v_p|^2)^{\frac{1}{2}}\} .$$

Returning to (9.1) we see that the Schwarz inequality gives as an upper limit to the bilinear form the value

$$\sqrt{\sum_{k=1}^{\infty} (\sum_{i=1}^{n} a_{ik} x_i)^2} \sqrt{\sum_{k=1}^{\infty} (\sum_{j=1}^{n} b_{kj} y_j)^2} .$$

We must now show that the fact that $\sum_{i\,j=1}^{\infty} a_{ij} x_i y_j$ is limited implies that $\sum_{k=1}^{\infty} (\sum_{i=1}^{n} a_{ik} x_i)^2 \leqq M^2$, where M is the upper bound of the bilinear form and independent of n.

By hypothesis

$$|\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_i y_j| \leqq M \sqrt{\sum_{i=1}^{n} x_i^2} \sqrt{\sum_{j=1}^{m} y_j^2} .$$

Since we may write the left-hand member of this inequality as $\sum_{j=1}^{m} (\sum_{i=1}^{n} a_{ij} x_i) y_j$, we may think of it as an ordinary linear form which becomes a linear form in an infinite number of variables as $m \to \infty$.

But if we assume that

$$|\sum_{i=1}^{\infty} A_i X_i| \leqq M$$

Diophantische Approximation. Leipzig, (1907), p. 95.

when $X_1^2 + X_2^2 + \cdots \leqq 1$, it follows that $\sum_{i=1}^{\infty} A_i^2 \leqq M^2$, because, by the Schwarz inequality,

$$\left| \sum_{i=1}^{\infty} A_i X_i \right| \leqq \sqrt{\sum_{i=1}^{\infty} A_i^2} \sqrt{\sum_{i=1}^{\infty} X_i^2} \ .$$

Moreover, for $X_i = A_i / \sqrt{A_1^2 + A_2^2 + \cdots}$ the equality sign prevails since the upper limit is actually attained.

Hence we get

$$\sum_{j=1}^{\infty} \left(\sum_{i=1}^{n} a_{ij} x_i \right)^2 \leqq M^2 \ .$$

It then follows that if M is the upper bound of A, and N that of B, the upper bound of C will not exceed $M N$.*

The two following theorems are also essential in the theory of limited bilinear forms:

Theorem 10. If $\| a_{ij} \|$ *is the matrix of a limited form and if* $\sum_{j=1}^{\infty} c_j^2 \leqq 1$, then $\sum_{i=1}^{\infty} x_i^2$ *converges, where* $x_i = \sum_{j=1}^{\infty} a_{ij} c_j$.

Proof: By the Schwarz inequality we have

$$\sum_{i=1}^{\infty} x_i^2 = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} c_j \right)^2 \ .$$

But we have just proved in the preceding theorem that this sum converges under the hypotheses assumed above.

Theorem 11. The form $\sum_{ij=1}^{\infty} a_{ij} x_i y_j$ *is limited provided* $\sum_{ij=1}^{\infty} a_{ij}^2$ *converges.*

Proof: The proof follows from two applications of the Schwarz inequality:

$$\left(\sum_{ij=1}^{\infty} a_{ij} x_i y_j \right)^2 = \left[\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ij} x_i \right) y_j \right]^2 \leqq \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ij} x_i \right)^2 \sum_{j=1}^{\infty} y_j^2$$

$$\leqq \sum_{ij=1}^{\infty} a_{ij}^2 \sum_{i=1}^{\infty} x_i^2 \leqq \sum_{ij=1}^{\infty} a_{ij}^2 \ .$$

We now turn to the problem of solving system (1.1), where we shall impose the conditions: (a) that the $\{b_i\}$ belong to Hilbert space; (b) that $\sum_{ij=1}^{\infty} |a_{ij}|$ converges; (c) that the solution $\{x_i\}$ shall also belong to Hilbert space

*E. Hellinger and O. Toeplitz: Grundlagen für eine Theorie der unendlichen Matrizen. *Math. Annalen.* vol. 69 (1910), pp. 293-301.

Let us now consider the infinite matrix

$$A = \| a_{ij} \| .$$

If it turns out that a second matrix $\Phi = \| \varphi_{ij} \|$ exists such that

$$\Phi A = I ,$$

where I is the unit matrix $\| \delta_{ij} \|$ in which $\delta_{ij} = 1, i = j$, and $\delta_{ij} = 0$, $i \neq j$, then Φ is called the *forward inverse* (or reciprocal) of A. Similarly, if $\Psi = \| \psi_{ij} \|$ is a matrix such that

$$A \Psi = I ,$$

then Ψ is the *backward inverse* (or reciprocal) of A.

Theorem 12. If backward and forward inverse matrices both exist for A, then they are identical.

Thus we have

$$\begin{aligned} \Phi A \Psi &= I \Psi = \Psi , \\ &= \Phi I = \Phi . \end{aligned}$$

Theorem 13. If there exists one and only one forward inverse matrix, then this is also the backward inverse.

Proof: Let us assume the contrary; namely, that Φ is unique, but has the properties:

$$\Phi A = I ,$$

$$A \Phi = B \neq I .$$

Then we have

$$\begin{aligned} A \Phi A &= A (\Phi A) = A I = A , \\ &= (A \Phi) A = B A . \end{aligned}$$

Hence we obtain, $A = I A = B A$, and $(B - I) A = 0$. It then follows, letting λ be any parameter, that we have

$$[\Phi + \lambda (B - I)] A = I .$$

Thus Φ is not unique and the theorem follows from the contradiction.

We next observe that if A has a unique inverse, then a unique solution of system (1.1) will be given formally by

$$x_j = \sum_{i=1}^{\infty} \varphi_{ji} b_i . \tag{9.2}$$

It remains for us to see under what conditions this solution actually exists and satisfies the restrictions imposed by the problem.

The following theorem due to O. Toeplitz* supplies a set of sufficient conditions:

Theorem 14. Let $A(x,x)$ be an infinite quadratic form with matrix A which satisfies the following conditions:

(1) *$A(x,x)$ is limited;*

(2) *$A(x,x)$ is positive definite;*

(3) *The roots of the equation*

$$|\mu I_n - A_n| = 0 \ , \qquad (9.3)$$

where I_n and A_n are the nth *principal minors of $|I|$ and $|A|$, do not have zero as a limit point.*

Then a unique inverse exists for A which is also limited.

Proof: Before considering the details of the proof, it might be illuminating to discuss the following example:

Consider the system

$$\frac{1}{n} x_n = c_n \ , \quad n = 1, 2, \cdots .$$

Since $\sum\limits_{n=1}^{\infty} 1/n^2$ converges, the matrix of the associated form is limited, but the inverse matrix, $\| i\,\delta_{ij} \|$, is clearly not limited. Hence the solutions, $x_n = n\,c_n$, do not necessarily belong to Hilbert space since $\sum\limits_{n=1}^{\infty} n^2\,c_n^2$ is not necessarily bounded when $\sum\limits_{n=1}^{\infty} c_n^2 \leqq 1$.

Equation (9.3) reduces to

$$(\mu - 1)(\mu - 1/2)(\mu - 1/3) \cdots (\mu - 1/n) = 0$$

and zero is seen to be a limit point for the roots as $n \to \infty$. Thus a limited inverse does not exist.

The details of the proof consist in the explicit construction of the inverse of the limited bilinear form

$$A(x,x) = \sum_{ij=1}^{\infty} a_{ij}\,x_i\,x_j \ ,$$

which we shall also assume to be positive definite and whose matrix satisfies condition (3) of the theorem.

Let us designate by $A_n(x,x)$ the nth segment of $A(x,x)$ and by A_n the matrix of the form. The variables x_i, belonging to Hilbert space, may, without loss of generality, be assumed to satisfy the conditions:

$$x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 + \cdots = 1 \ . \qquad (9.4)$$

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*Die Jacobische Transformation der quadratischen Formen von unendlichvielen Veränderlichen. *Göttinger Nachrichten,* (1907), pp. 101-109.

Adopting the notation that $A^{(r)}$ is the determinant of the matrix of the rth segment of $A(x,x)$ and $A_{ij}^{(r)}$ is the cofactor of the element a_{ij} in $A^{(r)}$, we now construct the following transformation due originally to C. G. J. Jacobi (1804-1851):*

$$v_r = b_1^{(r)}u_1 + b_2^{(r)}u_2 + b_3^{(r)}u_3 + \cdots + b_r^{(r)}u_r \ , \qquad r = 1, 2, 3, \cdots, n \ , \tag{9.5}$$

where we abbreviate

$$b_i^{(r)} = A_{ri}^{(r)}/[A^{(r-1)}A^{(r)}]^{\frac{1}{2}} \ . \tag{9.6}$$

It can then be proved that

$$\sum_{i=1}^{n} v_i^2 = \sum_{ij=1}^{n} \frac{A_{ij}^{(n)}}{A^{(n)}} u_i u_j \ ,$$

and since the matrix of this form is the reciprocal of A_n, we see that the matrix $B_n = \| b_i^{(j)} \|$ is related to A_n by the equation

$$B'_n B_n = A_n^{-1} \ . \tag{9.7}$$

Similarly, if B_n is the reciprocal of B_n, we have the equation

$$B_n^{-1}(B'_n)^{-1} = A_n \ . \tag{9.8}$$

It is now necessary to anticipate a theorem established in section 2, chapter 12. The roots of equation (9.3) we shall call the *characteristic numbers* associated with the matrix A_n. From (2.22) and (2.23) of chapter 12 we know that if $A_n(x,x)$ is a positive definite form we have

$$\text{Max}\, A_n(x,x) = M_n \ , \quad \text{Min}\, A_n(x,x) = m_n \ ; \tag{9.9}$$

$$\text{Max}\, A_n^{-1}(x,x) = 1/m_n \ , \text{Min}\, A_n^{-1}(x,x) = 1/M_n \ , \tag{9.10}$$

where M_n and m_n are respectively the largest and the smallest of the roots of equation (9.3).

This theorem applies equally well to the infinite form $A(x,x)$. Employing this fact and noting condition (3) of the theorem, we are able to show that $A^{(n)} > 0$ for all values of n. If this were not the case and if one determinant, let us say $A^{(r)}$, were zero, then $|\mu I - A_r|$ would vanish for $\mu = 0$. Hence a set of values, $x_1, x_2, \cdots, x_r$, would exist in Hilbert space for which we should have $A_r(x,x) = 0$. Setting all other values of x equal to zero, we should then have $A_{r+m}(x,x) = 0$, $m = 0, 1, 2, \cdots, \infty$. Hence the equation $|\mu I - A_n| = 0$ would have $\mu = 0$ as a limit point, contrary to the original assumption.

*Über eine elementare Transformation eines in Bezug auf jedes von zwei Variablen-Systemen linearen und homogenen Ausdrucks. *Journal für Math.*, vol. 53 (1857), pp. 265-270. Also *Gesammelte Werke*, vol. 3 (1884), pp. 583-590.

From this we conclude that the Jacobi transformation (9.5) exists and that the coefficients $b_i^{(j)}$ are real. Moreover from (9.10) we obtain

$$B_n(x,x) \leqq \sqrt{\text{Max } B'_n(x,x) B_n(x,x)} = \text{Max } A_n^{-1}(x,x) = 1/m_n \ .$$

From this we conclude that $B(x,x)$ is limited, hence $B'(x,x)$ is limited, and from theorem 9 above, their product is also limited.

In similar manner, employing condition (1) of the theorem, it is easily proved that $B^{-1}(x,x)$, $[B^{-1}(x,x)]'$ and their product are also limited forms.

The following two equations

$$A\, A^{-1} = B^{-1} (B')^{-1} B' B = I(x,x) \ ,$$

$$A^{-1} A = B' B\, B^{-1} (B')^{-1} = I(x,x) \ .$$

complete the proof of the theorem.

In the argument just set down it was shown that a unique limited inverse existed for a quadratic form of a special type, namely, one that was positive definite. It is possible from this conclusion, however, to derive both necessary and sufficient conditions that a general quadratic form have a unique inverse. These conditions are set forth in the following theorem:

Theorem 15. A real limited quadratic form $A(x,x)$ possesses a limited forward inverse if and only if $A' A$ does not have zero for the limit point of its characteristic numbers. The form also possesses a limited backward inverse if and only if $A\, A'$ does not have zero for a limit point of its characteristic numbers.

Proof: We shall first show that the condition is necessary. If there exists a forward inverse $X(x,x)$ with matrix $\| X_{ij} \|$, then we have $X\, A = I(x,x)$, that is to say,

$$\sum_{i=1}^{\infty} x_i^2 = \sum_{j=1}^{\infty} \Big(\sum_{i=1}^{\infty} X_{ij}\, x_i \sum_{k=1}^{\infty} a_{jk}\, x_k \Big) \ .$$

From the Schwarz inequality and the fact that the variables belong to Hilbert space, we then obtain

$$\sum_{i=1}^{\infty} x_i^2 \leqq \Big[\sum_{j=1}^{\infty} \Big(\sum_{i=1}^{\infty} X_{ij}\, x_i \Big)^2 \sum_{j=1}^{\infty} \Big(\sum_{k=1}^{\infty} a_{jk}\, x_k \Big)^2 \Big]^{\frac{1}{2}} \ ,$$

that is,

$$[\text{Max } (X\, X')]\, (A'A) \geqq 1 \ .$$

From the assumption that X exists, we know that Max $(X\, X')$ is a positive number; hence it follows that $A'\, A$ cannot be zero and

thus the spectrum of characteristic numbers cannot have zero for a limit point.

To show that the condition is sufficient, we need merely observe that if $A(x,x)$ is a given real, limited quadratic form, then the form

$$A' A = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ji} x_i \right)^2$$

is positive definite and limited. Hence it may be discussed under the conditions of theorem 14. If condition (3) of that theorem is fulfilled, then there must exist a unique inverse, Y, which is also limited. Hence we have

$$Y A' A = I(x,x) \quad ,$$

and from this observe that $Y A'$ is the desired inverse of A.*

We conclude by noting that theorem 15 is immediately extended to limited bilinear forms. It is also applicable to Hermitian forms provided we substitute $A \bar{A}'$ and $\bar{A}' A$ respectively for $A A'$ and $A' A$ in the theorem.

PROBLEMS†

1. Given the bilinear form

$$A(x,y) = \sum_{pq=1}^{\infty} a_{pq} x_p y_q \quad ,$$

where the variables belong to Hilbert space. If the series

$$k_p = \sum_{r=1}^{\infty} | a_{pr} | \quad , \quad m_q = \sum_{r=1}^{\infty} | a_{rq} |$$

converge and if $k_p \leqq \kappa$, $m_q \leqq \mu$, where κ and μ are constants, then the upper bound of $A(x,y)$ is at most equal to $\sqrt{\kappa \mu}$.

2. Show that the form considered in problem 1 is not necessarily limited by examining the special case

$$a_{pq} = (\log p \cdot \log q)/(p+q) \quad .$$

3. If $A(x,y)$ is a limited bilinear form, and if $| a_{pq} | < a$, then show that

$$\sum_{pq=1}^{\infty} f(a_{pq}) x_p y_q$$

is also limited provided the power series

$$f(x) = c_1 x + c_2 x^2 + \cdots$$

converges absolutely for $x = a$.

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*Another demonstration of theorem 15 has been given by E. Hilb: Über die Auflösung von Gleichungen mit unendlichvielen Unbekannten. *Sitzungsberichte der Phys.-Med. Sozietät, Erlangen*, vol. 40, (1908), pp. 84-89. See also F. Riesz: *Les systémes d'équations linéaires.* Paris (1913), pp. 89-94.

†The problems in this list are due to I. Schur: Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlichvielen Veränderlichen. *Journal für Mathematik*, vol. 140 (1911), pp. 1-28.

4. Show that the two forms

$$\sum_{pq=1}^{\infty} \frac{a_{pq}}{t-a_{pq}} x_p y_q \ , \quad \sum_{pq=1}^{\infty} \log \left(\frac{t}{t-a_{pq}} \right) x_p y_q \ ,$$

are limited provided the bilinear form $A(x,y)$ defined in problem 1 is limited and provided $| t | > | a_{pq} |$.

5. Let $f_p(t)$ and $g_p(t)$, $p = 1, 2, 3, \cdots$, be a set of arbitrary functions of integrable square over the interval $a \leqq t \leqq b$, such that

$$\int_a^b | f_p(t) |^2 \, dt \leqq \mu \ , \quad \int_a^b | g_p(t) |^2 \, dt \leqq \nu \ .$$

Now compute the constants

$$b_{pq} = \int_a^b f_p(t) g_q(t) \, dt \ ,$$

and construct the bilinear form

$$B(x,y) = \sum_{pq=1}^{\infty} a_{pq} \, b_{pq} \, x_p \, y_q \ .$$

If the form $A(x,y) = \sum a_{pq} \, x_p \, y_q$ has the upper bound α, then the upper bound of $B(x,y)$ is at most equal to $\frac{1}{2} \alpha(\mu + \nu)$.

6. If $A(x,\bar{x})$ and $B(x,\bar{x})$ are two positive definite Hermitian forms and if α, α' are respectively the largest and smallest characteristic numbers of $A(x,\bar{x})$ and β, β' the largest and smallest characteristic numbers of $B(x,\bar{x})$, prove that the characteristic numbers of the form

$$C(x,\bar{x}) = \sum_{pq=1}^{\infty} a_{pq} \, b_{pq} \, x_p \, \bar{x}_q$$

lie between $\alpha' \beta'$ and $\alpha \beta$.

7. Prove that the maximum value of the forms

$$P(x,y) = {\sum_{p,q=1}^{\infty}}' \frac{x_p \, y_q}{p-q} \ , \quad Q(x,y) = \sum_{p,q=1}^{\infty} \frac{x_p \, y_q}{p+q} \ ,$$

where $\sum'$ means the term $p = q$ is omitted from the sum, is at most equal to π .

8. Prove that the maximum value of the forms

$$P(x,y;\lambda) = \sum_{p,q=1}^{\infty} \frac{x_p \, y_q}{p-q+\lambda} \ , \quad Q(x,y;\lambda) = \sum_{p,q=1}^{\infty} \frac{x_p \, y_q}{p+q-1+\lambda} \ ,$$

is at most equal to π when λ is an integer and is at most $\pi / | \sin \lambda \pi |$ when λ is not an integer.

9. Prove that the form

$$A(x,y;\mu) = {\sum_{p,q=1}^{\infty}}' \frac{(pq)^{\frac{1}{2}(\mu-1)}}{p^\mu - q^\mu} x_p \, y_q \ , \quad \mu \geqq 1 \ ,$$

is limited and that its upper bound is at most equal to π/μ.

10. Extension of the Foregoing Theory to Hölder Space. Although the purpose of the present volume will be served by limiting the variables to Hilbert space, it is instructive to consider the more general problem of solving system (1.1) where the unknown quantities are assumed to belong to Hölder space, that is to say, where they satisfy the condition

$$\sum_{i=1}^{\infty} | x_i |^p \leqq M \ , \quad p > 1 \ . \tag{10.1}$$

The positive constant M may be set equal to 1 without limiting the generality of the problem. The quantity p is called the *exponent of the space.* It will be convenient also to introduce the ratio $q = p/(p-1)$, from which we have

$$\frac{1}{p}+\frac{1}{q}=1 \ . \tag{10.2}$$

Since p and q are separated by 2, except when $p = q = 2$, we may assume that $1 < p \leqq 2$, and hence that $q \geqq 2$.

The related problems of solving the set of equations (1.1) and of obtaining the reciprocal of the bilinear form

$$A(x, y) = \sum_{ij=1}^{\infty} a_{ij} \, x_i \, y_j \ , \tag{10.3}$$

under the assumption that the x_i and the y_i belong to Hölder spaces of exponents p and q respectively, have been extensively studied by F. Riesz.* Further contributions to the theory have been made by St. Bóbr† and L. W. Cohen.‡

The principal tools employed in these investigations have been the Hölder and Minkowski inequalities, which we have defined in the preceding section. A typical application is found in the following theorem due to E. Landau:§

Theorem 16. If an infinite set of variables $\{x_i\}$ belongs to a Hölder space of exponent p, then the linear form

$$\sum_{i=1}^{\infty} a_i \, x_i$$

converges for all values of x_i if and only if the set $\{a_i\}$ belongs to a Hölder space of index q.

**Les systèmes d'équations linéaires,* (*loc. cit.*), chapter 3.

†Eine Verallgemeinerung des v. Kochschen Satzes über die absolute Konvergenz der unendlichen Determinanten. *Mathematische Zeitschrift,* vol. 10, (1921), pp. 1-11.

‡A Note on a System of Equations with Infinitely Many Unknowns. *Bulletin of the American Math. Soc.,* vol. 36 (1930), pp. 563-572.

§Über einen Konvergenzsatz. *Göttinger Nachrichten,* (1907), pp. 25-27.

The proof of this theorem is obtained as an obvious derivation from the Hölder inequality.

St. Bóbr has extended von Koch's theorem on the convergence of an infinite determinant (see section 3) as follows:

Theorem 17. The determinant

$$A = | a_{ij} | \ ,$$

together with all its minors, converges absolutely provided

(1) *the product* $\prod_{i=1}^{\infty} a_{ii}$ *converges:*

(2) *there exists a number* $q \geqq 2$, *such that the double series*

$$\sum_{i=1}^{\infty} \{ \sum_{j=1}^{\infty} | a_{ij} |^{q} \}^{1/(q-1)} \ , \quad i \neq j \ , \tag{10.4}$$

converges.

The proof of this theorem is attained by an argument similar to that employed in section 3. The details will be omitted here.

Cohen in his study considered the problem of solving system (1.1) in which the coefficients a_{ij} of that system are replaced by $\delta_{ij} + a_{ij}$, that is to say, for a system, which has the following determinant:

$$\Delta = | \delta_{ij} + a_{ij} | \ .$$

The quantities $\{b_i\}$ which comprise the right hand members of the system are assumed to belong to a Hölder space of exponent p and the coefficients $\{a_{ij}\}$ are subject to the single condition:

$$\sum_{i=1}^{\infty} [\sum_{j=1}^{\infty} | a_{ij} |^{p/(p-1)}]^{(p-1)} < m \ , \quad 1 < p \leqq 2 \ , \tag{10.5}$$

where m is finite.

A system equivalent to (1.1) is first constructed as follows:

From (10.5) it follows that $\lim_{i=\infty} | a_{ii} | = 0$. Hence there exists a integer i_0 such that the following inequality holds:

$$\tfrac{1}{2} < 1/(1 + a_{ii}) < 2 \ , \quad \text{when } i > i_0 \ .$$

The original system is then replaced by the following equivalent one:

$$x_i + \sum_{j=1}^{\infty} c_{ij}\, x_j = e_i \ ; \ \delta_{ij} + c_{ij} = d_i (\delta_{ij} + a_{ij}) \ , \ e_i = d_i\, b_i \ , \tag{10.6}$$

where we write

$$d_i = 1 \text{ for } i \leqq i_0 \ , \quad d_i = 1/(1 + a_{ii}) \text{ for } i > i_0 \ .$$

Since the d_i are bounded and the $c_{ii} = 0$ for $i > i_0$, it follows that the infinite product $\Pi(\delta_{ii} + c_{ii})$ and the sum

$$\sum_{i=1}^{\infty} \{ \sum_{j=1}^{\infty} |c_{ij}|^q \}^{1/(q-1)} , \quad q \geqq 2$$

both converge. Hence the determinant of the system fulfills the conditions of St. Bóbr's theorem.

The solution of the equivalent system is then attained by Cramer's rule, since the minors converge absolutely, and the condition that the $\{b_i\}$ are in a Hölder space of exponent p is sufficient to establish the convergence of the solution thus found.

The following theorem may then be stated:

Theorem 18. If the determinant Δ does not vanish, if the quantities $\{b_i\}$ of (1.1) *belong to a Hölder space of exponent p, and if the coefficients $\{a_{ii}\}$ satisfy the inequality* (10.5), *then a solution of the original system may be found by solving by Cramer's rule the equivalent system* (10.6). *The solution will belong to a Hölder space of exponent p.*

It is instructive to examine the situation with respect to the transposed system

$$x_i + \sum_{j=1}^{\infty} a_{ji} x = b_i , \tag{10.7}$$

where the quantities $\{a_{ij}\}$ are subject to the inequality (10.5).

The pertinent theorem is immediately derived from a consideration of the following inequality established by an obvious application of the Minkowski inequality:

$$\sum_{j=1}^{\infty} \{ \sum_{i=1}^{\infty} | a_{ij} |^{q/(q-1)} \}^{(q-1)} \leqq \sum_{i=1}^{\infty} \{ \sum_{j=1}^{\infty} | a_{ij} |^{p(p-1)} \}^{(p-1)} .$$

Noting that inequality (10.5) imposes upon the coefficients of the transposed system an identical inequality in which p is replaced by q, we immediately derive the following:

Theorem 19. The statement of theorem 18 *also applies to the transposed system* (10.7) *provided the $\{b_i\}$ are assumed to belong to a Hölder space of exponent q. The solution will belong to a Hölder space of exponent q.*

The theorems which have been given above afford sufficient conditions for the existence of a solution of system (1.1). We shall conclude with a statement of the theorem of Riesz which specifies both a necessary and a sufficient condition for the existence of a solution:

Theorem 20. If in system (1.1) *the coefficients are subject to the condition that the following series converge:*

$$\sum_{j=1}^{\infty} | a_{ij} |^{p/(p-1)} , \quad (i = 1, 2, \cdots)$$

then in order for the system to have a solution belonging to a Hölder space of exponent $p > 1$, *it is both necessary and sufficient that there exists a positive quantity* m *such that*

$$| \sum_{i=1}^{n} m_i c_i | \leqq m \, \{ \sum_{j=1}^{\infty} | \sum_{i=1}^{n} m_i a_{ij} |^{p/(p-1)} \}^{(p-1)/p}$$

for all positive integers n *and for all values* m_i.

For a proof of this theorem the reader is referred to chapter 3 of *Les systèmes d'équations linéaires* of Riesz.

CHAPTER IV.

Operational Multiplication and Inversion

1. *Algebra and Operators.* Fundamental to any theory of operators is the *law of composition* or *operational multiplication.* The problem that confronts us in the establishment of such a law is essentially different from the equivalent problem in linear algebra. It seems worth while to indicate briefly the nature of this difference.

The construction of a linear algebra is based upon three concepts: (1) the existence of a set of unitary elements, $e_1, e_2, \cdots, e_n$, generally finite in number; (2) the existence of a field (A) of scalar multipliers, *independent* of the units, from which the general number is constructed by the addition of scalar products of the form A_1e_1, i.e.,

$$x = \sum_{i=1}^{n} A_i e_i$$

(3) the existence of a multiplication table for the unitary elements; that is to say,

$$e_i \times e_j = \sum_{k=1}^{n} d_{ijk} e_k \ ,$$

where the multipliers belong to (A).

The product of two such numbers, $x = \sum_{i=1}^{n} A_i e_i$ and $y = \sum_{j=1}^{n} B_j e_j$, is then found through use of the linearity postulate to be

$$A \times B = \sum_{ij}^{n} A_i e_i B_j e_j = \sum_{ij}^{n} A_i B_j e_i e_j = \sum_{ij}^{n} C_k e_k \ , C_k = \sum_{ij}^{n} A_i B_j d_{ijk} \ .$$

The problem of operational multiplication, on the other hand, is based (1) upon an infinite set of primary elements, $\cdots\cdot$, z^{-n}, z^{-n-1}, $\cdots\cdot$, $z^{-1}, 1, z, z^2$, $\cdots\cdot$, which form a group, $z^m\ z^n = z^{m+n}$, and (2) upon a class of functions, $\{A_i(x)\}$, the combination of which with the elementary operators is *not independent* of them. It is this dependence that so greatly complicates the problem of the multiplication of operators.

In spite of these essential differences, however, there does exist in many cases, particularly where the class of functions $\{A_i(x)\}$ is the class of constants, a formal analogy between operational symbols and the symbols of algebra. The source of this formal resemblance will be discussed in the ensuing pages.

Let us for the sake of ready reference designate the differential operator of infinite order by the symbol

$$A(x, z) = a_0(x) + a_1(x)z + a_2(x)z^2 + \cdots\cdots,$$

and the polar operator by the symbol

$$B(x, 1/z) = b_1(x)/z + b_2(x)/z^2 + b_3(x)/z^3 + \cdots\cdots .$$

It will also be convenient to refer to these occasionally as generatrix operators of *first* and *second* kind respectively.

2. The Generalized Formula of Leibnitz. It was first proved by Leibnitz* that the nth derivative of a product is given by

$$z^n \to uv = vu^{(n)} + nv'u^{(n-1)} + n(n-1)v''u^{(n-2)}/2! + \cdots + v^{(n)}u . \tag{2.1}$$

This formula is capable of an extraordinary generalization, being but a special case of the following one:†

$$\begin{aligned} F(x, z) \to uv = vF(x, z) \to u + v'F'_z(x, z) \to u/1! \\ + v''F_z''(x, z) \to u/2! + \cdots\cdots , \end{aligned} \tag{2.2}$$

where we have used the abbreviation $F_z^{(n)}(x, z) = \partial^n F(x, z)/\partial z^n$.

Under proper convergence assumptions, formula (2.2) is obviously obtained from (2.1) if $F(x, z)$ is a generatrix operator of first kind. That it also applies to integral operators will be apparent from a consideration of the expansion

$$\begin{aligned} z^{-n} \to uv &= \int_0^x \cdots \int_0^t u\, v\, dt^n = \int_0^x (x-t)^{n-1} uv\, dt/(n-1)! \\ &= \int_0^x (x-t)^{n-1} u(t) \{v(x) - (x-t)v'(x) \\ &\qquad + (x-t)^2 v''(x)/2! - \cdots\}dt/(n-1)! \\ &= v(x)z^{-n} \to u - v'(x)n\, z^{-n-1} \to u + n(n+1)v''\, z^{-n-2} \to u/2! \\ &\qquad - n(n+1)(n+2)v'''\, z^{-n-3} \to u/3! + \cdots . \end{aligned} \tag{2.3}$$

Since this is equivalent to formula (2.2) applied to the function z^{-n}, we see that it also applies, with proper convergence assumptions, to the general polar operator.

Moreover it will be observed that formula (2.2) also holds when we specialize $F(x, z) = z^{-\mu}$, μ not an integer, and $F(x, z) = z^{-\mu} \log z$. The first case follows from a slight generalization of (2.3), $(n-1)!$ being replaced by $\Gamma(n)$. The second case is established as follows:

---

*In a letter to Johann Bernoulli and later in his *Symbolismus memorabilis etc.*

†For this see Hargreave: *London Phil Trans.* (1848).

By definition we have

$$z^{-\mu}\log z \to uv = -\int_0^x (x-t)^{\mu-1}\{\log(x-t) - \psi(\mu)\}u(t)v(t)\,dt/\Gamma(\mu)$$

$$= -\int_0^x (x-t)^{\mu-1}\{\log(x-t) - \psi(\mu)\}u(t)\{\sum_{n=0}^{\infty} v^{(n)}(x)\,(x-t)^n$$

$$\times (-1)^n/n!\}\,dt/\Gamma(\mu) = -\int_0^x \{\log(x-t) - \psi(\mu)\}\,u(t)$$

$$\times \{\sum_{n=0}^{\infty} v^{(n)}(x)\,(x-t)^{\mu+n-1}(-1)^n/n!\}\,dt/\Gamma(\mu) \ .$$

But since $\psi(\mu) = \psi(\mu+1) - 1/\mu$, $\Gamma(\mu) = \Gamma(\mu+1)/\mu$, we can write

$$z^{-\mu}\log z \to uv = v(x)z^{-\mu}\log z - \int_0^x \sum_{n=1}^{\infty}\{\log(x-t) - \psi(\mu+n)$$

$$+ 1/\mu + 1/(\mu+1) + \cdots + 1/(\mu+n-1)\}\,v^{(n)}(x)(x-t)^{\mu+n-1}(-1)^n$$

$$\times u(t)\,dt\,\mu(\mu+1)\cdots(\mu+n-1)/\Gamma(\mu+n)\cdot n!$$

$$= v(x)z^{-\mu}\log z \to u(x) + \sum_{n=1}^{\infty}(-1)^n v^{(n)}(x)\{\mu(\mu+1)\cdots(\mu+n-1)$$

$$\times z^{-\mu-n}\log z \to u(x) - P_n(\mu)z^{-\mu-n-1} \to u(x)\}/n! \ ,$$

where we write

$$P_1(\mu) = 1, \quad P_n(\mu) = (\mu+1)\cdots(\mu+n-1) +$$
$$\mu(\mu+2)\cdots(\mu+n-1) + \cdots + \mu(\mu+1)\cdots + (\mu+n-2) \ .$$

But it is seen by explicit calculation that if $F(z) = z^{-\mu}\log z$, then $F^{(n)} = F_z^{(n)}$ equals the coefficient of $v^{(n)}(x)/n!$, which was the desired result.

3. Bourlet's Operational Product. Proceeding from the generalized formula of Leibnitz we are now able to derive the *product generatrix* obtained when an operator $X(x, z)$ operates upon a second operator, $F(x, z)$. This formula is due to Bourlet and plays a fundamental rôle in much of the theory that follows.

We shall designate the product of $X(x,z)$ into $F(x,z)$ by means of the symbol $X(x,z) \to F(x,z)$, or occasionally by $[X \cdot F](x,z)$.

We first prove the following fundamental theorem:

Theorem 1. If $X(x,z)$ and $F(x,z)$ are both operators of first kind (or both of second kind), then the operational product,

$$[X \cdot F](x,z) = X(x,z) \to F(x,z) \ ,$$

can be expanded formally into

$$
\begin{aligned}
X \to F = [X \cdot F] = F\,X + (\partial F/\partial x)\,(\partial X/\partial z) \\
+ (\partial^2 F/\partial x^2)\,(\partial^2 X/\partial z^2)/2! + \cdots \\
+ (\partial^n F/\partial x^n)\,(\partial^n X/\partial z^n)/n! + \cdots . \qquad (3.1)
\end{aligned}
$$

Proof: Let us assume first that both X and F have Taylor's expansions in z about the origin and let us denote them by

$$
\begin{aligned}
X(x,z) &= X_0(x) + X_1(x)z + X_2(x)z^2 + \cdots , \\
F(x,z) &= F_0(x) + F_1(x)z + F_2(x)z^2 + \cdots .
\end{aligned}
\qquad (3.2)
$$

It is then clear that we shall have

$$
\begin{aligned}
[X \cdot F] \to u(x) = X(x,z) \to \{F_0(x)\,u(x) + F_1(x)\,u'(x) \\
+ F_2(x)\,u''(x) + \cdots\} ,
\end{aligned}
$$

or making use of the generalized Leibnitz formula (2.2), we get

$$
\begin{aligned}
[X \cdot F] \to u(x) &= \sum_{j=0}^{\infty} \{F_0^{(j)}(x)/j!\}\{\partial^j X(x,z)/\partial z^j\} \to u(x) \\
&+ \sum_{j=0}^{\infty} \{F_1^{(j)}(x)/j!\}\{\partial^j X(x,z)/\partial z^j\} \to u'(x) \\
&+ \sum_{j=0}^{\infty} \{F_2^{(j)}(x)/j!\}\{\partial^j X(x,z)/\partial z^j\} \to u''(x) + \cdots \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \{F_1^{(j)}(x)/j!\}\{\partial^j X(x,z)/\partial z^j\} \to u^{(i)}(x) .
\end{aligned}
$$

Designating the differentiation of $u(x)$ by placing z^i on the left side of the arrow, we obtain

$$
\begin{aligned}
[X \cdot F] \to u(x) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \{z^i F_i^{(j)}(x)/j!\}\{\partial^j X(x,z)/\partial z^j\} \to u(x) \\
&= \sum_{j=0}^{\infty}\{\sum_{i=0}^{\infty} z^i F_i^{(j)}(x)\}\{\partial^j X(x,z)/\partial z^j\}/j! \to u(x) \\
&= \sum_{j=0}^{\infty} \{\partial^j F(x,z)/\partial x^j\}\{\partial^j X(x,z)/\partial z^j\}/j! \to u(x) \\
&= F(x,z)\,X(x,z) + (\partial F/\partial x)\,(\partial X/\partial z) \\
&\qquad + (\partial^2 F/\partial x^2)\,(\partial^2 X/\partial z^2)/2! + \cdots .
\end{aligned}
$$

We have thus established (3.1) for operators of first kind. But the formula has equal validity for operators of second kind because

polar operators are similarly included under the Leibnitz formula (2.2). The proof just given may be used with suitable modifications provided $X(x,z)$ and $F(x,z)$ are assumed to have Taylor's expansions in $1/z$ instead of z.

From the result of theorem 1 it is easily seen that, in general, the product of two operators is not commutative, since

$$X \to F \neq F \to X \ .$$

However, in the case of constant coefficients, that is to say, where $X_n(x)$ and $F_n(x)$ of (3.2) are constants, we have

$$X \to F = F \to X = X(z)\, F(z) \ ,$$

and hence we are able to conclude that *two linear uniform operators with constant coefficients are commutative.*

It will be found upon examination that the expansion (3.1) also holds when the functions contain terms of the form $z^{-\nu} \log z$. An explicit statement of this corollary follows:

Corollary: If $S(x,z)$ and $T(x,z)$ are operators the symbolic multiplication of which is given by the Bourlet product, then $z^{-\nu}$ $\log^n z\, S(x,z)$ and $z^{-\mu} \log^m z\, T(x,z)$ are also such operators.

Proof: If we abbreviate $\varphi(z) = z^{-\nu} \log^n z$ and $\varrho(z) = z^{-\mu} \log^m z$, our problem is to show that

$$\varphi(z) S \to \varrho T = \varrho[ST + (\partial T/\partial x)(\partial \varphi S/\partial z)/1! \\ + (\partial^2 T/\partial x^2)(\partial^2 \varphi S/\partial z^2)/2! + \cdots] \ .$$

To prove this we expand $C(x,z)$ and $T(x,z)$ in the series

$$S(x,z) = \sum_{n=-\infty}^{\infty} S_n(x) z^n, \text{ and } T(x,z) = \sum_{n=-\infty}^{\infty} T_n(x) z^n \ .^*$$

We can then write

$$A(x,z) \equiv (\varphi S) \to \sum_{n=-\infty}^{\infty} \varrho T_n(x) z^n = \sum_{n=-\infty}^{\infty} (\varphi S) \to \varrho T_n(x) z^n \ .$$

But since we have shown in section 2 the validity of the Leibnitz rule for φz^n, in particular, and hence formally for $\sum_{n=-\infty}^{\infty} S_n(x) \varphi z^n$, in general, we are able to write

$$A(x,z) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} (d^m T_n/dx^m)(\partial^m \varphi S/\partial z^m) \varrho\, z^n/m!$$

---

*For the generality of these expansions see section 14, chapter 2.

$$A(x,z) = \sum_{m=0}^{\infty} (\partial^m \varphi S/\partial z^m) \sum_{n=-\infty}^{\infty} (d^m T_n/dx^m)\, \varrho z^n/m!$$

$$= \varrho \sum_{m=0}^{\infty} (\partial^m S/\partial z^m)\, (\partial^m T/\partial x^m)/m!\ .$$

We thus attain the desired equation and the corollary is proved.

PROBLEMS

1. If $R = x\, e^{-z}$, show that

$$R^m = \frac{\Gamma(x+1)}{\Gamma(x-m+1)}\, e^{-mz}\ .$$

2. Prove that for the operator of problem 1,

$$R^m \rightarrow R^n = R^{m+n}\ .$$

3. If $P = x(1 - e^{-z})$, show that

$$P^n \rightarrow R^m = R^m \rightarrow (P+m)^n\ ,$$

where R is the operator of problem 1.

4. Prove that

$$F(P) \rightarrow R^m = R^m \rightarrow F(P+m)\ ,$$

where $F(x)$ is a polynomial and P and R are the operators defined in the preceding problems.

5. If $F(x)$ is a polynomial and if $F_n(x) = (1 - e^{-z})^n \rightarrow F(x)$, prove that

$$F(P \pm R) = F(P) \pm F_1(P)R + \frac{1}{2!} F_2(P)R^2 \pm \frac{1}{3!} F_3(P)R^3 + \cdots\ .$$

6. Prove that every linear difference equation with coefficients rational in x can be expressed in the form

$$[f_0(P) + f_1(P)R + f_2(P)R^2 + \cdot \quad + f_n(P)R^n] \rightarrow u(x) = f(x)\ ,$$

where $f_r(x)$, $r = 0, 1, 2, \cdots, n$, are polynomials and $f(x)$ is a known function.

[The operators employed in the six problems just given are due to G. Boole, who designated them by ρ and π respectively. For a systematic account of these operators see L. M. Milne-Thomson: On Boole's Operational Solution of Linear Finite Difference Equations. *Proc. Cambridge Phil. Soc.*, vol. 38 (1932), pp. 311-318; also Milne-Thomson: *The Calculus of Finite Differences*, (See *Bibliography*), chap. 14.]

7. Express the difference equation

$$x(x-1)\, u(x+1) - (x^2-1)\, u(x+1) + x\, u(x) = 1\ ,$$

in the form suggested by problem 6.

8. Prove the identity

$$J_n(x) = \frac{x^n}{2^{n-1}\,\Gamma(n+\tfrac{1}{2})\,\Gamma(\tfrac{1}{2})}\, (1+z^2)^{n-\frac{1}{2}} \rightarrow (\sin x/x)\ ,$$

where $J_n(x)$ is the nth Bessel function.

9. Prove the identity

$$-Y_n(x) = \frac{x^n}{2^{n-1}\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})}\,(1+z^2)^{n-\frac{1}{2}} \rightarrow (\cos x/x) \ ,$$

where $Y_n(x)$ is the Bessel function of second kind defined by

$$Y_n(x) = [J_n(x)\cos n\pi - J_{-n}(x)]/\sin n\pi \ .$$

{The identities given in problems 8 and 9 are due to J. C. Hargreave [See *Bibliography*: (Hargreave (2)] and to H. M. Macdonald: Note on Bessel Functions. *Proc. London Math. Soc.*, vol. 29 (1897-1898), pp. 110-115. See also G. N. Watson: *Theory of Bessel Functions*. Cambridge (1922), pp. 170-171.}

4. The Algebra of Functions of Composition. In the preceding section we found an expression of unusual generality for the composition of two operators X and F. If we now specialize our field to include only the class of operators defined by Volterra and Fredholm integrals we can construct an *algebra of functions of composition* which is fundamental to the theory of integral equations.

To exhibit the basis of this functional algebra let us define two operators by means of Volterra integrals:

$$V_1(x,z) \rightarrow u(x) = \int_a^x f(x,t)\, u(t)\, dt \ ,$$

$$V_2(x,z) \rightarrow u(x) = \int_a^x g(x,t)\, u(t)\, dt \ .$$

Forming the composition of these operators and employing the formula of Dirichlet* for the interchange of the order of integration, we shall then have

$$V_1(x,z) \rightarrow \{V_2(x,z) \rightarrow u(x)\} = \int_a^x f(x,t)\, dt \int_a^t g(t,y)\, u(y)\, dy$$

$$= -\int_a^x \psi(x,y)\, dy \ ,$$

where we abbreviate $\psi(x,y) = \int_x^y f(x,t)\, g(t,y)\, dt.$

It is thus seen that the composition of two operators defined by Volterra integrals is characterized by a *function of composition*, $\psi(x,y)$. A similar situation prevails for operators defined by Fredholm integrals, namely, operators of the form

$$F(x,z) \rightarrow u(x) = \int_a^b f(x,t)\, u(t)\, dt.$$

*See section 6, chapter 2.

In order to discover the analogies which this operational product possesses with the ordinary rules of algebra, Volterra was led to the development of an algebra of functions of composition, the main features of which we shall now present.

It is clear that the operation

$$\psi(x, y) = \int_{\beta(x)}^{\alpha(y)} f(x, t) g(t, y) \, dt$$

will, in general, define a new function $\psi(x, y)$. Obvious limitations as to integrability and the region of definition must, of course, be imposed. Since, however, we are now concerned only with the formal aspects of composition, these important considerations will be disregarded for the present.

Volterra has called $\psi(x, y)$ a *function of composition of first kind* for the case where $\alpha(y) = y$, $\beta(x) = x$, and a *function of composition of second kind* when $\alpha(y) = a$, $\beta(x) = b$, where a and b are both constants.

The functions $f(x, t)$ and $g(t, y)$ are called *permutable* provided they satisfy the relation

$$\int_{\beta(x)}^{\alpha(y)} f(x, t) g(t, y) \, dt = \int_{\beta(x)}^{\alpha(y)} g(x, t) f(t, y) \, dt \ ,$$

permutability of first and second kinds being defined for the cases $\alpha(y) = y$, $\beta(x) = x$, and $\alpha(y) = a$, $\beta(x) = b$ respectively. It is with permutability of first kind that we shall be concerned here.

Following the notation of Volterra, we employ the symbols

$$\psi(x, y) = \overset{*}{f} \overset{*}{g}(x, y) \ .$$

In cases where no ambiguity will result, this symbol will be modified by placing the asterisk between the two functions the composition of which is studied. In this capacity the asterisk plays a rôle analogous to multiplication. Thus we can represent the composition of f and g by the symbol

$$\psi(x, y) = f * g \ .$$

It will be seen directly from the definition that the composition is *associative*; i.e.,

$$\overset{*}{f}(\overset{*}{g} \overset{*}{h}) = (\overset{*}{f} \overset{*}{g}) \overset{*}{h}, \quad \text{or} \quad f * (g * h) = (f * g) * h \ .$$

Thus defining two new functions $\varphi(x, y) = g * h$ and $\psi(x, y) = f * g$, we have

$$f * (g * h) = \int_x^y f(x,s)\,\varphi(s,y)\,ds$$

$$= \int_x^y ds \int_s^y f(x,s)\,g(s,t)\,h(t,y)\,dt\ .$$

Interchanging the order of integration by use of Dirichlet's formula, we get

$$f * (g * h) = \int_x^y dt \int_x^t f(x,s)\,g(s,t)\,h(t,y)\,ds$$

$$= \int_x^y \psi(x,t)\,h(t,y)\,dt = (f * g) * h\ .$$

That composition is also *distributive,*

$$f * (g + h) = f * g + f * g\ ,$$

is immediately evident from the distributive nature of integration.

In order to establish a formal analogy with the processes of algebra it is necessary to impose the limitation of *permutability* upon the functions entering the domain of our calculations so that the *commutative* law may also hold, namely,

$$f * g = g * f\ .$$

We shall first prove the very important fact that functions formed by successive compositions of a function with itself will be members of a permutable set. This is seen to be a consequence of the next theorem.

Theorem 2. If $f(x,y)$ is a function integrable in the triangle $a \leqq t \leqq x \leqq b$, and if $f_h(x,y)$ is the h*th iterated function,*

$$f_h(x,y) = \int_x^y f(x,t)\,f_{h-1}(t,y)\,dt\ ,$$

then we shall have

$$f_h(x,y) = \int_x^y f_i(x,t)\,f_{h-i}(t,y)\,dt\ , \quad i = 1,\ 2,\ \cdots,\ h-1\ .$$

Proof: We see by definition that this is true for $i = 1$. We now assume that it is true for $i = k$ and establish it for $i = k + 1$. The proof then follows by induction.

Hence we have, by assumption,

$$f_h(x,y) = \int_x^y f_k(x,t) \int_t^y f(t,s)\, f_{h-k-1}(s,y)\, ds\, dt\ .$$

Applying to this iterated integral the Dirichlet formula for the interchange of the order of integration, we have

$$f_h(x,y) = \int_x^y f_{h-k-1}(s,y) \int_x^s f_k(x,t)\, f(t,s)\, dt\, ds$$

$$= \int_x^y f_{k+1}(x,t)\, f_{h-k-1}(t,y)\, dt\ ,$$

which is seen to establish the theorem for the subscript $k+1$.

We have assumed in this, however, that the theorem is also true for $i = h - 1$, and this fact must now be independently established. We again employ induction and consider the iterated integral

$$f_3(x,y) = \int_x^y f(x,t)\, f_2(t,y)\, dt$$

$$= \int_x^y f(x,t) \int_t^y f(t,s)\, f(s,y)\, ds\, dt\ .$$

Applying the Dirichlet formula to interchange the order of integration, we have

$$f_3(x,y) = \int_x^y f(s,y) \int_x^s f(x,t)\, f(t,s)\, dt\, ds$$

$$\int_x^y f_2(x,s)\, f(s,y)\, ds.$$

It then follows that we can write

$$\int_x^y f(x,t)\, f_2(t,y)\, dt = \int_x^y f_2(x,t)\, f(t,y)\, dt\ .$$

By successive repetitions of this argument we establish the general result,

$$f_n(x,y) = \int_x^y f(x,t)\, f_{n-1}(t,y)\, dt = \int_x^y f_{n-1}(x,t)\, f(t,y)\, dt\ .$$

If we cast theorem 2 into the symbolism of permutable functions we see that it is equivalent to the index law

$$f^m * f^n = \overset{*}{f}{}^{m+n}\ ,$$

where m and n are integers and denote the number of iterations used in generating the functions.

Another step in building the algebra is to give meaning to the symbol f°. By this we mean an element which is defined by the equations

$$f^\circ * f = f * f^\circ = f(x, y) \quad , \quad f^\circ = g^\circ = 1^\circ \ .$$

It is obvious that f° plays the rôle of unity in the algebra; its formal determination, however, is not possible. Thus the symbol cannot be regarded as a function of x and y since this would imply that it satisfied the two integral equations

$$\begin{aligned} f(x, y) &= \int_x^y f^\circ(x, t)\, f(t, y)\, dt \ , \\ f(x, y) &= \int_x^y f(x, t)\, f^\circ(t, y)\, dt \ . \end{aligned} \tag{4.1}$$

Regarding (4.1) as an integral equation in y with $f(t, y)$ as the known kernal, we know from the theory of the inversion of a Volterra integral that a solution of (4.1) is possible in general only when $f(x, x) = 0$, $f(y, y) \neq 0$,* But these conditions are incompatible with each other.

In order to avoid this difficulty we shall regard $f^0 = 1^0$ merely as an element which combines with any other function to produce that function.

Another symbol, f^{-1}, of similar nature, may be defined as an element which in composition with $f(x, y)$ plays the rôle of f^0 . That is to say

$$f * f^{-1} = f^\circ \ .$$

Following the model established by E. V. Huntington,† G. C. Evans has formulated these results into a set of postulates which define the algebra of permutable functions.‡ Thus the postulates of addition may be stated as follows:

A 1. $f + g$ exists in the system.

A 2. $(f+g)+h = f+(g+h)$.

A 3_1. If $f + g = f + h$, then we have $g = h$.

A 3_2. If $g + f = h + f$, then we have $g = h$.

A 4. If $af = ag$, a being a positive integer, we conclude that $f = g$.

*See: *A Survey of Methods for the Inversion of Integrals of Volterra Type*, Indiana University Study, Nos. 76, 77, p. 9.

†The Fundamental Law of Addition and Multiplication in Elementary Algebra. *Annals of Mathematics*, vol. 8 (1906), pp. 1-44.

‡Sopra l'algebra delle funzioni permutabili. *Memorie della reale accademia dei Lincei*, vol. 8 (1911), pp. 695-710.

Similarly the postulates of multiplication (composition) become:

$M\,1.$ $f * g$ exists in the system.

$M\,2.$ $(f * g) * h = f * (g * h)$.

$M\,3_1.$ If $f * g = f * h$, where f is a non-zero function, then $g = h$.

$M\,3_2.$ If $g * f = h * f$, where f is a non-zero function, then $g = h$.

$M\,4_1.$ $f * (g + h) = f * g + f * h$.

$M\,4_2.$ $(g + h) * f = g * f + h * f$.

$M\,5.$ $f * g = g * f$.

Since these postulates correspond to the postulates of ordinary algebra, it follows that the identities of the latter, in so far as they depend upon these postulates, are carried over to permutable functions. We thus have the index law, the binomial theorem for integral coefficients, the factorization law for polynomials, etc., holding in the algebra of permutable functions.

5. *Selected Problems In the Algebra of Permutable Functions.*

A few problems will illustrate the power inherent in the notation of the algebra of permutable functions.

Example 1. Given that $K(x, y)$ is a known function integrable and bounded in an interval (ab), let us solve the equation

$$k(x, y) = K(x, y) + \lambda k * K \ .$$

Writing this in the symbolic form

$$k * 1^\circ = K * 1^\circ + \lambda k * K \ ,$$

we may proceed by algebraic methods. Thus we get

$$k * (1^\circ - \lambda K) = K * 1^\circ \ ,$$

$$k = K * (1^\circ - \lambda K)^{-1}$$

$$= K + \lambda \overset{*}{K}{}^2 + \lambda^2 \overset{*}{K}{}^3 + \cdots \ .$$

The function $k(x, y)$ is referred to as the *resolvent kernel* of $K(x, y)$.

Example 2. Given a function $g(x, y)$ integrable and bounded in a region (ab), let us find the solution of the equation

$$g(x, y) = f(x, y) + \overset{*}{f}{}^2/2! + \overset{*}{f}{}^3/3! + \ldots \ .$$

This equation may be written in the form

$$g * 1^\circ = f * 1 + \overset{*}{f}{}^2/2! + \overset{*}{f}{}^3/3! + \cdots ,$$

or, in algebraic symbols, as

$$g * 1^\circ = e^{\overset{*}{f}} - 1 .$$

The solution from algebraic analogy will then be

$$f * 1^0 = \log(1^0 + \overset{*}{g}) ,$$

or
$$f(x, y) = g(x, y) - \overset{*}{g}{}^2/2 + \overset{*}{g}{}^3/3 - \cdots .$$

The convergence of the right hand member of this equation, under the assumption that $g(x, y)$ is integrable and bounded in the interval (ab), follows from the general inequality

$$|\overset{*}{g}{}^n| \leq G^n (b-a)^n/n! ,$$

where $|g(x, y)| \leqq G$ in (ab) .

Example 3. If $k_1(x, y)$ is the resolvent kernel of $K(x, y)$ and $k_2(x, y)$ is the resolvent of $-K(x, y)$, prove that $-\overset{*}{K}{}^2$ is the resolvent of $k_1 * k_2$.*

Because of the relationship between the kernels, we have

$$\begin{aligned} k_1 &= K + K * k_1 , \\ k_2 &= - K - K * k_2 . \end{aligned} \tag{5.1}$$

Noting the permutability of the functions involved and making use of identities (5.1), we have

$$\begin{aligned} (K * k_1) * (K * k_2) &= K * [k_1 * (K * k_2)] \\ &= K * k_1 * (-K - k_2) = -K * (k_1 * K) - K * (k_1 * k_2) \\ &= -K * (K * k_1) - k_1 * (K * k_2) \\ &= K * (K - k_1) + k * (K + k_2) = \overset{*}{K}{}^2 + k_1 * k_2 . \end{aligned}$$

Since this result may be written in the form

$$-\overset{*}{K}{}^2 = k_1 * k_2 - \overset{*}{K}{}^2 * (k_1 * k_2) ,$$

the truth of the theorem is demonstrated.

*This theorem, which is due to Evans, is stated in slightly different form by him. His equation connecting the resolvent with its kernel is $k + K = K{*}k$, whereas the definition adopted in this study is $k = K + K{*}k$. Evans' statement concludes that $k{*}k$ is the resolvent of $\overset{*}{K}{}^2$. See: *Atti dei Lincei,* vol. 20 (2) (1911), pp. 453-460.

PROBLEMS.

1. If k_1 is the resolvent kernel of K, and k_2 is the resolvent of $-K$, prove that $(k_1 + k_2)/2$ is the resolvent of $\overset{*}{K^2}$. [Evans: *Atti dei Lincei*, vol. 20 (2) (1911), pp. 688-694.]

2. If $k_1 + K_1 = K_1 * k_1$ and $k_2 + K_2 = K_2 * k_2$, and if K_1 and K_2 are permutable with each other, prove that $K + k = K*k$, where $K = K_1 + K_2 - K_1*K_2$ and $k = k_1 + k_2 - k_1*k_2$. [Evans: *Atti dei Lincei*, vol. 20 (2) (1911), pp. 688-694.]

3. Prove that the function $V(\lambda; x, y) = \lambda f + \lambda^2 \overset{*}{f^2}/2! + \cdots + \lambda^n \overset{*}{f^n}/n! + \cdots$ is a solution of the equation

$$V(\lambda + \mu; x, y) = V(\lambda; x, y) + V(\mu; x, y) + \int_x^y V(\lambda; x, t)\, V(\mu; t, y)\, dt \; .$$

6. The Calculation of a Function Permutable with a Given Function. One of the principal problems in the algebra of permutable functions is that presented by the calculation of a function $X(x, y)$ which is permutable with a given function $F(x, y)$.

In section 10, chapter 2, we have solved this problem for the case where $F(x, y) = 1$. We now turn to the general case, which may be stated as follows:

We seek a function $X(x, y)$ such that

$$X * F = F * X \; ,$$

where $F(x, y)$ is assumed to satisfy the canonical conditions

$$F(x, x) = 1 \; , \quad (\partial F/\partial x)_{y=x} = (\partial F/\partial x)_{y=x} = 0 \; . \quad (6.1)$$

If $F(x, y)$ is not in this form, the following transformation is sufficient to derive from $F(x, y)$ a function of the canonical type:

$$F_1(x_1, y_1) = a(x_1)\, b(y_1)\, F[m(x_1), m(y_1)] \; , \qquad (6.2)$$

where we use the abbreviations

$$a(x) = e^{-\int^x [(\partial F/\partial x)_{y=x}/F(x,x)]dx} \; ,$$

$$b(x) = e^{\int^x [(\partial F/\partial x)_{y=x}/F(x,x)]dx}/F(x, x) \; ,$$

$$m(x_1) = x, \quad m(y_1) = y \; , \quad m'(x_1) = 1/F(x, x) \; .$$

For example, suppose that we have $F(x, y) = x + y$. We then compute $a(x) = x^{-1/2}$, $b(x) = x^{-1/2}/2$. It follows from the fact that

$$dm(x_1)/dx_1 = dx/dx_1 = 1/F(x, x) \; ,$$

that we get

$$x_1 = \int^x F(x, x)\, dx \; ,$$

or in the present instance, $x_1 = x^2$. We thus obtain

$$m(x_1) = x^{1/2}\ , \quad a(x_1) = (x_1)^{-1/4}\ , \quad b(y_1) = (y_1)^{-1/4}/2\ ,$$

and the new function becomes

$$F(x, y) = (x^{1/2} + y^{1/2})/2(xy)^{1/4}\ ,$$

which is seen to be in canonical form.

Since the composition of $F_1(x_1, y_1)$ with

$$X_1(x_1, y_1) = a(x_1)b(y_1)X[m(x_1), m(y_1)]$$

is equivalent to

$$\begin{aligned} F_1 * X_1(x_1, y_1) & \\ &= a(x_1)b(y_1)\int_{x_1}^{y_1} F[m(x_1), m(t)]\ X[m(t), m(y_1)]\ dt \\ &= a(x_1)\ b(y_1)\ F * X[m(x_1), m(y_1)]\ , \end{aligned}$$

we see that the transformation defined above conserves composition. Hence if $X_1(x_1, y_1)$ is determined we can compute $X(x, y)$ by the transformation inverse to (6.2).

Assuming that the function $F(x, y)$ is canonical and that in addition the derivative $\partial^2 F/\partial x\, \partial y$ exists and is continuous, we now seek to determine the function $X(x, y)$ permutable with $F(x, y)$.

We introduce the new function

$$\varphi(x, y) = F * X = X * F\ .$$

Taking derivatives with respect to x and y we obtain the equations

$$\begin{aligned} \partial\varphi/\partial x &= -X(x, y) + \int_x^y F'_x(x, t)\ X(t, y)\ dt\ , \\ \partial\varphi/\partial y &= X(x, y) + \int_x^y X(x, t)\ F'_y(t, y)\, dt\ . \end{aligned} \tag{6.3}$$

If we employ the abbreviations

$$f_1 = F'_x + \overset{*}{F}{}'^2_x + \overset{*}{F}{}'^3_x + \overset{*}{F}{}'^4_x + \cdots\ ,$$

$$f_2 = F'_y - \overset{*}{F}{}'^2_y + \overset{*}{F}{}'^3_y - \overset{*}{F}{}'^4_y + \cdots\ ,$$

it is clear that the solution of the integral equations (6.3) may be written

$$X(x, y) = -\partial\varphi/\partial x - \int_x^y f_1(x,)\{\partial\varphi(t, y)/\partial t\}\ dt\ , \tag{6.4}$$

$$X(x, y) = \partial\varphi/\partial y - \int_x^y \{\partial\varphi(x, t)/\partial t\} f_2(t, y)\, dt \ .$$

Integrating the first of these equations by parts we obtain

$$X(x, y) = -\partial\varphi/\partial x - \varphi(t, y)\, f_1(x, t)\Big|_x^y - \int_x^y f_{12}(x, t)\varphi(t, y)\, dt \ ,$$

where we use the abbreviation $f_{12} = -\partial f_1/\partial y$.

Since $\varphi(y, y) = 0$ and $f_1(x, x) = 0$ by (6.1), this reduces to

$$X(x, y) = -\partial\varphi/\partial x - \int_x^y f_{12}(x, t)\, \varphi(t, y)\, dt \ . \qquad (6.5)$$

Similarly, the second equation becomes

$$X(x, y) = \partial\varphi/\partial y - \int_x^y \varphi(x, t)\, f_{21}(t, y)\, dt \ , \qquad (6.6)$$

where we abbreviate $f_{21} = -\partial f_2/\partial x$.

But we can demonstrate that $f_{12} = f_{21}$, a fact first proved by J. Pérès in his notable exploration of this field.* If we make the abbreviation $H = \partial^2 F/\partial x\, \partial y$, we see from (6.1) that

$$F_x'(x, y) = \int_x^y H(x, t)\, dt = H * 1$$

and

$$F_y'(x, y) = -\int_x^y H(t, y)\, dt = -1 * H \ .$$

Consequently we have

$$f_1 = [H + H * 1 * H + H * 1 * H * 1 * H + \cdots] * 1 \ ,$$

$$f_2 = -1 * [H + H * 1 * H + H * 1 * H * 1 * H + \cdots] \ ,$$

from which it follows that

$$f_{12} = f_{21} = - [H + H * 1 * H + H * 1 * H * 1 * H + \cdots] \ .$$

If we substitute the value $f_{12} = f_{21} = f(x, y)$ in equations (6.5) and (6.6) and subtract the second equation from the first, we obtain as the integro-differential equation for the determination of $\varphi(x, y)$:

$$\partial\varphi/\partial x + \partial\varphi/\partial y = \int_x^y \{\varphi(x, s)\, f(s, y) - \varphi(s, y)\, f(x, s)\}\, ds \ . \qquad (6.7)$$

If we recall that the solution of the partial differential equation

$$\partial\varphi/\partial x + \partial\varphi/\partial y = p(x, y)$$

*Sur les fonctions permutables de première espèce de M. Vito Volterra. Paris (1915).

may be written in the form

$$\varphi(x, y) = \varphi_0(y-x) + \int_u^v p(t-u, t+u)\, dt ,$$

where $2u = y-x$, $2v = y+x$, and $\varphi_0(y-x)$ is an arbitrary function, then equation (6.7) can be written as the following integral equation:

$$\varphi(x, y) = \varphi_0(y-x) + \int_u^v dt \, \{\int_{t-u}^{t+u} \varphi(t-u, s)\, f(s, t+u)\, ds$$
$$- \int_{t-u}^{t+u} \varphi(s, t+u)\, f(t-u, s)\, ds\} . \quad (6.8)$$

Making the transformation $s = r + t - u$ upon the first integral and $s = -r + t + u$ upon the second, we may then write

$$\varphi(x,y) = \varphi_0(y-x) + \int_u^v dt \int_0^{2u} \{\varphi(t-u, r+t-u)\, f(r+t-u, t+u)$$
$$-\varphi(-r+t+u, t+u)\, f(t-u, -r+t+u)\}\, dr . \quad (6.9)$$

Setting $\varphi_0(y-x) = \int_0^{2u} q(s)\, ds$, where $q(s)$ is an arbitrary function, we solve (6.9) by successive approximations:

$$\varphi_0(x, y) = \varphi_0(y-x) = \int_0^{2u} q(s)\, ds ,$$

$$\varphi_1(x, y) = \varphi_0(y-x) + \int_u^v dt \int_0^{2u} \varphi_0(r)\, [f(r+t-u, t+u)$$
$$- f(t-u, -r+t+u)]\, dr$$

$$= \int_0^{2u} q(s)\, ds + \int_u^v dt \int_0^{2u} [f(r+t-u, t+u)$$
$$- f(t-u, -r+t+u)]\, dr \int_0^r q(s)\, ds .$$

Interchanging the order of integration in the last two integrals, we have

$$\varphi_1(x, y) = \int_0^{2u} q(s)\, ds + \int_u^v dt \int_0^{2u} q(s)\, ds \int_s^{2u} \{f(r+t-u, t+u)$$
$$- f(t-u, -r+t+u)\}\, dr$$

$$= \int_0^{2u} q(s)\, ds + \int_0^{2u} q(s)\, ds \int_u^v dt \int_s^{2u} \{f(r+t-u, t+u)$$
$$- f(t-u, -r+t+u)\}\, dr .$$

Continuing this process we are able to put the solution of equation (6.8) in the form

$$\varphi(x, y) = \int_0^{2u} q(s)\{A_0(s; x, y) + A_1(s; x, y) + A_2(s; x, y) + \cdots\}\, ds\ , \qquad (6.10)$$

where

$$A_0 = 1\ , \quad A_1(s; x, y) = \int_u^v dt \int_s^{2u} \{f(r+t-u, t+u) - f(t-u, -r+t+u)\}dr\ ,$$

and in general,

$$A_n(s; x, y) = \int_u^v dt \int_s^{2u} \{A_{n-1}(s; t-u, r+t-u) f(r+t-u, t+u) - A_{n-1}(s; t+u-r, t+u)\, f(t-u, t+u-r)\}\, dr\ .$$

When this value of $\varphi(x, y)$ is introduced into equation (6.6) we obtain all the functions $X(x, y)$ permutable with $F(x, y)$; that is to say

$$X(x, y) = q(y-x) + \int_0^{y-x} q(s)\, A'_y(s; x, y)\, ds - \int_x^y f(t, y)\, dt \int_0^{t-x} q(s)\, A(s; x, t)\, ds\ ,$$

where $A(s; x, y) = \sum_{i=0}^{\infty} A_i(s; x, y)$.

The second integral, by an interchange of the order of integration, may be written in the form

$$\int_0^{y-x} q(s)\, ds \int_{s+x}^{y} A(s; x, t)\, f(t, y)\, dt\ .$$

Hence the desired function $X(x, y)$ becomes

$$X(x, y) = q(y-x) + \int_0^{y-x} q(x)\, [A'_y(s; x, y) - \int_{s+x}^{y} A(s; x, t)\, f(t, y)\, dt]\, ds\ .$$

As an example let us indicate the method of calculating the function permutable with $F(x, y) = 1 - (x^2 - y^2)^2/8$.

Since this is already in canonical form we calculate $H = \partial^2 F/\partial x\, \partial y = xy$. From this we find without excessive difficulty the sequence of functions:

$$H * 1 * H = xy(x^2 - y^2)^2/4 \cdot 2!\ ,$$

$$H * 1 * H * 1 * H = xy(x^2 - y^2)^4/4^2 \cdot 4! \ ,$$

$$H * 1 * H * 1 * H * 1 * H = xy(x^2 - y^2)^6/4^3 \cdot 6! \ ,$$

. . . .

Hence $f(x,y) = -[xy + xy(x^2 - y^2)^2/4 \cdot 2!$

$$+ xy(x^2 - y^2)^4/4^2 \cdot 4! + \cdots] = -xy \cosh \{(x^2 - y^2)/2\} \ .$$

The solution of equation (6.8) then proceeds by the method indicated above. The first two approximations are

$$A_0 = 1 \text{ and } A_1(s;x,y) = 2xS_yS_{s-x}/(s-2u) + 2yS_xS_{s+y}/(s-2u)$$
$$- 4S_xS_yS_s/(s-2u)^2 \ ,$$

where we employ the abbreviation $S_f = \sinh\{(s-2u)\ f/2\}$.

7. The Transformation of Pérès. In the last section we derived an equation of the form,

$$X(x,y) = q(y-x) + \int_0^{y-x} q(t)\psi(t;x,y)\,dt \ ,$$

which determined all functions permutable with a given function $F(x,y)$. We shall abbreviate the right hand member of this equation by the symbol $\Omega(q)$ and shall seek to discover values of $\psi(t;x,y)$ such that the equation

$$\Omega(p) * \Omega(q) = \Omega(p * q) \qquad (7.1)$$

is satisfied.

Any transformation $X(x,y) = \Omega(q)$ which satisfies (7.1) is said to *conserve composition* and is called a *transformation of Pérès* since it was J.Pérès who first indicated its important rôle in the theory of permutable functions. We shall prove the following theorem:

Theorem 3. If $n(x,y)$ is an arbitrary function finite and integrable in a region (ab), and if $m(x,y)$ is defined in terms of it by the series

$$m(x,y) = -n(x,y) + \overset{*}{n}{}^2(x,y) - \overset{*}{n}{}^3(x,y) + \cdots \ ,$$

then the transformation $\Omega(q)$ defined by the equation

$$\Omega(q) = (1^0 + m) * q * (1^0 + n)$$

is a transformation of Pérès.

From the definition of $m(x,y)$ and $n(x,y)$ it is clear that they satisfy the equation

$$(1^0 + m) * (1^0 + n) = (1^0 + n) * (1^0 + m) = 1^0 \ .$$

Hence we have

$$\Omega(p) * \Omega(q) = (1^0 + m) * p * (1^0 + n) * (1^0 + m) * q * (1^0 + n)$$
$$= (1^0 + m) * p * q * (1^0 + n) = \Omega(p * q) \ .$$

Theorem 4. If $m(x,y)$ and $n(x,y)$ are defined as in theorem 3, then $\varphi(t;x,y)$ is given by the equation

$$\varphi(t;x,y) = n(x+t,y) + m(x,y-t) + \int_x^{y-t} m(x,s)\, n(s+t,y)\, ds \ . \tag{7.2}$$

In order to prove this theorem we first explicitly evaluate the terms in equation (7.1). After some rather long but straightforward transformations, equation (7.1) reduces to the form

$$\int_0^{y-r} p(r)\,dr \int_0^{y-x-r} q(s)\{\varphi(r+s;x,y) - \varphi(s;x+r,y) - \varphi(r;x,y-s)$$
$$- \int_{x+r}^{y-s} \varphi(r;x,t)\, \varphi(s;t,y)\,dt\}ds = 0 \ .$$

From this we derive the relation

$$\varphi(r+s;x,y) = \varphi(s;x+r,y) + \varphi(r;x,y-s)$$
$$+ \int_{x+r}^{y-s} \varphi(r;x,t)\, \varphi(s;t,y)\,dt \ , \tag{7.3}$$

which defines the desired function $\varphi(t;x,y)$.

Replacing x by 0 and r by x and employing the abbreviation $\varphi(x;0,y) = n(x,y)$, we reduce this equation to

$$n(x+s,y) = \varphi(s;x,y) + n(x,y-s) + \int_x^{y-s} n(x,t)\, \varphi(s;t,y)\,dt \ .$$

Since this is a Volterra equation of second kind in which the unknown function is $\varphi(s;x,y)$, its solution is obtained by the ordinary methods and is found to be

$$\varphi(s;x,y) = n(x+s,y) - n(x,y-s)$$
$$+ \int_x^{y-s} \{n(t+s,y) - n(t,y-s)\}\, m(x,t)\,dt \ .$$

By means of the identity

$$m(x,y) + n(x,y) + \int_x^y m(x,t)\, n(t,y)\, dt = 0 \ ,$$

this equation is immediately seen to be equivalent to (7.2).

8. The Permutability of Functions Permutable With a Given Function. By means of the theory of the transformation of Pérès we are able to prove the following useful theorem:

Theorem 5. All functions permutable with a given function $F(x,y)$ are permutable with each other.

The proof of this theorem for functions defined by a transformation of Pérès is immediate. Thus let us suppose that we have

$$X(x,y) = \Omega(p) \ , \quad Y(x,y) = \Omega(q) \ .$$

From the fact that $p(y-x)$ is permutable with $q(y-x)$, since they belong to the group of the closed cycle, it follows that

$$X * Y = \Omega(p) * \Omega(q) = \Omega(p * q) = \Omega(q * p)$$

$$= \Omega(q) * \Omega(p) = Y * X \ .$$

Hence, in order to prove our theorem, it is necessary to show that all functions permutable with a given function can be defined by a transformation of Pérès. As a matter of fact we shall prove that there exists a function $\varphi(t;x,y)$ which forms the nucleus of a transformation of Pérès such that the function can be expressed in the form

$$F(x,y) = \Omega(1) = 1 + \int_0^{y-x} \varphi(t;x,y)\,dt \ .$$

It is then clear that all functions permutable with $F(x,y)$ are given by the formula

$$X(x,y) = \Omega(p) \ .$$

We first consider the equation

$$F(x,y) = 1 + 1 * f * 1 + 1 * f * 1 * f * 1 + \cdots ,$$

whose solution is easily found in the form

$$f(x,y) = -[H + H * 1 * H + H * 1 * H * 1 * H + \cdots] \ , \tag{8.1}$$

where we use the abbreviation $H = \partial^2 F / \partial x\, \partial y$.

In order that $F(x,y)$ should be in the desired form it is necessary that we should have

$$\int_0^{y-x} \varphi(t;x,y)\,dt = 1 * f * 1 + 1 * f * 1 * f * 1 + \cdots$$

$$\int_0^{y-x} \psi(t;x,y)\,dt = \int_0^{y-x} dr \int_0^r f(x+s,y+s-r)\,ds$$
$$+ \int_0^{y-x} dr \int_0^r ds \int_x^{y-r} B(s;x,t)\, f(s+t,y+s-r)\,dt \ ,$$

where

$$B(s;x,t) = \sum_{n=1}^{\infty} \int_0^s ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1\, f_{s_1} * f_{s_2} * \cdots * f_{s_n}(x,y) \ ,$$

in which we use the abbreviation $f_s(x,y) = f(x+s,y+s)$.

But it is not difficulty to prove that $B(s;x,y)$ thus defined is a solution of the equation

$$B(s;x,y) = \int_0^s f(x+r,y+r)\,dr$$
$$+ \int_0^s dr \int_x^y B(r;x,t)\, f(r+t,y+r)\,dt \ . \qquad (8.2)$$

We have thus established the relationship

$$\psi(t;x,y) = B(t;x,y-t) \ . \qquad (8.3)$$

It remains for us to prove that the function $\psi(t;x,y)$ thus defined is the nucleus of a transformation of Pérès.

Returning to equation (7.3) and making in it the change of variables $x = \xi - r$, $y = \eta + s$, we have

$$\psi(r+s;\, \xi-r,\, \eta+s) = \psi(s;\, \xi,\, \eta+s) + \psi(r;\, \xi-r,\, \eta)$$
$$+ \int_\xi^\eta \psi(r;\xi-r,t)\, \psi(s,t,\eta+s)\,dt \ . \qquad (8.4)$$

If we let $r = s = 0$, this reduces to,

$$\psi(0;\xi,\eta) + \int_\xi^\eta \psi(0;\xi,t)\, \psi(0;t,\eta)\,dt = 0 \ ,$$

which proves that $\psi(0;x,y) = 0$.

If we abbreviate $\psi(r;\xi,\eta) = B(r;\xi,\eta-r)$ we can write (8.4) in the form

$$B(r+s;\, \xi-r,\, \eta-r) = B(s;\, \xi,\, \eta) + B(r;\, \xi-r,\, \eta-r)$$
$$+ \int_\xi^\eta B(r;\, \xi-r,\, t-r)\, B(s;\, t,\, \eta)\,dt \ .$$

Forming the derivative of this equation with respect to s and then letting $s = 0$, we get

$$B_s'(r;\, \xi-r,\, \eta-r) = B_s'(0;\, \xi,\, \eta)$$
$$+ \int_\xi^\eta B(r;\, \xi-r,\, t-r)\, B_s'(0;\, t,\, \eta)\,dt \ , \qquad (8.5)$$

where we mean by B_s' the derivative with respect to the first variable. Let us now write $B_s'(0;\xi,\eta) = f(\xi,\eta)$ where $f(\xi,\eta)$ is an arbitrary function, and make the transformation $x = \xi - r$, $y = \eta - r$, $t = s + r$. Equation (8.5) then reduces to

$$B_s'(r;x,y) = f(x+r,y+r) + \int_x^y B(r;x,s)\, f(s+r,y+r)\, ds \ .$$

Integrating both sides of this equation with respect to the first variable we obtain finally

$$B(r;x,y) = \int_0^r f(x+r,y+r)\, dr$$
$$+ \int_0^r dr \int_x^y B(r;x,s)\, f(s+r,y+r)\, ds \ .$$

If $f(x,y)$ is identified with (8.1), this equation with slight change in notation is seen to be equivalent to (8.2), which establishes the desired identification of $\varphi(t;x,y)$ in (8.3) with the nucleus of a transformation of Pérès.

It is interesting to note that the permutability of functions permutable with a given function was first conjectured by Volterra and later proved by E. Vessiot.* Volterra and Pérès also established the theorem, the proof the the latter being reproduced above.†

The proof of Vessiot is elegant because of its simplicity. It depends essentially upon the fact that the function

$$X(x,y) = a_1 F(x,y) + a_2 \overset{*}{F}{}^2 + a_3 \overset{*}{F}{}^3 + \cdots \ , \qquad (8.6)$$

where the a_i are arbitrary constants subject only to the restriction that the series converges uniformly, is permutable with $F(x,y)$ and with any other function of the same form.

Since (8.6), because of the infinite number of arbitrary parameters upon which it depends, has the same degree of generality as the closed form

$$X(x,y) = q(y-x) + \int_0^{y-x} q(t)\, \varphi(t;x,y)\, dt \ , \quad (8.7)$$

we conclude that (8.7) is also permutable with any other function similarly defined.

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*Sur les fonctions permutables et les groups continus de transformations fonctionnelles linéaires. *Comptes Rendus*, vol. 154 (1918), pp. 682-684.

†Volterra: Teoria della potenze, dei logaritmi e delle funzioni di composizione. *Memorie dell' accademia dei Lincei*, vol. 11 (1916), pp. 167-269. Pérès: Sur certaines transformations fonctionnelles et leur application a la théorie des fonctions permutables. *Annales de l'école normale superieure*, vol. 36 (3) (1919), pp. 37-50.

9. *Permutable Functions of Second Kind.* The problem presented by permutability of second kind is essentially different from that of permutability of first kind and does not possess the intimate connection with the theory of operators, at least as developed in this book, which is exhibited by the latter. Permutability of second kind has aspects of a theory of linear substitutions and can be most effectively discussed from its algebraic basis. For this reason we shall give only a casual résumé of one or two results which are most directly useful in the theory of operators.

Let us first note that the operation of second kind is both *associative* and *distributive.* When we consider its *commutativity,* however, we must, as in the case of composition of first kind, impose conditions.

We shall first prove the theorem:

Theorem 6. If $F(x,y)$ is any function integrable in the square $a \leq x \leq b$, $a \leq y \leq b$, then the function

$$\Phi(x,y) = F(x,y) - \{\int_a^b F(x,y)\,dx - \int_a^b F(x,y)\,dy\}/(b-a)$$

is permutable with unity.

Proof: The proof is immediate from the equation

$$\int_a^b \Phi(x,y)\,dx = -\int_a^b \int_a^b F(x,y)\,dy\,dx/(b-a) = \int_a^b \Phi(x,y)\,dy \ .$$

The next theorem shows the connection of composition of second kind with the theory of linear substitutions.

Theorem 7. If we define

$$A(x,y) = \sum_{ij=1}^{n} a_{ij}\, f_i(x)\, g_j(y)$$

and

$$B(x,y) = \sum_{ij=1}^{n} b_{ij}\, f_i(x)\, g_j(y) \ ,$$

where $f_i(x)$ and $g_j(y)$ are functions integrable over the interval (ab), and if

$$\int_a^b \int_a^b f_i(x)\, g_j(y)\,dy\,dx = c_{ij} \ ,$$

then in order that $A(x,y)$ be permutable with $B(x,y)$, it is necessary and sufficient that we have

$$A\,C\,B = B\,A\,C \ ,$$

where we abbreviate

$$A = ||a_{ij}|| \ , \ B = ||b_{ij}|| \ , \ C = ||c_{ij}|| \ .$$

Proof: Forming the composition of $A(x,t)$ with $B(t,y)$, we have

$$\int_a^b A(x,t)\, B(t,y)\, dt = \sum_{ijkl}^{n} a_{ij}\, b_{kl} \int_a^b f_i(x)\, g_j(t)\, f_k(t)\, g_l(y)\, dt$$

$$= \sum_{ijkl}^{n} a_{ij}\, c_{jk}\, b_{kl}\, f_i(x)\, g_l(y) \ .$$

Similarly the composition of $B(x,t)$ with $A(t,y)$ yields

$$\int_a^b B(x,t)\, A(t,y)\, dt = \sum_{ijkl}^{n} a_{ij}\, c_{il}\, b_{kl}\, f_k(x)\, g_j(y) \ .$$

If we form the matrix

$$M = ||\mu_{il}|| \ ,$$

where

$$\mu_{il} = \sum_{jk}^{n} a_{ij}\, c_{jk}\, b_{kl} \ ,$$

we see that M is identically equal to the matrix product, that is,

$$M = A\, C\, B \ .$$

Similarly, the matrix $N = ||\nu_{kj}||$, where

$$\nu_{kj} = \sum_{il} b_{kl}\, c_{il}\, a_{ij} \ ,$$

is equal to

$$N = B\, C\, A \ .$$

Since it is both necessary and sufficient for the permutability of $A(x,y)$ with $B(x,y)$ that $M = N$, we derive the theorem.

10. The Inversion of Operators (*Bourlet's Theory*). We next consider Bourlet's method of finding the inverse of an operator

$$S(u) = f(x) \ , \qquad (10.1)$$

where $f(x)$ is an arbitrary function and S an operator of generatrix $F(x,z)$. This is formally equivalent to the problem of solving an arbitrary linear functional equation and it is with the formal aspects that we shall mainly be concerned. That the inversion is not always possible is easily seen from the following example:

$$S(u) \equiv u - xu'/1! + x^2u''/2! - x^3u'''/3! + \cdots = f(x) \ ,$$

where $f(x)$ is not a constant.

Since $S(u) \equiv u(0) =$ a constant, it is clear that no solutions will exist except for the special case $f(x) =$ a constant. However, the equation

$$S(u) = a$$

will obviously be satisfied with the function

$$u(x) = a + xg(x) \ ,$$

where $g(x)$ is any function analytic in the neighborhood of $x = 0$.

The problem of inverting the equation (10.1) is clearly that of finding an operator of generatrix $X(x,z)$ such that

$$X(x,z) \to F(x,z) = [X \cdot F](x,z) = 1 \ ,$$

because it is at once evident that

$$u(x) = X(x,z) \to f(x)$$

will be a formal solution of the original equation.

Definition: The function $X(x,z)$ is called the *resolvent generatrix* of the original equation.

As an elementary example, consider the integral equation

$$\int_0^x u(x)\,dx = f(x) \ , \quad f(0) = 0 \ .$$

Referring to section 6, chapter 2, we see that the generatrix function is

$$F(x,z) = (1 - e^{xz})/z \ .$$

Substituting this in (3.1) we immediately obtain

$$X \to F = X(1 - e^{-xz})/z + \frac{\partial X}{\partial z} e^{-xz} - \frac{1}{2!} z \frac{\partial^2 X}{\partial z^2} e^{-xz} + \cdots$$

$$+ (-1)^{n-1} \frac{1}{n!} z^{n-1} \frac{\partial^n X}{\partial z^n} e^{-xz} + \cdots$$

$$= [X(x,z) - e^{-xz} X(x,0)]/z \ ,$$

which, for the determination of $X(x, z)$, must be set equal to 1.

Choosing $X(x,0)$ as some arbitrary function of x, i.e., $g(x)$, we solve for $X(x,z)$ and thus obtain

$$X(x,z) = z + g(x) e^{-xz} \ .$$

Hence the desired solution of the integral equation will be

$$u(x) = \frac{d}{dx} f(x) + g(x) e^{-xz} \to f(x) \ .$$

Since by hypothesis $f(0) = 0$, the last term vanishes.

We now consider the essential feature of the Bourlet theory, which is concerned with the number of zeros of the generatrix function regarded as a function of z. This we shall state in the following theorem:

Theorem 8. If the generatrix function associated with the linear, uniform, and complete operator S has exactly m zeros in z in the finite plane, where m is zero or a positive integer, then the equation

$$S(u) = 0 \qquad (10.2)$$

will have m solutions provided the generatrix function is not of the form $F(x,z) = e^{-(x-c)z}$.

Proof: *Case 1.* Suppose that $F(x,z)$ has no z-zero in the finite plane. Then, since $F(x,z)$, by theorem 3, chapter 2, is at most of genus one, it must be of the form

$$F(x,z) = a(x)\, e^{g(x)z} .$$

We can then write equation (10.2) as

$$F(x,z) \to u \equiv a(x)\, e^{g(x)z} \to u = a(x)\, u(g+x) = 0 . \qquad (10.3)$$

It is now clear that if $g(x) + x \neq$ constant, equation (10.2) will not have a solution other than $u(x) \equiv 0$.

But if $g(x) + x \equiv c$, where c is a constant, then any function $u(x)$ which vanishes for $x = c$ will be a solution. But this solution corresponds to the equation

$$e^{-(x-c)z} \to u(x) = 0 ,$$

which we have already excluded.

Case 2. Suppose that $F(x,z)$ has m zeros in the finite plane. Since the generatrix is at most of genus one, it must be of the form

$$F(x,z) = e^{g(x)z}\, P(x,z) ,$$

where $P(x,z)$ is a polynomial in z of degree m.

But from (3.1) we know that

$$e^{g(x)z} \to X(x,z) = e^{g(x)z}\, X(x+g, z) .$$

Hence we can choose the function $X(x,z)$ so that

$$X(x+g, z) = P(x,z) ,$$

from which we derive

$$X(x,z) = P[h(x), z] ,$$

where $h(x)$ is the inverse function of $x + g(x)$; i.e., $h(x)$ satisfies the equation

$$h(x) + g[h(x)] = x \ .$$

It follows from this that $F(x,z)$ is the generatrix of the symbolic product of the operators with generatrices $e^{g(x)z}$ and $P[h(x,),z]$.

Thus we see that the solution of equation (10.2) is reduced to finding the solutions of the ordinary equation of infinite order m.

$$P[h(x), z] \to u(x) = 0 \ .$$

In this manner we establish the theorem.

It will be noticed that we have not treated the case where the generatrix function has an infinite number of zeros in the z-plane. This problem was studied by Bourlet who made the following observation:*

"Il resterait à le prouver pour le cas où le nombre des zéros est infini, en montrant que, dans ce cas (en supposant, bien entendu, la transmutation inverse complète), le nombre des constantes arbitraires est aussi infini. La chose parait vraisemblable et il serait très intéressant de la prouver, car elle montrerait que les *transformations* (*ou substitutions*) sont les seules transmutations additives, uniformes, complètes, telles que la transmutation inverse soit de même nature.

"Je n'ai, malheureusement, pas encore pu prouver cette proposition dans toute sa généralité. Elle est évident dans le cas des coefficients constants."

The theorem thus conjectured by Bourlet is unfortunately not true, as he himself pointed out in a subsequent paper by means of an explicit example furnished to him by S. Pincherle.† What some of these difficulties are will appear in subsequent chapters of this book.

As an example illustrating theorem 8 let us consider the integral equation

$$u(x) = \lambda \int_0^1 (x + t)\, u(t)\, dt \ .$$

By a simple calculation we compute the generatrix function

$$F(x,z;\lambda) = 1 - \lambda \int_0^1 (x + t)\, e^{(t-x)z}\, dt$$
$$= e^{-zx}(e^{xz} - \lambda x e^z/z - \lambda e^z/z + \lambda e^z/z^2 + \lambda x/z - \lambda/z^2) \ .$$

Expanding the function in parentheses as a power series in z, we obtain

*See *Bibliography*: Bourlet (1), p. 184.
†See *Bibliography*: Bourlet (2), p. 337.

$$F(x,z;\lambda) = [(1-\frac{1}{2}\lambda - \lambda x) + (x - \frac{1}{2}\lambda x - \frac{1}{3}\lambda)z + \cdots]e^{-xz} .$$

For the determination of the principal numbers, the first two coefficients are set equal to zero, that is,

$$(1-\frac{1}{2}\lambda) - \lambda x = 0 ,$$

$$-\frac{1}{3}\lambda + (1-\frac{1}{2}\lambda)x = 0 .$$

In order that these equations may be consistent, we equate the determinant

$$\Delta(\lambda) = \begin{vmatrix} (1-\frac{1}{2}\lambda) & -\lambda \\ -\frac{1}{3}\lambda & (1-\frac{1}{2}\lambda) \end{vmatrix}$$

to zero, and hence define the principal numbers as the roots of the equation

$$\Delta(\lambda) = 1 - \lambda - \lambda^2/12 = 0 .$$

The generatrix function may then be written

$$F(x,z) = e^{q(x)z} z^2 .$$

Since the second factor is independent of x, the solution of the integral equation is reduced to the solution of the differential equation

$$\frac{d^2u}{dx^2} = 0 ;$$

that is to say, $u(x) = ax + b$.

Another example showing the general efficacy of this theorem in application will be found in section 6 of chapter 9 where the Euler differential equation of infinite order is discussed.

11. The Method of Successive Substitutions. A number of important applications, notably in the theory of integral equations, have been made of the *method of successive substitutions*. In order to describe its formal aspects, let us discuss the inversion of the linear functional equation

$$S(u) + \varphi(x)u(x) = f(x) , \qquad (11.1)$$

where S is a general linear operator and $\varphi(x)$ and $f(x)$ are known functions. If we indicate by S^{-1} the reciprocal of S, that is to say,

$S^{-1} \to S = 1$, then the solution of (11.1) can be expressed formally in two ways:

$$u(x) = S^{-1} \to f(x) - S^{-1} \to [\varphi S^{-1} \to f] + S^{-1} \to [\varphi S^{-1} \to \{\varphi S^{-1} \to f\}] - \cdots , \tag{11.2}$$

$$u(x) = f(x)/\varphi(x) - (1/\varphi)[S \to (f/\varphi)] + (1/\varphi)[S \to (1/\varphi)\{S \to (f/\varphi)\}] - \cdots . \tag{11.3}$$

The proof that both of these expansions furnish formal solutions of equation (11.1) is immediately obtained by operating on both members with S. Thus we get in the first case

$$\begin{aligned} S(u) &= f - \varphi\{S^{-1} \to f - S^{-1} \to (\varphi S^{-1} \to f) + S^{-1} \to [\varphi S^{-1} \to (\varphi S^{-1} \to f)] - \cdots\} \\ &= f - \varphi u , \end{aligned}$$

and similarly in the second case,

$$\begin{aligned} S(u) &= S \to (f/\varphi) - S \to \{(1/\varphi)\, S \to (f/\varphi)\} + S \to \{(1/\varphi)\, S \to [(1/\varphi)\, S \to (f/\varphi)]\} - \cdots \\ &= f - \varphi u . \end{aligned}$$

It will be noticed that in this formal development we have omitted reference to the complementary function, which takes account of the solution of the homogeneous equation

$$S(u) + \varphi(x)\, u(x) = 0 . \tag{11.4}$$

If we let $c_i(x)$ be any solution of the equation

$$S \to c_i(x) = 0 , \tag{11.5}$$

then the corresponding complementary solution of (11.4) may be obtained formally from the series

$$u_i(x) = c_i(x) - S^{-1} \to (c_i\varphi) + S^{-1} \to [\varphi S^{-1} \to (c_i\varphi)] - S^{-1} \to \varphi\{S^{-1} \to \varphi[S^{-1} \to (c_i\varphi)]\} + \cdots ,$$

which we may designate by the functional symbol $F(c_i,\varphi)$.

Because of the linearity of the operators, it is seen that if we write

$$c(x) = \sum_{i=1}^{m} A_i c_i(x) ,$$

where the A_i are constants and the $\{c_i(x)\}$ form a complete set of solutions of (11.5), then

$$u_0(x) = F(c,\varphi)$$

is the complementary function of (11.1).

As an example, let us consider the inversion of the equation

$$(xz^2 + z + x) \to u(x) = x^2 .$$

If we write $S = xz^2 + z$, we readily find that

$$S^{-1} = \int_c^x [e^{(s-x)z}/sz]\, ds .$$

If we introduce $-\infty$ as the lower limit of this integral, we can write

$$S^{-1} = e^{-xz} z^{-1} \int_{-\infty}^x (e^{sz}/s)\, ds$$

$$= e^{-xz} z^{-1} (C + \log xz + xz + \frac{1}{2}\frac{x^2z^2}{2} + \frac{1}{3}\frac{x^3z^3}{3} + \cdots) ,$$

where C is Euler's constant.

Noting that $e^{-xz}\, z^{-1} \to f(x) = 0,$

$$e^{-xz}\, z^{-1} \log z \to f(x) = -\lim_{x=0} \int_0^x [\log\ (x-t) + C]\ f(t)\, dt = 0,$$

$$e^{-xz}\, z^n \to f(x) = f^{(n)}(0) ,$$

we then find

$$S^{-1} \to x^n = x^{n+1}/(n+1)^2 ,$$

$$S^{-1} \to (x^n \log x) = x^{n+1} \log x/(n+1)^2 - 2x^{n+1}/(n+1)^3 .$$

Employing (11.2) we obtain

$$u_0(x) = x^3/3^2 - x^4/(3\cdot 5)^2 + x^5/(3\cdot 5\cdot 7)^2 - \cdots .$$

Noting that the equation $(xz^2 + z) \to u(x) = 0$ has the solutions $c_1(x) = A$, $c_2(x) = B \log x$, we then get

$$u_1(x) = A[1 - x^2/2^2 + x^4/(2\cdot 4)^2 - x^6/(2\cdot 4\cdot 6)^2 + \cdots] ,$$

$$u_2(x) = B\{\log x[1 - x^2/2^2 + x^4/(2\cdot 4)^2 - x^6/(2\cdot 4\cdot 6)^2 + \cdots]$$

$$+ x^2/2^2 - \frac{3}{2} x^4/(2\cdot 4)^2 + (1 + \frac{1}{2} + \frac{1}{3})\ x^6/(2\cdot 4\cdot 6)^2 - \cdots\} .$$

The general solution of the differential equation is thus found to be

$$u(x) = u_0(x) + u_1(x) + u_2(x) .$$

This example is selected from a number used by W. O. Pennell [See *Bibliography*: Pennell (1)] to illustrate a method of symbolic division, which depends essentially upon the algorithms given in formulas (11.2) and (11.3). Since this method has intrinsic interest, we shall illustrate it by inverting the simple equation

$$(z+x) \rightarrow u(x) = 1 \ .$$

In Pennell's notation the solution is symbolically written $u(x) \doteq 1/(z+x)$, to which meaning is given by the formal operational device:

$$\begin{array}{l|l|ll}
\underline{z+x} & 1 & 1/z - x^2/z + x^4/(3z) & \text{(a)} \\
 & \underline{1+x^2} & x - \frac{1}{3}x^3 + x^5/(3\cdot 5) & \text{(b)} \\
 & -x^2 & & \\
 & \underline{-x^2 - \frac{1}{3}x^4} & & \\
 & \qquad x^4/3 + x^5/(3\cdot 5) & &
\end{array}$$

A solution of the form $u(x) = x - x^3/(1\cdot 3) + x^5/(1\cdot 3\cdot 5) - x^7/(1\cdot 3\cdot 5\cdot 7) + \cdots$ is thus attained. Since the division otherwise follows ordinary algebraic rules, it seems necessary only to call attention to the fact that line (b) is obtained by performing the indicated integration of line (a). It will also be noticed that the inversion is the one which would be attained by an application of formula (11.2).

If the positions of x and z are reversed in the division it is possible to obtain a second solution, as follows:

$$\begin{array}{l|l|l}
\underline{x+z} & 1 & \underline{1/x + 1/x^3 + (1\cdot 3)/x^5 + (1\cdot 3\cdot 5)/x^7} \\
 & \underline{1 - 1/x^2} & \\
 & \quad 1/x^2 & \\
 & \quad \underline{1/x^2 - 3/x^4} & \\
 & \qquad 3/x^4 & \\
 & \qquad \underline{3/x^4 - (1\cdot 3\cdot 5)/x^6} &
\end{array}$$

We notice that the solution thus obtained, $u(x) = 1/x + 1/x^3 + (1\cdot 3)/x^5 + (1\cdot 3\cdot 5)/x^7 + \cdots$, is completely divergent. It is, however, summable along the imaginary axis and we have the asymptotic expansion

$$iu(xi) \backsim e^{\frac{1}{2}x^2} \int_x^\infty e^{-\frac{1}{2}t^2}\, dt \ .$$

A deeper insight into these difficulties will be gained from the discussion in later chapters.

PROBLEMS

1. Writing the equation

$$(z^2 + a\,z + b) \to u(x) = e^{kx}$$

in the form

$$u(x) = (1/b)\, e^{kx} + S(z) \to u(x) \ ,$$

where $S(z) = -(1/b)(z^2 + a\,z)$, solve by the method of successive substitutions.

2. Solve the equation

$$u(x) = x - \int_0^1 (x-t)\, u(t)\,dt \ .$$

by the method of successive substitutions.

3. Obtain by the method of successive substitutions the solution of the equation

$$u(x) = a + bx - \lambda \int_0^x (x-t)\, u(t)\,dt \ .$$

Show that the solution is given by

$$u(x) = \frac{\sqrt{\lambda\, a^2 + b^2}}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\, x + c) \ , \quad \text{where } c = \text{arc sin} \frac{\sqrt{\lambda}\, a}{\sqrt{\lambda\, a^2 + b^2}} \ .$$

12. Some Further Properties of the Resolvent Generatrix. We shall describe below a few specific properties of the resolvent generatrix which are useful in application.

(a) *If we assume that* $F(x,z)$ is of the form

$$F(x,z) = 1 - \lambda g(x,z) \ , \qquad (12.1)$$

then the following function,

$$X(x,z) = 1 + \lambda g(x,z) + \lambda^2 g^{(2)}(x,z) + \lambda^3 g^{(3)}(x,z) + \cdots \ , \qquad (12.2)$$

where $g^{(m)}(x,z)$ *means the generatrix of the* nth *power of the opertor, is easily seen to be a formal solution of the equation*

$$X(x,z) \to F(x,z) = 1 \ . \qquad (12.3)$$

To prove this we substitute (12.1) and (12.2) in equation (12.3) and thus obtain

$$X \to F = 1 + \lambda(g-g) + \lambda^2[g^{(2)} - g^2 - (\partial g/\partial x)\,(\partial g/\partial z)$$
$$- (\partial^2 g/\partial x^2)\,(\partial^2 g/\partial z^2)/2! - \cdots] + \lambda^3[g^{(3)} - gg^{(2)}$$

$$- (\partial g/\partial x)(\partial g^{(2)}/\partial z) - (\partial^2 g/\partial x^2)(\partial^2 g^{(2)}/\partial z^2)/2! - \cdots]$$

$$+ \cdots .$$

Since we have by definition

$$g^{(m)}(x,z) = gg^{(m-1)} + (\partial g/\partial x)(\partial g^{(m-1)}/\partial z) + (\partial^2 g/\partial x^2)(\partial^2 g^{(m-1)}/\partial z^2)/2! + \cdots ,$$

the multiplier of λ^m is seen to vanish for every value of m, and (12.2) is thus the formal solution of equation (12.3).

(*b*) *Let* $X(x,z)$ *be expanded in a power series in* z,

$$X(x,z) = b_0(x) + b_1(x)z + b_2(x)z^2 + \cdots . \qquad (12.4)$$

Then $X(x,z) \to x^i$, *i a positive integer or zero, is a solution of the equation*

$$F(x,z) \to u(x) = x^i .$$

Designating this solution by $u_i(x)$ we see from equation (14.2) of chapter 2 that the coefficients of z^r in (12.4) and hence $X(x,z)$ itself are formally expansible in terms of a set of special solutions of $F(x,z) \to u(x) = f(x)$, namely, $u_i(x)$, $i = 0, 1, 2, 3, \cdots$. Since the coefficients of (12.4) are $b_r(x) = (\partial^r X/\partial z^r|_{z=0})/r!$, we obtain the identity

$$(\partial^r X/\partial z^r)|_{z=0} = u_r(x) - rxu_{r-1} + r(r-1)x^2 u_{r-2}/2! + \cdots \pm x^r u_0 . \qquad (12.5)$$

One of the first studies based upon the functions $u_i(x)$ was made by A. Hurwitz* who considered the equation $(e^z - 1) \to u(x) = g(x)$. In this case the functions $u_i(x)$ are given explicitly by $u_i(x) = B_{i+1}(x)/(i+1)$, where $B_n(x)$ is the nth Bernoulli polynomial obtained from the expansion,

$$te^{xt}/(e^t - 1) = \sum_{n=0}^{\infty} B_n(x)t^n/n! \ .\dagger$$

If $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, then the function $u(x) = a_0 u_0(x) + a_1 u_1(x) + a_2 u_2(x) + \cdots$, is the formal solution of the original equation.

I. Sheffer‡ has extended this idea for the operator,

$$F(x,z) = zA(z) + B(z) + xzB(z) \ , \quad B(0) \neq 0 \ ,$$

to which corresponds the resolvent,

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*Sur l'intégrale finie d'une fonction entière. *Acta Mathematica*, vol. 20 (1896-97), pp. 285-312.

†See Davis: *Tables of the Higher Mathematical Functions*, vol. 2.

‡See *Bibliography*: Sheffer: (1), p. 351.

$$X(x,z) = (e^{-xz}/z) \int_{-\infty}^{z} e^{xt} X(z) Y(t)\, dt \ ,$$

where we employ the abbreviation:

$$X(z)\ Y(t) = e^{\int_z^t [A(z)/B(z)]dz} / [zB(t)] \ .$$

Sheffer proved that the generating function,

$$f(s,x) = \sum_{i=0}^{\infty} u_i(x) s^i/i! = \sum_{i=0}^{\infty} \{X(x,z) \to (xs)^i/i!\} \ ,$$

is uniformily convergent provided $s \leq \sigma < \rho$, where ρ is the absolute value of the smallest zero of $B(z)$. Assuming that the symbols Σ and $X(x,z)$ may be interchanged and noting the fact that $X(x,z) \to e^{sx} = X(x,s)e^{sk}$, we easily derive Sheffer's result:

$$f(s,x) = \int_{-\infty}^{s} e^{xt} X(s) Y(t)\, dt/s \ .$$

(c) *It is occasionally useful to know how the equation*

$$F(x,z) \to u(x) = f(x) \tag{12.6}$$

may be reduced to a differential equation of infinite order with constant coefficients,

$$\varphi(z) \to u(x) = g(x) \ ,$$

where we write

$$\varphi(z) = \varphi_0 + \varphi_1 z + \varphi_2 z^2 + \cdots + \varphi_m z^m + \cdots \ .$$

To make this transformation we first define the following set of operators, of which $X_0(x,z)$, the resolvent generatrix, is a special member:

$$X_0 \to F = 1 \ , X_1 \to F = z \ , \quad X_2 \to F = z^2 \ , \cdots, \quad X_n \to F = z^n \ , \cdots \ ,$$
$$X_\varphi \to F = \varphi(z) \ .$$

By means of the Bourlet operational product these operators are seen to be related in the following manner:

$$\begin{aligned} X_n(x,z) &= z^n \to X_0 \ , \\ X_\varphi(x,z) &= \varphi_0 X_0 + \varphi_1 X_1 + \varphi_2 X_2 + \cdots \ , \\ &= \varphi(z) \to X_0 \ , \\ &= \varphi(z) X_0(x,z) + \varphi'(z)\,(\partial X_0/\partial x) \\ &\quad + \varphi''(z)\,(\partial^2 X_0/\partial x^2)/2! + \cdots \ . \end{aligned} \tag{12.7}$$

It is obvious that (12.7) transforms (12.6) into a differential equation of infinite order with constant coefficients, namely,

$$\varphi(z) \to u(x) = X_\varphi(x, z) \to f(x) \ . \tag{12.8}$$

13. The Inversion of Operators by Infinite Differentiation. A method of inverting operators which possesses useful applications may be described as follows:

Let us assume that we wish to solve the equation

$$F(x,z) \to u(x) = f(x) \ . \tag{13.1}$$

Operating on both sides successively with z, z^2, z^3, etc., we obtain the following system of equations:

$$\begin{aligned}
&\{zF + \partial F/\partial x\} \to u(x) = z \to f(x) \ , \\
&\{z^2F + 2z\,\partial F/\partial x + \partial^2 F/\partial x^2\} \to u(x) = z^2 \to f(x) \ , \\
&\{z^3F + 3z^2\partial F/\partial x + 3z\partial^2 F/\partial x^2 + \partial^3 F/\partial x^3\} \to u(x) = z^3 \to f(x) \ , \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned} \tag{13.2}$$

These equations taken together with the original one form a system which may now be solved algebraically for $u(x)$ in terms of $f(x)$ and its derivatives. The resulting operator is the resolvent generatrix for the original equation, assuming, naturally, that the system can be solved under some existential criteria.

A more explicit specification of this method may be obtained if we specialize $F(x,z)$ as the operator

$$[1 + a_0(x)] + a_1(x)z + a_2(x)z^2 + \cdots + a_n(x)z^n + \cdots \ . \tag{13.3}$$

The system which we wish to invert then becomes the following:

$$\begin{aligned}
&[1 + a_0(x)]u(x) + a_1(x)\,u'(x) + a_2(x)\,u''(x) + \cdots \\
&\qquad\qquad + a_{n-1}(x)\,u^{(n-1)}(x) + \cdots = f(x) \ , \\
&[(1 + a_0)u]' + (a_1u')' + (a_2u'')' + \cdots + (a_{n-1}u^{(n-1)})' \\
&\qquad\qquad + \cdots = f'(x) \ , \\
&[(1 + a_0)u]'' + (a_1u')'' + (a_2u'')'' + \cdots + (a_{n-1}u^{(n-1)})'' \\
&\qquad\qquad + \cdots = f''(x) \ , \\
&[(1 + a_0)u]^{(n-1)} + (a_1u')^{(n-1)} + (a_2u'')^{(n-1)} + \cdots + (a_{n-1}u^{(n-1)})^{(n-1)} \\
&\qquad\qquad + \cdots = f^{(n-1)}(x) \ , \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned} \tag{13.4}$$

If we designate by $D(n,x)$ the determinant of the first n equations of this system after we have suppressed terms of order greater than n—1, we can give its explicit form as follows:

$$D(n,x) = \tag{13.5}$$

$$\left.\begin{array}{lllll} \cdot\, a_0 , & a_1 & ,\cdots, & a_2 & , \quad a_{n-1} \\ \quad , & 1+a_0+a'_1 & ,\cdots, & a_1+a'_2 & , \quad a_{n-2}+a'_{n-1} \\ \quad , & 2a'_0+a''_1 & ,\cdots, & 1+a_0+2a'_1+a''_2 & , a_{n-3}+2a'_{n-2}+a''_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ {}^{(n-1)}, & {}_{n-1}C_1a_0{}^{(n-2)}+a_1{}^{(n-1)} & ,\cdots, & {}_{n-1}C_2a_0{}^{(n-3)}+{}_{n-1}C_1a_1{}^{(n-2)}+a_2{}^{(n-1)} & 1+\sum_{r=0}^{n-1} {}_{n-1}C_ra_r{}^{(r)} \end{array}\right| .$$

If we represent the cofactors of the elements of the first column by $D_1(n,x)$, $D_2(n,x)$, $\cdots$, $D_n(n,x)$, the formal segmentary resolvent operator in the sense of the *method of segments* as developed in section 7, chapter 3, may be written

$$X_n(x,z) = [D_1(n,x) + zD_2(n,x) + \cdots + z^{n-1}D_n(n,x)]/D(n,x) \; . \tag{13.6}$$

The limit

$$X(x,z) = \lim_{n=\infty} X_n(x,z)$$

is obviously the formal resolvent operator for the equation (13.1).

A sufficient condition for the existence of $X(x,z)$ is readily obtained from theorem 8 of chapter 3. If we make use of the abbreviation

$$\sigma = |a_0| + |a_1| + |a_2| + \cdots + |a_n| + \cdots ,$$

and interpret $\sigma^{(n)}$ to mean

$$|a_0{}^{(n)}| + |a_1{}^{(n)}| + |a_2{}^{(n)}| + \cdots + |a_n{}^{(n)}| + \cdots ,$$

it is clear upon summation of the absolute values of the terms of (13.4) that the sums $S_i = \sum_{k=1}^{n} |a_{ik}|$ of theorem 8 become in this case the sums:

$$\begin{aligned} &\sigma \;, \quad \sigma_1 = \sigma + \sigma' &&= e^{-x}\frac{d}{dx}(e^x\sigma) \;, \\ &\sigma_2 = \sigma + 2\sigma' + \sigma'' &&= e^{-x}\frac{d^2}{dx^2}(e^x\sigma) \;, \\ &\sigma_3 = \sigma + 3\sigma' + 3\sigma'' + \sigma''' &&= e^{-x}\frac{d^3}{dx^3}(e^x\sigma) \;, \\ &\sigma_n = \sigma + n\sigma' + n(n-1)\,\sigma''/2! + \cdots &&= e^{-x}\frac{d^n}{dx^n}(e^x\sigma) \;. \end{aligned}$$

Hence we have the following result:

The resolvent operator $X(x,z)$ *corresponding to* (13.3) *exists and can be obtained by the method of segments, provided*

$$e^{-x}\frac{d^n}{dx^n}(e^x\sigma) < 1 \ , \quad n = 1, 2, \cdots \ .$$

14. The Permutability of Linear Differential Operators. We have spoken in previous sections of the theory of permutable functions in the sense of the definitions of composition. It is now of interest to consider the problem of permutability for linear differential operators, the ensuing discussion being derived mainly from the investigations of J. L. Burchnall and T. W. Chaundy (See *Bibliography*).*

We shall consider two linear differential operators, $P(z)$ and $Q(z)$, where P is of order m and Q is of order n, subject to the condition that they shall be permutable, namely that

$$P \to Q = Q \to P \ .$$

Let us first consider some examples.

Example 1. We shall assume that $P(z)$ is of first degree and $Q(z)$ is of second degree; that is,

$$P = z + a(x) \ , \quad Q = z^2 + \beta(x)z + \gamma(x) \ .$$

Computing the *alternant*, $\Delta(z)$, of P and Q, we get

$$\Delta(z) \equiv P \to Q - Q \to P = \beta' z + \gamma' - (2z + \beta)a' - a'' \ .$$

Then if $\Delta \equiv 0$, we shall have

$$\beta' = 2a' \ ,$$

$$\gamma' = \beta a' + a'' \ .$$

These equations are easily integrated and one obtains

$$\beta = 2a + A \ , \tag{14.1}$$

$$\gamma = a^2 + a' + Aa + B \ ,$$

where A and B are arbitrary constants.

Now since

$$P^{(2)} = (z + a)^2 + a' \ ,$$

we may write with the help of (14.1)

$$Q = P^{(2)} + AP + B \ .$$

*The reader is also referred to E. L. Ince: *Ordinary Differential Equations.* London (1927), pp. 128-132.

Since any Q written in this form is obviously permutable with P, we see that this equation expresses both a necessary and a sufficient condition for the permutability of P and Q, when P is a linear and Q a quadratic function of z.

Example 2. As a second example consider the two quadratic operators

$$P = z^2 + \alpha(x)z + \beta(x) \ , \quad Q = z^2 + \gamma(x)z + \delta(x) \ .$$

Computing the alternant, we obtain

$$\Delta \equiv P \to Q - Q \to P = (2\gamma' - 2\alpha')z^2$$
$$+ (\alpha\gamma' + 2\delta' + \gamma'' - \alpha'\gamma - 2\beta' - \alpha'')z + \alpha\delta' + \delta'' - \gamma\beta' - \beta'' \ .$$

From the permutability condition $\Delta = 0$, we obtain

$$\gamma' = \alpha' \ , \quad \alpha\gamma' + 2\delta' + \gamma'' = \alpha'\gamma + 2\beta' + \alpha'' \ ,$$
$$\alpha\delta' + \delta'' = \gamma\beta' + \beta'' \ .$$

This system of equations may be integrated and yields the following relationships;

$$\begin{aligned} \gamma &= \alpha + 2A \ , \\ \delta &= \beta + A\alpha + B \ , \qquad\qquad (14.2) \\ \tfrac{1}{2}A\alpha^2 + A\alpha' &= 2A\beta + C \ , \end{aligned}$$

where A, B, and C are arbitrary constants.

Let us now consider the operators $P - HI$ and $Q - KI$, where I is the identical operator $I \to f(x) = f(x)$. We now form the eliminant, E, between the equations

$$P - HI \equiv z^2 + \alpha z + \beta - H = 0 \ ,$$
$$z \to (P - HI) \equiv z^3 + \alpha z^2 + \alpha' z + \beta z + \beta' - Hz = 0 \ ,$$
$$Q - KI \equiv z^2 + \gamma z + \delta - K = 0 \ ,$$
$$z \to (Q - KI) \equiv z^3 + \gamma z^2 + \gamma' z + \delta z + \delta' - Kz = 0 \ ,$$

and thus obtain

$$\begin{aligned} E \equiv\ & (\gamma - \alpha)\alpha(\delta - \beta + H - K) + (\delta' - \beta')(\gamma - \alpha) \\ & - (\gamma - \alpha)^2(\beta - H) - (\delta - \beta + H - K)(\gamma' + \delta \\ & - \alpha' - \beta + H - K) = 0 \ . \end{aligned}$$

Making use of the relations given in (14.2), we are able to eliminate all the functions of x and thus to obtain the following equation between the constants H and K:

$$H^2 - 2HK + K^2 + (2B - 4A^2)\, H - 2BK + B^2 - 2AC = 0 \ . \tag{14.3}$$

If one now takes account of the fact that $P \to u = Hu$ and $Q \to u = Ku$, it is evident that P and Q may be substituted for H and K in (14.3). One thus reaches the identity

$$P^{(2)} - 2P \to Q + Q^{(2)} + (2B - 4A^2)\, P - 2BQ + B^2 - 2AC = 0.$$

The results obtained in the preceding examples are capable of the following generalization:

Theorem 9. If P and Q are permutable differential operators of order m and n respectively,

$$P \to Q = Q \to P \ ,$$

then they will satisfy an operational identity of the form

$$f(P,Q) \equiv 0 \ ,$$

where $f(H,K)$ is a polynomial of degree n in H and m in K.

Proof: Consider the differential equation

$$(P - HI) \to u(x) = 0 \ , \tag{14.4}$$

where I is the identical operator and H is an arbitrary constant. Let $u_1, u_2, \cdots, u_m$ be a fundamental set of solutions of this equation.

Let us also assume that Q is an operator permutable with P so that if u_i is any member of the fundamental set we shall have

$$\{(P - HI) \to Q\} \to u_i = \{Q \to (P - HI)\} \to u_i = 0 \ .$$

Hence $Q \to u_i(x)$ is a solution of (14.4) and by setting i successively equal to 1, 2, 3, etc., we obtain the relations

$$\begin{aligned} Q \to u_1 &= a_{11}u_1 + a_{12}u_2 + \cdots + a_{1m}u_m \ , \\ Q \to u_2 &= a_{21}u_1 + a_{22}u_2 + \cdots + a_{2m}u_m \ , \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ Q \to u_m &= a_{m1}u_1 + a_{m2}u_2 + \cdots + a_{mm}u_m \ . \end{aligned}$$

If K is any constant which satisfies the equation,

$$|a_{ij} - K\delta_{ij}| = 0 \ .$$

where δ_{ij} is the Kronecker symbol which is zero when $i \neq j$ and 1 when $i = j$, then there exists a solution $u(x)$ such that

$$Q \to u = \text{Ku} \ .$$

Since there are obviously m such values of K for each value of H, we can find for every H a set of m systems of the form

$$\begin{aligned}(P - HI) &\to u = 0 \ ,\\ (Q - KI) &\to u = 0 \ ,\end{aligned} \tag{14.5}$$

which have a common solution.

Similarly for every value of K, there exists n values of H for which the equations of (14.5) have a common solution.

Thus between the numbers H and K there exists an algebraic relation of the form

$$f(H,K) = 0 \ ,$$

which is of degree n in H and m in K.

Also, since $P \to u = Hu$ and $Q \to u = Ku$, it follows that

$$f(P,Q) \to u = f(H,K) = 0 \ . \tag{14.6}$$

The order of this equation is clearly equal to mn.

Let us now show that the equation

$$f(P,Q) = 0 \ , \tag{14.7}$$

holds identically.

Let us first observe that every solution common to (14.5) is also a solution of (14.6). Now let $\{H_i\}$ be a set of k distinct numbers to which corresponds the set of k solutions $\{U_i\}$ common to the system (14.5) in which the corresponding values of K have been inserted. The functions $U_1, U_2, \cdots, U_k$ are linearly independent for if they were not then there would exist a set of constants $\{c_i\}$ such that

$$c_1U_1 + c_2U_2 + \cdots + c_kU_k = 0 \ .$$

Operating on the left-hand member of this equation with P, $P^{(2)}, \cdots, P^{(k-1)}$, we then obtain the system

$$c_1H_1{}^jU_1 + c_2H_2{}^jU_2 + \cdots + c_kH_k{}^jU_k = 0 \ , \ j = 1, 2, \cdots, \ k-1 \ .$$

But since the values of H_i are distinct, the determinant

$$|c_iH_i{}^j| \ , \ i = 1, 2, \cdots, k; \ j = 0, 1, \cdots, , k-1$$

does not vanish and consequently the values of the c_i are zero.

Since this conclusion holds for any choice of the values of H there will exist an infinite number of linearly distinct functions $U_1, U_2, \cdots,$ which satisfy (14.7). But since this equation is of order mn, it cannot possess more than mn such solutions and hence must hold identically.

The actual determination of the equation (14.7) is most easily accomplished by forming the eliminant for the set of equations

$$\begin{aligned} z^r &\to (P - HI) = 0 \ , \quad r = 0, 1, \cdots, n-1 \ ,\\ z^s &\to (Q - KI) = 0 \ , \quad s = 0, 1, \cdots, m-1\end{aligned}$$

PROBLEMS.

1. Prove that the operator $z+\alpha(x)$ is permutable with the operator $z^2+\beta(x)z+\gamma(x)$ if and only if $z^2+\beta(x)z+\gamma(x) \equiv [z+\alpha(x)+A] \rightarrow [z+\alpha(x)+B]$, where A and B are constants.

2. Show that the operators

$$z^2 - 2x^{-2} \text{ and } z^3 - 3x^{-2}z + 3x^{-3}$$

are permutable.

3. If $P = z^2+\alpha(x)z+\beta(x)$ and $Q = z^3+\gamma(x)z^2+\delta(x)z+\varepsilon(x)$, show that

$$P \rightarrow Q - Q \rightarrow P = F(x,z)$$

where $F(x,z) = (2\gamma' - 3\alpha')z^3 + (2\delta' + \alpha\gamma' + \gamma'' - 3\beta' - 2\alpha'\gamma - 3\alpha'')z^2$

$$+ (2\varepsilon' + \alpha\delta' + \delta'' - 2\beta'\gamma - \alpha'\delta - 3\beta'' - \alpha''\gamma - \alpha''')z + \alpha\varepsilon'$$
$$+ \varepsilon'' - \beta'\delta - \beta''\gamma - \beta''' .$$

4. For the operators of problem 3 show that $P \rightarrow Q - Q \rightarrow P = 0$, provided

$$\gamma = \frac{3}{2}\alpha + A\ , \qquad \delta = \frac{3}{8}\alpha^2 + \frac{3}{4}\alpha' + \frac{3}{2}\beta + A\alpha - \frac{B}{2}\ ,$$

$$\varepsilon = \frac{3}{4}\alpha\beta - \frac{1}{16}\alpha^3 + \frac{1}{8}\alpha'' + \frac{3}{4}\beta' + A\beta + \frac{B}{4}\alpha + \frac{C}{2}$$

where A, B and C are arbitrary constants.

5. If P and Q are the operators of problem 2, show that

$$P^{(3)} \equiv Q^{(2)} .$$

15. A Class of Non-permutable Operators. The demands of the quantum theory of radiation have recently turned attention to a class of non-permutable operators, the interchange of factors being subject to the equation

$$QP - PQ = c\ , \qquad (15.1)$$

where c is a constant.

As we have said in the first chapter, such operators were first studied by Charles Graves as early as 1857. They were introduced into the quantum theory by W. Heisenberg and have been extensively studied among others by P. A. M. Dirac, N. H. McCoy, W. O. Kermack and W. H. McCrea. (See *Bibliography*).

It is obvious that in multiplication the respective positions of the operators is of primary importance. Following a suggestion due to McCoy we shall say that an operator is in *normal form* when all the factors involving Q alone follow all the factors involving P alone. This definition is arbitrary since we might as easily have had the factors involving Q precede the factors involving P. However, if $F(P,Q)$ is an operator in normal form, then one can easily show that

$F(Q,-P)$ is its equivalent in which all the factors involving P follow all the factors involving Q.

For example, we have

$$PQP = P(PQ + c) = P^2 Q + c P \ ,$$
$$= (QP - c)P = Q P^2 - c P \ .$$

One of the simplest examples of non-permutable operators is furnished by the specialization: $Q = x$, $P = z$, $c = 1$. We see that $QP = \vartheta$, where ϑ is the operator introduced in section 12, chapter 2.

We shall first prove the following theorem, due to Charles Graves.

Theorem 10. If $F(Q)$ and $G(P)$ are functions expansible as power series about the origin and if Q and P are non-permutable operators which belong to the class defined by (15.1), *then we have the following formal expansion:*

$$F(Q)G(P) = G(P)F(Q) + c\,G'(P)F'(Q) + \frac{c^2}{2!}G''(P)F''(Q)$$
$$+ \cdots + \frac{c^n}{n!}G^{(n)}(P)F^{(n)}(Q) + \cdots \qquad (15.2)$$

Proof: We first observe that

$$QP = PQ + c \ ,$$
$$QP^2 = (QP)P = (PQ+c)P = P(PQ+c) + cP = P^2Q + 2cP \ ,$$

and hence in general

$$QP^n = P^n Q + nc\,P^{n-1} \ .$$

From this we immediately derive

$$Q\,G(P) = G(P)Q + c\,G'(P) \ . \qquad (15.3)$$

We shall now show that

$$Q^n G(P) = G(P)Q^n + c\,G'(P)\,{}_nC_1\,Q^{n-1} + c^2\,G''(P)\,{}_nC_2\,Q^{n-2}$$
$$+ \cdots + c^n\,G^{(n)}(P)\,{}_nC_n \ , \qquad (15.4)$$

where ${}_nC_r$ is the rth binomial coefficient.

Using induction we assume that this expansion is true and then compute

$$Q^{n+1}G(P) = Q\,G(P)Q^n + c\,Q\,G'(P)\,{}_nC_1\,Q^{n-1} + \cdots \ ,$$

which by (15.3) becomes

$$Q^{n+1}G(P) = [G(P)Q + c\,G'(P)]\,Q^n + c[G'(P)Q$$
$$+ c\,G''(P)]\,{}_nC_1\,Q^{n-1} + c^2[G''(P)Q$$
$$+ c\,G^{(3)}(P)]\,{}_nC_2\,Q^{n-2} + \cdots$$

$$Q^{n+1}G(P) = G(P)Q^{n+1} + c\,{}_{n+1}C_1\,G'(P)\,Q^n + c^2\,{}_{n+1}C_2\,G''(P)Q^{n-1} + \cdots .$$

Since by (15.3) the expansion was true for $n = 1$, it must hold for all other positive values of n.

The expansion (15.2) is derived in an obvious manner from (15.4).

Using this theorem, one may obtain immediately the following:

$$Q^2 P^3 = P^3 Q^2 + 6\,c\,P^2 Q + 6\,c^2 P ,$$

$$e^Q e^P = e^c e^P e^Q .$$

In the application of non-permutable operators the normal form of functions of $Q + P$ is of importance. We shall first prove the following theorem: [See *Bibliography*: Kermack and McCrea (2), p. 223.]

Theorem 11. If we adopt the notation

$$(P + Q)^{(n)} = P^n + {}_nC_1\,P^{n-1}\,Q + {}_nC_2\,P^{n-2}\,Q^2 + \cdots + Q^n ,$$

then the expansion

$$(P + Q)^n = (P+Q)\,(P+Q)\cdots(P+Q)$$

is given by

$$\begin{aligned}(P+Q)^n = {} & (P+Q)^{(n)} + (\tfrac{1}{2}c)\,(2!/1!)\,{}_nC_2(P+Q)^{(n-2)} \\ & + (\tfrac{1}{2}c)^2(4!/2!)\,{}_nC_4(P+Q)^{(n-4)} \\ & + (\tfrac{1}{2}c)^3(6!/3!)\,{}_nC_6(P+Q)^{(n-6)}) + \cdots . \qquad (15.5)\end{aligned}$$

Proof: We first observe that

$$(P+Q)\,(P+Q)^{(n)} = P(P+Q)^{(n)} + (P+Q)^{(n)}\,Q + cn(P+Q)^{(n-1)} ,$$

which may be derived immediately from theorem 10 or proved by direct multiplication and use of (15.1).

Now assume that (15.5) is true and then consider

$$\begin{aligned}(P+Q)^{n+1} = {} & (P+Q)\,(P+Q)^n \\ = {} & P(P+Q)^{(n)} + (P+Q)^{(n)}Q + cn(P+Q)^{(n-1)} \\ & + (\tfrac{1}{2}c)\,(2!/1!)\,{}_nC_2[P(P+Q)^{(n-2)} \\ & \qquad + (P+Q)^{(n-2)}\,Q + c(n-2)\,(P+Q)^{(n-1)}] \\ & + (\tfrac{1}{2}c)^2(4!/2!)\,{}_nC_4\,[P(P+Q)^{(n-4)} \\ & \qquad + (P+Q)^{(n-4)}\,Q + c(n-4)\,(P+Q)^{(n-3)} \\ & + \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\end{aligned}$$

$$(P+Q)^{n+1} = (P+Q)^{(n+1)} + (\tfrac{1}{2}c)[2n+(2!/1!)\,{}_nC_2](P+Q)^{(n-1)}$$
$$+ (\tfrac{1}{2}c)^2[2(2!/1!)\,{}_nC_2(n-2)+(4!/2!)\,{}_nC_4](P+Q)^{(n-3)}$$
$$+ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$+ (\tfrac{1}{2}c)^{r+1}[2\,{}_nC_{2r}(n-2r)\frac{2r!}{r!}$$
$$+ {}_nC_{2r+2}\frac{(2r+2)!}{(r+1)!}](P+Q)^{(n-2r-1)}$$
$$+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

But from the identity

$$2\,{}_nC_{2r}(n-2r)\frac{2r!}{r!} + {}_nC_{2r+2}\frac{(2r+2)!}{(r+1)!} = {}_{n+1}C_{2r+2}\frac{(2r+2)!}{(r+1)!} \;,$$

it is clear that the above expansion reduces to (15.5) with n replaced by $n+1$. Hence since (15.1) is obviously true for $n = 1$, we may use induction to prove it for all higher values of n.

From theorem 11 we derive the following general proposition:

Theorem 12. If $F(P)$ is a function developable as a power series in P, and if

$$F(P+Q) = \sum_{n=0}^{\infty} F_n(P+Q)^n \;, \quad F[(P+Q)^{(1)}] = \sum_{n=0}^{\infty} F_n(P+Q)^{(n)} \;,$$

where $(P+Q)^n$ and $(P+Q)^{(n)}$ have the same significance as in theorem 11, *then we shall have formally*

$$F(P+Q) = F[(P+Q)^{(1)}] + (\tfrac{1}{2}c)F''[(P+Q)^{(1)}]$$
$$+ (\tfrac{1}{2}c)^2\frac{1}{2!}F^{(4)}[(P+Q)^{(1)}]$$
$$+ (\tfrac{1}{2}c)^3\frac{1}{3!}F^{(6)}[(P+Q)^{(1)}] + \cdots \,. \tag{15.6}$$

Proof: Making use of the results of theorem 11, we get

$$F(P+Q) = \sum_{n=0}^{\infty} F_n(P+Q)^{(n)} + (\tfrac{1}{2}c)\sum_{n=0}^{\infty} F_n\frac{2!}{1!}\,{}_nC_2(P+Q)^{(n-2)}$$
$$+ (\tfrac{1}{2}c)^2\sum_{n=0}^{\infty} F_n\frac{4!}{2!}\,{}_nC_4(P+Q)^{(n-4)} + \cdots \,.$$

This sum is seen to reduce formally to (15.6) through the identity

$$\sum_{n=0}^{\infty} F_n\frac{(2r)!}{r!}\,{}_nC_{2r}\,P^{n-2r} = \frac{1}{r!}F^{(2r)}(P) \;.$$

Corollary: Under the assumptions of theorem 12, *we have*

$$e^{P+Q} = e^{\frac{1}{2}c}\, e^P\, e^Q \ .$$

It is a curious fact that non-permutable operators of the kind just discussed are basic to Dirac's expression of the quantum theory. The connection is made through an extension to quantum mechanics of the theory of *contact transformations* as it appears in the foundations of classical dynamics.

It is obviously impossible to sketch adequately the theory of contact transformations in a brief section, but the formal connection between such transformations and the quantum theory may be indicated. For the basic theory of contact transformations and their central position in analytical mechanics, the reader is referred to E. T. Whittaker: *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge, (1904), chap. 11, and E. Cartan: *Leçons sur les invariants intégraux*, Paris (1922). An account of the extension of this theory to the phenomena of quantum mechanics will be found in Dirac's treatise [see *Bibliography*: Dirac, chapter 5] and in Weyl's theory of groups [see *Bibliography*: Weyl (1)].*

Let $(q_1, q_2, \cdots, q_n; p_1, p_2, \cdots, p_n)$ be a set of $2n$ variables, which may be defined in terms of a second set of $2n$ variables $(Q_1, Q_2 \cdots, Q_n; P_1, P_2, \cdots, P_n)$ by means of a set of $2n$ equations. Then if the equations are such that the differential form

$$dF = \sum_{i=1}^{n} [P_i\, dQ_i - p_i\, dq_i]$$

is a perfect differential when it is expressed in terms of p_i, q_i and their differentials, the transformation is called a *contact* (or *canonical*) *transformation.*

For example, the equations

$$P_i = -q_i \ , \quad p_i = Q_i \ , \quad i = 1, 2, \cdots, n$$

define a contact transformation since

$$dF = -\sum_{i=1}^{n} (q_i\, dp_i + p_i\, dq_i) = -d\Big(\sum_{i=1}^{n} q_i\, p_i\Big) \ .$$

If we employ the abbreviation $[u, v]$, called a *Poisson bracket*, which is defined as follows:

$$[u, v] = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right) \ .$$

then it can be proved that the following conditions must be satisfied

*The reader is also referred to an excellent work by E. Bloch: *L'ancienne et la nouvelle théorie des quanta.* Paris (1930), 417 p., in particular chapters 11 and 18.

provided the variables $(Q_1, Q_2, \cdots, Q_n; P_1, P_2, \cdots, P_n)$ and $(q_1, q_2, \cdots, q_n; p_1, p_2, \cdots, p_n)$ are connected by a contact transformation:

$$[P_i, P_j] = 0 \ , \ [Q_i, Q_j] = 0 \ , \ [Q_i, P_j] = \delta_{ij} \ . \qquad (15.7)$$

In order to preserve the formal aspects of molar dynamics so that *"classical mechanics may be regarded as the limiting case of quantum mechanics"*, Dirac was led to introduce operators p and q, which satisfied equations (15.7). Hence the basic *commutation rule* of Heisenberg appeared in the form

$$qp - pq = c\,[q, p] \ , \quad c = ih/(2\pi) \ ,$$

where $h = 6.547 \ 10^{-27}$ erg secs, is Planck's constant.

In the Heisenberg theory the operators p and q are matrices and hence the third condition of (15.7) became

$$[q, p] = I \ ,$$

where I is the unit matrix.

PROBLEMS

1. Prove that if f is a function of P and Q, partial derivatives may be defined as follows:

$$fP - Pf = c\frac{\partial f}{\partial Q}$$

$$Qf - fQ = c\frac{\partial f}{\partial P} \ .$$

The following problems were communicated to the author by N. H. McCoy:

2. Prove that

$$\exp(P^m + Q) = \exp\left[\sum_{i=1}^{m+1} c^{i-1} \frac{m!}{i!\,(m+1-i)!} P^{m+1-i}\right] e^Q \ .$$

3. Prove that

$$\exp(Q^m + P) = e^P \exp\left[\sum_{i=1}^{m+1} c^{i-1} \frac{m!}{i!\,(m+1-i)!} Q^{m+1-i}\right] \ .$$

4. If $\phi(P)$ is any polynomial in P, show that

$$\exp[\phi(P) + Q] = \exp[A(P)] \exp Q \ ,$$

where we abbreviate

$$A(P) = \phi(P) + \frac{c}{2!}\frac{\partial \phi(P)}{\partial P} + \frac{c^2}{3!}\frac{\partial^2 \phi(P)}{\partial P^2} + \cdots \ .$$

5. Prove that if a is a constant, then

$$e^{aPQ} = \sum_{n=0}^{\infty} K^n P^n Q^n / n! \ ,$$

where we abbreviate

$$K = (e^{ca} - 1)/c \ .$$

6. Establish the identity

$$(PQ)^n = \sum_{k=1}^{n} \sum_{i=1}^{k} \frac{c^{n-k}(-1)^{k-i}\, i^n}{i!\,(k-i)!} P^k Q^k \ .$$

7. $e^{P^2+Q^2} = (\sec 2c)^{\frac{1}{2}}\, e^{(\frac{1}{2}c^{-1}\tan 2c)P^2}\, e^{-(c^{-1}\log\cos 2c)PQ} \times e^{(\frac{1}{2}c^{-1}\tan 2c)Q^2}$.

8. Express $e^{P^2+Q^2}$ in normal form.

16. Special Examples Illustrating the Application of Operational Processes. We shall conclude this chapter with three examples which will illustrate the usefulness of symbolic processes in applied problems. These examples might be multiplied many times, as one readily surmises from the fact that operational methods are essentially invoked in any problem which is formulated in terms of a functional equation. The Heaviside calculus, developed in chapter 7, is an outstanding example of such an application. The problems discussed in this section were selected mainly to illustrate the varied fields to which symbolic methods may be applied.

Development of the Disturbing Function of Planetary Motion

One of the most striking examples of the usefulness of operators is found in the extraordinary complexities of the problem of developing the coefficients of the disturbing function of planetary motion. This application, the essentials of which have general interest, was first given by S. Newcomb in 1880;* the procedure was later recast in a more satisfactory form by K. P. Williams.†

Suppose that we have two planets moving in elliptic orbits with semi-major axes a and a', eccentricities ε and ε', and perihelia at angular distances w and w' from the selected node. Let r and r', $r < r'$, be the radii vectors, f and f' the true anomalies (the angular distances of the planets from their respective perihelia measured at the common focus in the sun), and V and V' the true angular distances of the planets from their common node, that is,

$$V = w + f \ , \quad V' = w' + f' \ .$$

It is customary to represent the mean values of V and V' by λ and λ', and the mean values of f and f' by g and g'. Thus in the discussion

*A Method of Developing the Perturbative Function of Planetary Motion. *American Journal of Mathematics*, vol. 3 (1880), pp. 193-209. See also: *Astronomical Papers of the American Ephemeris*, vol. 3 (1891), pp. 1-200; vol. 5 (1895), pp. 1-48; 301-378.

†The Symbolic Development of the Disturbing Function. *American Journal of Mathematics*, vol. 51 (1929), pp. 109-122.

of circular orbits the quantities V, V', f, f', and r, r' are replaced by λ, λ', g, g', and a, a' respectively. We shall also employ the further abbreviations $\alpha = a/a'$ and $\sigma = \sin \frac{1}{2}J$, where J is the acute angle between the planes of the two orbits.

If we represent by Δ_0 the distance apart of the planets at a given time $t = t_0$, then this function may be written

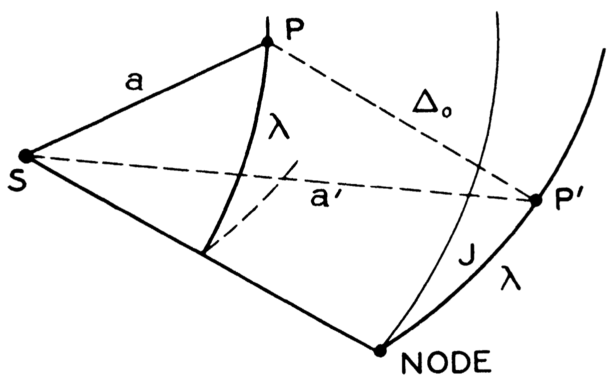
$$\Delta_0 = (r^2 + r'^2 - 2\,r\,r' \cos W)^{\frac{1}{2}} ,$$

where $\cos W$ is given by

$$\cos W = (1 - \sigma^2) \cos(V - V') + \sigma^2 \cos(V + V') .$$

The disturbing function may then be written

$$R = \Delta_0^{-1} - (r/r'^2) \cos W .$$



Since, however, the second term may be absorbed into the coefficients of the first by certain well known modifications it is sufficient to consider only the development of the first term.

The problem of the disturbing function, then, is that of the explicit expansion of the reciprocal distance, Δ_0^{-1}, in terms of some one of the parameters involved. It is obviously not desirable for us to carry out in detail any one of these expansions since such expansions are of technical interest to the astronomers and may be found explicitly given in the references cited above. However, the operational method by which the obvious computational difficulties were modified is of general interest and merits description here.

The problem just described may be stated more generally as follows: Let us consider a function, $F(A,B)$, in which the parameters A and B are power series in a third variable, h, that is,

$$A = \alpha_0 + \alpha_1\, h + \alpha_2\, h^2/2! + \alpha_3\, h^3/3! + \cdots ,$$

$$B = \beta_0 + \beta_1\, h + \beta_2\, h^2/2! + \beta_3\, h^3/3! + \cdots .$$

If we now employ the abbreviations, $p = \partial/\partial x$ and $q = \partial/\partial y$, it is clear that we may write the Taylor's expansion of $F(A,B)$ about the origin in the symbolic form

$$F(A,B) = e^{Ap+Bq} \to F(x,y)\,|_{x,y=0} \; .$$

Let us now adopt the following abbreviations:

$$\begin{aligned}
\vartheta &= A\,p + B\,q \;, \quad e^{\vartheta} = G(h) \;, \\
\vartheta_0 &= \vartheta(0) = \alpha_0\,p + \beta_0\,q \;, \\
\vartheta_1 &= \vartheta'(0) = \alpha_1\,p + \beta_1\,q \;, \\
\vartheta_2 &= \vartheta''(0) = \alpha_2\,p + \beta_2\,q \;, \\
&\cdots\cdots\cdots
\end{aligned}$$

Expanding the function $G(h)$ as a power series,

$$G(h) = G(0) + G'(0)\,h + G''(0)\,h^2/2! + \cdots \;,$$

we seek an expression for the coefficients in terms of the elementary operators ϑ_r. Let us designate $G^{(r)}(0)$ by D_r. We then note that

$$G'(h) = \vartheta'\,e^{\vartheta} = \vartheta'\,G(h) \;,$$

and hence, from the expansion of Leibnitz, that

$$G^{(n+1)}(h) = \sum_{r=0}^{n} {}_nC_r\,\vartheta^{(r+1)}\,G^{(n-r)}(h) \; .$$

Letting $h = 0$ in this expression and replacing the derivatives by their operational equivalents, we obtain

$$\begin{aligned}
D_1 &= \vartheta_1 \;, \\
D_2 &= D_1\,\vartheta_1 + \vartheta_2 \;, \\
D_3 &= D_2\,\vartheta_1 + 2\,D_1\,\vartheta_2 + \vartheta_3 \;, \\
D_4 &= D_3\,\vartheta_1 + 3\,D_2\,\vartheta_2 + 3\,D_1\,\vartheta_3 + \vartheta_4 \;, \\
&\cdots\cdots\cdots \\
D_n &= D_n\,\vartheta_1 + {}_nC_1\,D_{n-1}\,\vartheta_2 + {}_nC_2\,D_{n-2}\,\vartheta_3 + \cdots + \vartheta_n \; .
\end{aligned}$$

Hence we may write

$$\begin{aligned}
G(h) &= e^{D_0}[1 + D_1\,h + D_2\,h^2/2! + D_3\,h^3/3! + \cdots] \\
&= e^{D_0}\,e^{(Dh)} \;,
\end{aligned}$$

where (Dh) is the symbolic representation of the expression within the brackets.

From this it follows that

$$F(A,B) = G(h) \to F(x,y)\,|_{x,y=0}$$

$$F(A,B) = e^{D_0} e^{(Dh)} \to F(x, y)|_{x,y=0}$$
$$= F(a_0, \beta_0) e^{(Dh)} \to F(x, y)|_{x,y=0} .$$

We may illustrate the application of this symbolic method in the simplest case, where both orbits are circles and $\sigma = 0$. We then write

$$a' \Delta_0^{-1} = (1 + A + B)^{-\frac{1}{2}} ,$$

where we abbreviate $A = a^2$, $B = -2\, a \cos(\lambda - \lambda')$.

The development according to a can now be effected as follows:

We have $\vartheta_0 = 0$, $\vartheta_1 = -2 \cos(\lambda - \lambda')\ q = \nu\, q$, where we abbreviate

$$\nu = -(z + 1/z) \ , \quad z = e^{(\lambda - \lambda') i} ,$$

$$\vartheta_2 = 2\, p \ , \quad \vartheta_3 = \vartheta_4 = \cdots = 0 .$$

From these values we then compute

$$D_1 = \nu\, q \ , \quad D_2 = \nu^2 q^2 + 2\, p \ , \quad D_3 = \nu^3 q^3 + 6\, \nu\, p\, q \ ,$$
$$D_4 = \nu^4 q^4 + 12\, \nu^2 p\, q^2 + 12\, p^2 , \cdots .$$

Then from the development

$$F(x, y) = (1+x+y)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}(x + y) + \frac{3}{8}(x + y)^2$$
$$- \frac{5}{16}(x + y)^3 + \frac{35}{128}(x + y)^4 - \cdots$$

we readily compute

$$D_1 \to F = -\tfrac{1}{2}\, \nu = \cos(\lambda - \lambda') \ ,$$

$$D_2 \to F = \frac{3}{4} \nu^2 - 1 = \tfrac{1}{2}[1 + \cos 2(\lambda - \lambda')] \ ,$$

$$D_3 \to F = -\frac{15}{8} \nu^3 + \frac{9}{2} \nu = \frac{3}{4}[5 \cos 3(\lambda - \lambda') + 3 \cos(\lambda - \lambda')] \ ,$$

$$D_4 \to \mathrm{F} = \frac{3}{16}(35\, \nu^4 - 120\, \nu^2 + 48) = \frac{3}{8}[35 \cos 4(\lambda - \lambda')$$
$$+ 20 \cos 2(\lambda - \lambda') + 9] .$$

It will be observed that, if we abbreviate $\varphi = \lambda - \lambda'$, the expressions just attained may be written

$$D_1 \to F = P_1(\cos \varphi) \ , \quad D_2 \to F = 2!\, P_2(\cos \varphi) \ ,$$
$$D_3 \to F = 3!\, P_3(\cos \varphi) \ , \quad D_4 \to F = 4!\, P_4(\cos \varphi) \ ,$$

where $P_n(\cos \varphi)$ is the nth Legendrian polynomial.

Hence we have attained the classical expansion for the reciprocal of the distance between the two planets. The general case is the extension of these results to the more complex problem.

In order to show the efficacy of the method in the general problem, let us now assume that $\sigma \neq 0$ and that both orbits are elliptical. It will be sufficient for our purpose to neglect the ellipticity of the outer orbit and to obtain the development of the disturbing function in terms of the coefficient ε. In order to attain this development let us first write

$$\begin{aligned} a' \Delta_0^{-1} = \sum_{m=-\infty}^{\infty} \tfrac{1}{2} A_m \cos m(\lambda' - \lambda) \\ + \sigma^2 B_m \cos [(m+1)\lambda' - (m-1)\lambda] \\ + \sigma^4 C_m \cos [(m+2)\lambda' - (m-2)\lambda] \\ + \cdots \end{aligned}$$

where A_m, B_m, C_m, etc. are functions of σ^2 and a.*

If λ' and λ are replaced respectively by $w' + g'$ and $w + g$ and if the coefficients are represented by $\tfrac{1}{2}N$, we may then write

$$a' \Delta_0^{-1} = \sum \tfrac{1}{2} N \cos (\nu w' + \mu w + \nu g' + \mu g) \ .$$

In order to avoid notation extraneous to the problem in hand, let us further abbreviate this expression by writing

$$a' \Delta_0^{-1} = \sum \tfrac{1}{2} N \cos \mu g \ .$$

Introducing, now, the eccentricity of the inner planet, we replaced a by r and g by f. Newcomb found it more convenient to use the logarithms of r and a, so we shall write

$$\log r = \log a + A$$

$$f = g + B$$

where we have

$$A = \sum_{m=1}^{\infty} \alpha_m \varepsilon^m/m! \ , \quad B = \sum_{m=1}^{\infty} \beta_m \varepsilon^m/m! \ .$$

If we employ the abbreviations

$$c(m) = e^{img} + e^{-img} \ , \quad s(m) = e^{img} - e^{-img} \ , \quad i = \sqrt{-1} \ ,$$

we find typical values of the coefficients to be

$$\alpha_1 = -\tfrac{1}{2} c(1) \ , \quad \alpha_2 = -(3/4) c(2) + \tfrac{1}{2} \ ,$$

$$\alpha_3 = -(17/8) c(3) + (9/8) c(1), \cdots ,$$

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*For these explicit values see Newcomb: *Astronomical Papers*, vol. 5, p. 339 et seq.

$$\beta_1 = -i\,s(1) \ , \quad \beta_2 = -(5/4)\,i\,s(2) \ ,$$
$$\beta_3 = -(13/4)\,i\,s(3) + (3/4)\,i\,s(1), \cdots \ .$$

We note also the following combining forms of the symbols $c(m)$ and $s(m)$:

$$c(m)c(n) = c(m+n) + c(m-n) \ ,$$
$$s(m)s(n) = c(m+n) - c(m-n) \ .$$

Referring, now, to the general theory we specialize the operators ϑ_i by identifying the x and y of p and q with log a and g respectively and the a_i and β_i with the coefficients just written down.

Noting that p operates only on N and q only on cos $\mu\, g$, we have for the explicit form of ϑ_1 the following:

$$\begin{aligned}
\vartheta_1 \to \tfrac{1}{2} N \cos \mu\, g &= \vartheta_1 \to N\, c(\mu) \\
&= [-\tfrac{1}{2}\, c(1)p - i\,s(1)\,q] \to N\, c(\mu) \\
&= [-\tfrac{1}{2}\, c(1)c(\mu)p + \mu\, s(1)s(\mu)] \to N \\
&= [c(\mu+1)\,(-\tfrac{1}{2}p + \mu) + c(\mu-1)\,(-\tfrac{1}{2}p - \mu)] \to N \ .
\end{aligned}$$

In similar manner we find for $\vartheta_2 \to \tfrac{1}{2} N \cos \mu\, g$

$$\begin{aligned}
\vartheta_2 \to \tfrac{1}{2} N \cos \mu\, g = \{&c(\mu+2)\,[-(3/4)p + (5/4)\mu] \\
&+ c(\mu)\,\tfrac{1}{2}\, p + c(\mu-2)\,[-(3/4)p - (5/4)\mu]\} \to N \ .
\end{aligned}$$

The other primary operators can be similarly constructed and from these, by ordinary multiplication and subsequent operation upon $N\ c(\mu)$, the values of $D_i \to N_i\ c(\mu)$ may be computed. Thus one will have

$$\begin{aligned}
D_1 \to N\, c(\mu) &= \vartheta_1 \to N\, c(\mu) \ , \\
D_2 \to N\, c(\mu) &= (\vartheta_1{}^2 + \vartheta_2) \to c(\mu) \\
&= [c(\mu+2)\,\{(1/4)p^2 \\
&\qquad + [-\mu - (3/4)]p + [\mu^2 + (5/4)\mu]\} \\
&\quad + c(\mu)\,\{\tfrac{1}{2}\, p^2 + \tfrac{1}{2}\, p - 2\,\mu^2\} \\
&\quad + c(\mu-2)\,\{(1/4)p^2 \\
&\qquad + [\mu - (3/4)]\, p + \mu^2 - (5/4)\mu\}] \to N \ .
\end{aligned}$$

The *operators of Newcomb,* represented by the symbol $\Pi_j{}^s$, are the coefficients of $c(\mu+j)$ in the expansion $D_s/s!$. Their efficacy is immediately evident when we see that each single term $\tfrac{1}{2}\ N$ cos $\mu\ g$ of the expansion of $a'\,\Delta_0{}^{-1}$ is expanded into a series with terms of the form

$$[\Pi_j{}^s \to \tfrac{1}{2} N]\ \varepsilon^s \cos(\mu g + j g) \ .$$

The complexities of the expansion when the second eccentricity is introduced are similarly resolved. For further details the reader is referred to the original papers.

The Propagation of Electromagnetic Waves

The *equations of Maxwell* for the propagation of electromagnetic disturbances *in vacuo* are conveniently written in the form

$$a\varkappa \frac{\partial E}{\partial t} = \text{curl}\, M \ , \quad a\mu \frac{\partial M}{\partial t} = -\,\text{curl}\, E \qquad (16.1)$$

$$\text{div}\, E = 0 \ , \quad \text{div}\, M = 0 \ ,$$

where $E = (E_x, E_y, E_z)$ and $M = (M_x, M_y, M_z)$ are respectively the vectors of the electric and magnetic forces, $\varkappa$ is the dielectric constant, μ the magnetic permeability, and $a = 1/c$, where c is the velocity of light. By curl E we mean the vector

$$\text{curl}\, E = \left[\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) , \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) , \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \right] ,$$

and by div $E = 0$ the equation

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \ .$$

Similar definitions hold for curl M and div $M = 0$.

L. Silberstein (see *Bibliography*) has attained an elegant operational solution of the first two equations of (16.1) in the following manner:

Let us assume solutions of the form

$$\begin{aligned} E_t &= E_0 + t\, E'_0 + \frac{t^2}{2!} E''_0 + \cdots \\ M_t &= M_0 + t\, M'_0 + \frac{t^2}{2!} M''_0 + \cdots \end{aligned} \qquad (16.2)$$

where the coefficients of the powers of t are functions of x, y, z.

We now assume further that $\partial/\partial t$ is permutable with the curl, that is, that

$$\frac{\partial}{\partial t}\, \text{curl} = \text{curl}\, \frac{\partial}{\partial t} \ .$$

Hence, substituting (16.2) in the first equations of (16.1), we readily obtain the following relations between the coefficients:

$$E_0^{(2m-1)} = (-1)^{m+1} v^{2m} (a\mu) \operatorname{curl}^{(2m-1)} \to M_0 , \quad (m = 1, 2, \cdots, \infty)$$

$$E_0^{(2m)} = (-1)^m v^{2m} \operatorname{curl}^{(2m)} \to E_0 \ ;$$

$$M_0^{(2m-1)} = (-1)^m v^{2m} (a\varkappa) \operatorname{curl}^{(2m-1)} \to E_0 \ ,$$

$$M_0^{(2m)} = (-1)^{m+1} v^{2m} \operatorname{curl}^{(2m)} \to M_0 \ ,$$

where we employ the abbreviation

$$v^2 = (a^2 \varkappa \mu)^{-1} \ .$$

If these values be substituted in (16.2), the following symbolic solution is then obtained:

$$E_t = \{1 - \frac{1}{2!} (vt \operatorname{curl})^2 + \frac{1}{4!} (vt \operatorname{curl})^4 - \cdots \} \to E_0$$

$$+ (\mu/\varkappa)^{\frac{1}{2}} \{ (vt \operatorname{curl}) - \frac{1}{3!} (vt \operatorname{curl})^3 + \cdots \} \to M_0 \ ,$$

$$M_t = \{1 - \frac{1}{2!} (vt \operatorname{curl})^2 + \frac{1}{4!} (vt \operatorname{curl})^4 - \cdots \} \to M_0$$

$$- (\varkappa/\mu)^{\frac{1}{2}} \{ (vt \operatorname{curl}) - \frac{1}{3!} (vt \operatorname{curl})^3 + \cdots \} \to E_0 .$$

It is obvious that a further simplification may be achieved by writing

$$\begin{aligned} E_t &= \{\cos(vt \operatorname{curl})\} \to E_0 + (\mu/\varkappa)^{\frac{1}{2}} \{\sin(vt \operatorname{curl})\} \to M_0 \ , \\ M_t &= \{\cos(vt \operatorname{curl})\} \to M_0 - (\varkappa/\mu)^{\frac{1}{2}} \{\sin(vt \operatorname{curl})\} \to E_0 \ . \end{aligned} \qquad (16.3)$$

It will be seen from the permutability of the curl with itself, that this operator may be handled as if it were an algebraic quantity. *Hence a calculus in which the curl is replaced by the letter p would have essentially all the formal properties of the Heaviside calculus.* We could, for example, invert equations (16.3) and thus obtain

$$E_0 = \{\cos(vt \operatorname{curl})\} \to E_t - (\mu/\varkappa)^{\frac{1}{2}} \{\sin(vt \operatorname{curl})\} \to M_t \ ,$$

$$M_0 = \{\cos(vt \operatorname{curl})\} \to M_t + (\varkappa/\mu)^{\frac{1}{2}} \{\sin(vt \operatorname{curl})\} \to E_t \ .$$

This striking property of the curl has been employed by E. P. Northrop (see *Bibliography*) to obtain the solution of the first two equations of (16.1) in integral form.

The Propagation of a Population By Fission

In his well known volume on *Natural Inheritance,* London (1889), Sir Francis Galton proposed the problem of determining the

probable number of surnames, out of an initial total of N in a stable population, which would become extinct in r generations. A mathematical solution for this problem was attained by H. W. Watson. More recently T. H. Rawles (see *Bibliography*) has applied operational methods to a closely related problem which we shall discuss here. This problem may be stated as follows:

Suppose that a population consists of N individuals divided into two classes, which we designate by x and y, and suppose that in the initial state there exist P members of the x class and $Q = N - P$ members of the y class. Let us further suppose that each time a member of the population dies another divides so that the population remains constant. The problem is to determine the probable distribution of the members in the rth generation.

It is obvious that in the first generation we shall have the following possible cases:

(a) A member of the x class dies and a member of the y class divides; (b) a member of the x class dies and a member of the x class divides; (c) a member of the y class dies and a member of the y class divides; (d) a member of the y class dies and a member of the x class divides.

If the respective probabilities are assumed to be determined by the number of members in each class, then it is clear that the respective probabilities, designated by p_a, p_b, etc., for the four contingencies will be the following:

$$p_a = \frac{P}{N}\frac{Q}{N-1}, \quad p_b = \frac{P}{N}\frac{P-1}{N-1}, \quad p_c = \frac{Q}{N}\frac{Q-1}{N-1}, \quad p_d = \frac{Q}{N}\frac{P}{N-1}.$$

Let us call the function

$$S_0 = x^P y^Q$$

the initial state of the population, and let us define the operator

$$\Delta = \frac{1}{N(N-1)}\left\{ x^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}\right)\right\}.$$

We then observe that

$$\Delta \to S_0 = p_a\, x^{P-1} y^{Q+1} + (p_b + p_c)\, x^P y^Q + p_d\, x^{P+1} y^{Q-1}$$

determines the probable distribution of the population in the first generation. Hence the coefficient of $x^K y^{N-K}$ in the function

$$S_r = \Delta^r \to S_0$$

will give the probability for the distribution of K members of the x class and $N-K$ members of the y class in the rth generation.

Now consider the function

$$\Sigma = S_0 + e\, S_1 + e^2 S_2 + \cdots,$$

which converges uniformly for $x, y \leqq 1$ and $e < 1$. It is, in fact, a homogeneous polynomial of degree N in x, y with coefficients which are functions of e, that is,

$$\Sigma = \sum_{n=0}^{N} A_n(e)\, x^n y^{N-n} \; . \tag{16.4}$$

It is immediately seen from the definitions that Σ satisfies the following equation

$$\Sigma - e\, \Delta \rightarrow \Sigma = S_0 \; . \tag{16.5}$$

If Σ can be found by the inversion of this equation, then the probable distribution of the population in the rth generation will be given by the coefficient of e^r.

We accordingly seek an explicit determination for the functions $A_n(e)$. Hence substituting (16.4) in (16.5) and equating coefficients, we obtain the system

$$\begin{aligned} A_n - \lambda\, \{A_{n-1}(n-1)(m+1) &+ A_n[n(n-1) + m(m-1)] \\ &+ A_{n+1}(n+1)(m-1)\} = \delta_{Pn} \; , \quad (n = 0, 1, \cdots, N) \; , \end{aligned} \tag{16.6}$$

where we abbreviate $\lambda = e/[N(N-1)]$, $m = N-n$, and δ_{Pn} is the Kronecker symbol.

The determinant of this system, which we represent by $D(\lambda)$, can be factored into rational factors, and will be found to reduce to the following product

$$D(\lambda) = \prod_{k=0}^{N} (1 - \lambda q_k) \; ,$$

where we abbreviate $q_k = (N+k-1)(N-k)$.

Solving system (16.6) by Cramer's rule for $A_s(e)$, we obtain

$$A_s(e) = C_s(\lambda)/D(\lambda) \; , \tag{16.7}$$

where $C_s(\lambda)$ is the determinant $D(\lambda)$ in which the s column has been replaced by the right-hand members of (16.6), that is, by δ_{tP}.

Expanding the right-hand member of (16.7) into a series of partial fractions, we obtain the following explicit solution

$$A_s(e) = \sum_{k=1}^{N} B_{ks}/(1 - \lambda\, q_k) \; , \tag{16.8}$$

where we abbreviate

$$B_{ks} = \frac{C'_s(1/q_k)}{\prod_{j \neq k}(1 - q_j/q_k)} \; ,$$

in which $C'_s(\lambda)$ denotes $C_s(\lambda)$ with the factor $1 - \lambda\, N(N-1)$ removed.

Expanding the partial fractions of (16.8) into series in λ and replacing λ by $e/[N(N-1)]$, we obtain the desired expression for $A_s(e)$, namely,

$$A_s(e) = \sum_{r=0}^{\infty} \sum_{k=1}^{N} a_{ks}(r) e^r \; , \tag{16.9}$$

where we abbreviate

$$a_{ks}(r) = B_{ks}\{q_k/[N(N-1)]\}^r \; .$$

The quantity

$$a_s(r) = \sum_{k=1}^{N} a_{ks}(r)$$

is the probability that the population, originally in the state of P of the x class and $N - P$ of the y class, has s members of the x class and $N - s$ members of the y class in the rth generation.

PROBLEMS

1. Assuming that two planets are moving in circular orbits, which are inclined at an angle J to one another, compute three terms of the disturbing function as a power series in the ratio of their radii.

2. Compute the value of $D_3 \rightarrow N\,c(\mu)$.

3. Evaluate the equations (16.3) on the assumption that the vectors satisfy the equations

$$\operatorname{curl} E_0 = E_0 \; , \quad \operatorname{curl} M_0 = M_0 \; .$$

4. Determine the probability, $E(r)$, that in a population of N members, all different, the members of the rth generation will be descendents of a single one of the original members. (Rawles).

Hint: Let $P = 1$ and note that $E(r) = N\,a_N(r)$, since $E(r)$ is N times the probability that the rth generation is descended from a particular one of the original members. Show that

$$B_{kN} = (-1)^{k-1} \frac{[(N-1)!]^2\,(2k-1)}{(N-k)!\,(N+k-1)!} ,$$

and hence that

$$a_N(r) = \sum_{k=1}^{N} B_{kN} \left[\frac{(N+k-1)\,(N-k)}{N(N-1)} \right]^r .$$

5. Prove that the population tends to a state in which all the members are descended from one of the original members. (Rawles)

Hint: Show that $\lim_{r=\infty} E(r) = 1$.

6. Prove that the mean number of generations for a population to reach the state in which it consists of descendents of one of the original members is $(N-1)^2$. (Rawles)

Hint: Compute $\sum_{r=1}^{\infty} r[E(r) - E(r-1)]$.

CHAPTER V

Grades Defined by Special Operators

1. Definition. In much of the work which has preceded we have been concerned with formal aspects of our subject. It is now our purpose to introduce a concept which is deeply imbedded in the difficult convergence problem of the theory of operators, i. e., the concept of the *grade* (*Stufe*) of a function defined by a linear operator.*

By the *grade* of the function $f(x)$ as it is related to the operator S we shall mean the limit,

$$L = \limsup_{n \to \infty} L_n \ ,$$

where we define,

$$L_n = |S^n \to f(x)|^{1/n} \ .$$

As an example let us consider the Volterra transformation,

$$S = \int_a^x K(x,t) \ e^{(t-x)z} \, dt \ .$$

It is well known that if $f(x)$ is a function integrable and bounded in the interval $a \leq x \leq b$, then the following inequality holds:

$$|S^n(f)| \leqq F \ K^n (b-a)^n/n \ ! \ ,$$

where F is the maximum value of $f(x)$ in the interval and K the maximum value of $K(x,t)$ in its triangle of definition.†

We are thus able to infer that $L = 0$ and hence that the series,

$$S^0(f) + \lambda \, S(f) + \lambda^2 S^2(f) + \lambda^3 S^3(f) + \cdots + \lambda^n S^n(f) + \cdots$$

is an entire function of λ of genus not greater than one.

Similarly, if we consider the Fredholm transformation

$$S = \int_a^b K(x,t) \ e^{(t-x)z} \, dt \ ,$$

we know that the series

$$1 + \lambda \, S(f) + \lambda^2 S^2(f) + \lambda^3 S^3(f) + \cdots + \lambda^n S^n(f) + \cdots$$

*The term *grade* as a translation of the original designation *Stufe* was suggested to the author by J. D. Tamarkin. The limit L has also been called *degree* and *exponential value,* the latter being used by I. Sheffer [See *Bibliography*: *Sheffer* (3)].

†For this inequality see the author's study: *The Theory of the Volterra Integral Equation of Second Kind.* Indiana University Studies, (1930), p. 9.

converges only when $\lambda < |\lambda_0|$, where λ_0 is the smallest zero of $D(\lambda)$.*
We are thus able immediately to infer that $L = 1/|\lambda_0|$.

Of special concern to us will be the elementary operators $1/z$ and z. Clearly the first of these is a special case of the Volterra transformation and hence for it we have *the simple result that* $L = 0$. For the second operator, however, the matter is quite different as we can see from the following special cases:

$$(1) \quad f(x) = e^{ax} , \qquad L = a \; ;$$

$$(2) \quad f(x) = \text{a polynomial} , \quad L = 0 \; ;$$

$$(3) \quad f(x) = 1/x , \qquad L = \infty .$$

We shall discuss these results more thoroughly in the next section.

2. The Grade of an Unlimitedly Differentiable Function. O. Perron in 1921 (see *Bibliography*) stated several theorems relating to the grades of unlimitedly differentiable functions, by which we mean the limit,

$$L = \limsup_{n-\infty} L_n ,$$

where we define, $L_n = |f^{(n)}(x)|^{1/n}$.

From the examples which we have given in the preceding section it is clear that L may be either finite or infinite. In the former case $f(x)$ has been called a function of exponential type.†

If L is infinite or zero we shall indicate the mode of approach of L_n to these values by means of the customary symbol $L_n = O[\varphi(n)]$, where $\varphi(n)$ is a positive function that approaches ∞ in the first case or 0 in the second.

It will appear that the analytic properties of $f(x)$ can be characterized in a measure by means of the function $\varphi(n)$. To show this let us expand $f(x)$ about the point $x = a$,

$$f(x) = c_0 + c_1(x—a) + c_2(x—a)^2 + \cdots + c_n(x—a)^n + \cdots ,$$

where $c_n = f^{(n)}(a)/n!$. The assumption that $f(x)$ is an *entire function* is equivalent to the assumption that

$$\lim_{n=\infty} F_n = |f^{(n)}(a)/n!|^{1/n} = 0 .$$

Using Stirling's formula, $n! \backsim n^n e^{-n}(2\pi n)^{\frac{1}{2}}$, this condition becomes, $\lim_{n=\infty} F_n \backsim e\, L_n/(2\pi)^{1/2n} n^{1+1/2n} = 0$, from which we conclude that

*See chapter 1, section 10.

†This term is due to G. Pólya: Analytische Fortsetzung und konvexe Kurven. *Mathematische Annalen*, vol. 89 (1923), pp. 179-191. Functions of exponential type have been specially studied by R. D. Carmichael: Functions of Exponential Type, *Bulletin Amer. Math. Soc.*, vol. 40 (1934), pp. 241-261.

if $f(x)$ is an entire function, then $L_n = O[\varphi(n)]$, where $\varphi(n)$ is of lower order than n.

Similarly if $f(x)$ is an analytic function expansible about $x = a$ within a circle of radius ϱ on the circumference of which it has a branch point, pole, or essential singularity we have from Cauchy's theorem,

$$\lim_{n \to \infty} F_n = |f^{(n)}(a)/n!|^{1/n} = 1/\varrho .$$

Making use of Stirling's formula and noting the constant limit we infer that $L_n = O(n)$.

We also note that if $f(x)$ is analytic within a circle of radius r and center a, and if $|f(x)| < M$ on the circumference of the circle, then by Cauchy's integral formula

$$f^{(n)}(x) = \frac{n!}{2\pi i} \int_c \frac{f(t)}{(t-x)^{n+1}}\, dt$$

we shall have

$$|f^{(n)}(a)| < n!\, M/r^n . \qquad (2.1)$$

This is known as *Cauchy's inequality.*

The class of quasi-analytic functions is characterized similarly by the fact that $\sum_{n=k}^{\infty} 1/\varphi(n)$ is divergent. We obtain the k-th class of Denjoy, for example, if we have $L_n = O(n \log n \log_2 n \cdots \log_k n)$ where $\log_2 n = \log \log n$, $\log_3 n = \log \log \log n$, etc.*

In order to sharpen our criterion for the case of entire functions let us recall the definitions of Laguerre and Borel. If in the function,

$$f(x) = e^{Q(x)} \prod_{n=1}^{\infty} (1 - x/a_n)\, e^{x/a_n + x^2/2a_n^2 + \cdots x^k/k\, a_n^k} ,$$

$Q(x)$ is a polynomial of degree q and k is the smallest integer for which the series, $\sum_{n=1}^{\infty} 1/|a_n|^{k+1}$ converges, then p, the larger of the two numbers q and k, is called the *genus* of the entire function $f(x)$. The smallest value ϱ for which the series $\sum_{n=1}^{\infty} 1/|a_n|^{\rho+\varepsilon}$, $\varepsilon > 0$, converges, is called the *real order* of the function $f(x)$.

From Poincaré's theorem that the quantity $f^{(n)}(0)\ \Gamma[(n + h + 1)/(p + 1)]/n!$ tends toward zero as $n \to \infty$, provided $f(x)$ is a function of genus p and h is any positive number, we are able to infer that for every entire function of genus p, $L_n = [\varphi(n)]$, where

*See A. Denjoy: *Comptes Rendus,* vol. 173 (1921), p. 1329; W. J. Trjitzinsky: *Annals of Mathematics,* vol. 30 (1928-29), pp. 526-546.

$\varphi(n) = n^{(p-1)/p}$.* For functions of genus zero this criterion becomes $a^n(L_n)^n \to 0$ for any finite value of a.†

A theorem which applies to the function,

$$g(x) = \prod_{n=1}^{\infty} (1 - x/a_n)\, e^{x/a_n + x^2/2a_n^2 + \cdots + x^k/k\, a_n^k} , \tag{2.2}$$

can be derived from the known inequality,

$$|g^{(n)}(0)/n!|^{1/n} < (n/e\sigma)^{1/\sigma} ,$$

where σ is the real order, ϱ, of the function $g(x)$, augmented by ε.

Replacing $n!$ by its Stirling's approximation we are able to infer that $L_n = O[\varphi(n)]$, where $\varphi(n) < n^{1-1/\sigma}\, e^{-1-1/\sigma}\, \sigma^{-1/\sigma}$.

For convenience these results may be summarized as follows:

(1) If $f(x)$ is an entire function of genus p, then $L = O[\varphi(n)]$, where $\varphi(n) = n^{(p-1)/p}$.

(2) If $f(x)$ is an entire function of genus zero, then the limit $a^n(L_n)^n \to 0$ for any finite value of a.

(3) If $f(x)$ is an entire function of form (2.2) and $\sigma = \varrho + \varepsilon$, where ϱ is the real order of $f(x)$, then we have the inequality, $\varphi(n) < n^{1-1/\sigma}\, e^{-1-1/\sigma}\, \sigma^{-1/\sigma}$.

(4) If $f(x)$ is an analytic function with a regular singularity in the finite plane, then $\varphi(n) = n$.

(5) If $f(x)$ is quasi-analytic of Denjoy class k, then $\varphi(n) = n \log n \log_2 n \cdots \log_k n$, where $\log_2 n = \log\log n$, $\log_3 n = \log\log\log n$, etc.

PROBLEMS

1. Determine the grades of the following functions: $\cosh x$, $\log x$, x^x, $\Gamma(x)$, $x^{\frac{1}{2}}$, e^{x^2}, $e^{\log x}$, $e^{-\log x}$, $e^{m \log x}$, $1/\Gamma(x)$, $\sec x$, $J_0(x)$, $\cos(\sqrt{x})$.

2. Discuss the grade of the function

$$u(x) = \int_a^b e^{-xt}\, g(t)\, dt ,$$

where a and b are finite and $g(t)$ is a function of finite grade.

3. Discuss the grade of the function

$$u(x) = \int_0^{\infty} e^{-xt}\, g(t)\, dt ,$$

where $g(t)$ is a function of finite grade.

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*See É Borel: *Leçons sur les fonctions entières*. Paris, (1921), Chapter III; also Poincaré: *Bull. de la Société Mathématique de France*, vol. 11 (1882-83), pp. 136-144.

†For an independent proof of this see Ritt: [*Bibliography*, Ritt (1)], p. 34.

4. Let $b_0, b_1, b_2, \cdots$ be a set of complex numbers such that

$$\lim_{n\to\infty} |(b_n)^{1/n}| = 0 .$$

If then we have

$$F(z) = b_0 - \frac{b_1}{1!} z + \frac{b_2}{2!} z^2 - \frac{b_3}{3!} z^3 + \cdots$$

prove that $F(z) \to f(x)$ is a regular analytic function in the entire region of existence of the analytic function $f(x)$. [See G. Pólya: *Bibliography*, p. 191.]

5. Prove the following theorem:

If in the integral

$$u(x) = \int_L e^{xt} Y(t)\, dt$$

L is a Cauchy circuit about the pole $t = a$ of $Y(t)$, then the grade of $u(x)$ is equal to $|a|$; if L is a Pochhammer circuit (see section 4, chapter 8) about the two points $t = a$ and $t = b$, then the grade of $u(x)$ is equal to the larger of the two numbers $|a|$ and $|b|$; if L is a Laurent circuit (see section 7, chapter 2), then the grade of $u(x)$ is infinite.

6. If the function

$$f(x) = f_0 + f_1 x/1! + f_2 x^2/2! + f_3 x^3/3! + \cdots$$

is of finite grade q, show that the series

$$g(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots$$

converges uniformly in the circle $|x| < 1/q$.

3. Functions of Finite Grade. In many of the important applications of the theory of operators it is convenient to assume that the functions considered are of finite grade which we shall designate by the letter q. We shall state below a number of theorems relating to such functions, the first four of which are essentially due to Perron.

Theorem 1. If $f(x)$ is of grade q, then its mth *derivative, $f^{(m)}(x)$, and its* mth *integral $\int_a^x \cdots \int_a^x f(t)\, dt^m$ are of grade q.*

Theorem 2. If $f(x)$ is of grade q, then $f(ax)$ is of grade $|a|q$.

Proof: Since $d^n f(ax)/dx^n = a^n f^{(n)}(ax)$, it follows that

$$\lim_{n=\infty} |d^n f(ax)/dx^n|^{1/n} = |a| q .$$

Theorem 3. If $f_1(x)$ and $f_2(x)$ are of grades q_1 and q_2 respectively, where $q_2 \geqq q_1$ then $a_1 f_1(x) + a_2 f_2(x)$ will be of grade not greater than q_2.

Proof: Since $f_1(x)$ and $f_2(x)$ are of grades q_1 and q_2, $q_2 > q_1$, there will exist a positive function $M(n)$, such that $\lim_{n=\infty} [M(n)]^{1/n} = 1$, for which the following inequalities hold:

$$|f_1^{(n)}(x)| < M(n) q_1^n \ , \ |f_2^{(n)}(x)| < M(n) q_2^n \ .$$

From this we obtain the inequality,

$$|a_1 f_1^{(n)}(x) + a_2 f_2^{(n)}(x)| < M(n) \ q_2^n (|a_1| + |a_2|) \ ,$$

and hence establish the theorem.

Theorem 4. If $f_1(x)$ is of grade q_1 and $f_2(x)$ of grade q_2 then $f_1(x) f_2(x)$ is at most of grade $q_1 + q_2$.

Proof: By the rule of Leibnitz we have

$$[f_1(x) f_2(x)]^{(n)} = f_1^{(n)} f_2 + {}_nC_1 f_1^{(n-1)} f_2' + {}_nC_2 f_1^{(n-2)} f_2'' + \cdots \ ,$$

where ${}_nC_m$ is the mth binomial coefficient.

Since f_1 and f_2 are of grades q_1 and q_2 respectively, they satisfy the inequalities stated in the proof of theorem 3.

Hence we get,

$$|[f_1(x) f_2(x)]^{(n)}| < M^2(n) (q_1 + q_2)^n \ ,$$

and from this inequality the theorem is at once derived.

That the grade $q_1 + q_2$ may actually be attained by the product function is seen from the example, $f_1 = \sin x$, $f_2 = \cos x$, where $q_1 = q_2 = 1$. Then we have $f_1 f_2 = \sin x \cos x = \frac{1}{2} \sin 2x$, for which $q = 2$.

Theorem 5. If $f(x)$ is of grade q, then there will exist positive constants M, ε and n', all independent of n, such that

$$|f^{(n)}(x)| < M(q + \varepsilon)^n \ ,$$

for all values of n greater than n'.

Proof: Since by hypothesis $\limsup_{n=\infty} |f^{(n)}(x)|^{1/n} = q$, there exists a positive function $M(n)$, $\lim_{n=\infty} [M(n)]^{1/n} = 1$, such that

$$|f^{(n)}(x)| \leq M(n) q^n \ .$$

But since $\lim_{n=\infty} [M(1 + \varepsilon/q)^n]^{1/n} = 1 + \varepsilon/q > 1$, this function for a properly chosen value of M will dominate $M(n)$ when n exceeds n'. The theorem is a consequence of this fact.

Theorem 6. If $f(x)$ is a function of grade q and if $F(z)$ is an operator with constant coefficients which possesses a Laurent expansion about $z = 0$ within an annulus that contains $z = q$ within it or

upon its interior boundary, then the function $F(z) \to f(x)$ is at most a function of grade q.

Proof: Expanding $F(z)$ in its Laurent series in the annulus described we shall have,

$$F(z) = (a_0 + a_1 z + a_2 z^2 + \cdots) + (b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n} + \cdots) \ .$$

Operating upon the function $F(z) \to f(z)$ with z^n and taking the absolute value of the result we obtain,

$$|z^n \to \{F(z) \to f(x)\}| = |F(z) \to \{z^n \to f(x)\}| \tag{3.1}$$
$$\leqq \{|a_0 f^{(n)}| + |a_1 f^{(n-1)}| + \cdots\} |b_1 f^{(n-1)}| + |b_2 f^{(n-2)}| + \cdots + |b_n f|$$
$$+ \left| \int_a^x [b_{n+1} + b_{n+2}(x-t) + b_{n+3}(x-t)^2/2! + \cdots] f(t)\,dt \right| \ .$$

From theorem 1 we know that if x be restricted to some interval (ab) we can find a sequence of positive numbers M_n, $n = 0, 1, 2, \cdots$, such that $\lim_{n\to\infty} [M_n]^{1/n} = 1$, for which the following inequality holds:

$$|f^{(n)}(x)| \leqq M_n q^n \ .$$

Let us assume, moreover, that $M_n \leqq M_{n+1}$ so that the sequence $\{M_n\}$ is non-decreasing and thus has no null elements. Under this condition we know that $\lim_{n=\infty} [M_n]^{1/n} = 1$ implies the limit $\lim_{n=\infty} M_{n+p}/M_n = 1$ for every value of p.

Replacing the various terms of (3.1) by these dominating values and using the abbreviation $m_p(n) = M_{n+p}/M_n$, we then get,

$$|z^n \to \{F(z) \to f(x)\}| \leqq q^n M_n [I_1(n) + I_2(n)] + I_3(n) \ ,$$

where we abbreviate,

$$I_1(n) = \sum_{r=0}^{\infty} |a_r| m_r(n) q^r \ , \quad I_2(n) = \sum_{r=1}^{n} |b_r| m_{-r}(n) q^{-r} \ ,$$
$$I_3(n) = \int_a^x \sum_{m=0}^{\infty} |b_{n+m+1}| (x-t)^m f(t)\,dt/m! \ .$$

From the condition, $\lim_{n=\infty} m_r(n) = 1$, which holds for every value of r, we are able to conclude that $m_r(n)$ is dominated by $K\,q^{\delta r}$, where K is a constant independent of n and r, and δ is an arbitrarily small positive quantity. Since, moreover, q is an *interior point* of the annulus of convergence of the Laurent expansion of $F(z)$, the series

$$I_1 = K \sum_{r=1}^{\infty} |a_r|\ q^{(1+\delta)r}$$

converges and we have the inequality, $I_1(n) \leqq I_1$, for all values of n.

Moreover we obtain from $M_n \leqq M_{n+1}$, the inequality $m_{-r} \leqq 1$, $0 \leqq r \leqq n$, and hence we conclude that

$$I_2(n) \leqq \sum_{r=1}^{\infty} |b_r| \, q^{-r} = I_2 \ ,$$

for all values of n.

Finally, since $\lim_{r\to\infty} |b_r| = 0$ and since $f(t)$ is bounded in (a,b), we can write,

$$I_3(n) \leqq FB \int_a^b e^{(x-t)} \, dt = I_3 \ ,$$

where $|f(t)| < F$ and $|b_r| < B$.

Taking the nth root of the inequality,

$$|z^n \to \{F(z) \to f(x)\}| \leqq q^n M_n (I_1 + I_2) + I_3 \ ,$$

and noting the inequality, $[A^n + B^n]^{1/n} < A + B$, for A and B positive, we obtain, $\lim_{n=\infty} |z^n \to \{F(z) \to f(x)\}|^{1/n} \leqq q$, from which the theorem follows as an immediate consequence.

A result of more extensive character than the one just proved may be established for operators introduced by differential equations of Laplace type. This result follows:

Theorem 7. If an operator $F(x,z)$ can be represented in the form,

$$F(x,z) = e^{-xz} \int_0^z e^{rt} \, Y(t) \, dt \ ,$$

then the function defined by $F(x,z) \to f(x)$ exists and is of grade q provided $f(x)$ is of grade q and $Y(z)$ is analytic throughout the interior of the circle $\varrho = R$, where R is greater than q.

Proof: Since $f(x)$ is of grade q it is dominated by a function of the type $P(x)\, e^{qx}$ where $P(x)$ is an entire function of genus zero. We shall, therefore, consider the operation $F(x,z) \to P(x)\, e^{qx}$.

Let us first write the identity,

$$F(x,z) \to (uv) = uF(x,z) \to v + u'F'(x,z) \to v \qquad (3.2)$$
$$+ u''F''(x,z) \to v/2! + \cdots \ ,$$

where the differentiation refers to the variable z.

Specializing this formula by setting $u = P(x)$ and $v = e^{qx}$ we get

$$F(x,z) \to Pe^{qx} = \{PF(x,q) + P'F_q'(x,q)$$
$$+ P''F_q''(x,q)/2! + \cdots\}\, e^{qx} . \qquad (3.3)$$

Employing the abbreviation $D = d/dq$ we also note that $D^n \to F(x,q) = e^{-qx}(D-x)^n \to I(x,q)$, where we set,

$$I(x,q) = \int_0^q e^{xt}\, Y(t)\, dt \ . \qquad (3.4)$$

Substituting this in (3.3) we obtain,

$$F(x,z) \to Pe^{qx} = \sum_{n=0}^{\infty} P^{(n)}(x)\,(D-x)^n \to I(x,q)/n! \ . \qquad (3.5)$$

But we see from (3.4) that $D \to I(x,q) = e^{qx}Y(q)$, $D^2 \to I(x,q) = e^{qx}(D+x) \to Y(q)$, and in general,

$$D^n \to I(x,q) = e^{qx}(D+x)^{n-1} \to Y(q) \ . \qquad (3.6)$$

Making use of this we are able to derive,

$$\begin{aligned}(D-x)^n &\to I(x,q)\\ &= e^{qx}\{(D+x)^{n-1} - nx(D+x)^{n-2} + n(n-1)x^2(D+x)^{n-3}/2!\\ &\quad - \cdots \pm nx^{n-1}\} \to Y(q) + (-1)^n x^n I(x,q)\\ &= e^{qx}\{[D^n - (-1)^n x^n]/(D+x)\} \to Y(q) + (-1)^n x^n I(x,q) \ .\end{aligned}$$

But since $Y(q)$ is analytic its nth derivative, by (4) section 2, is dominated by $M_n a^n n!$ where $\lim_{n=\infty} [M_n]^{1/n} = 1$. Hence the function $\{[D^n - (-1)^n x^n]/(D+x)\} \to Y(q)$ is dominated by $M_n A^n n!$ where A is suitably chosen. But since $P(x)$ is an entire function of genus zero we can find a set of values m_n such that $|P^{(n)}(x)| < m_n/Q^n$, where Q is arbitrary and $\lim_{n=\infty} [m_n]/^{1/n} = 1$. From this we see that the series expansion of $F(x,z) \to P(x)e^{qx} - P(0)\, I(x,q)$ is dominated by the majorant,

$$\sum_{n=0}^{\infty} M_n m_n (A/Q)^n, \ Q > A \ , \quad \lim_{n=\infty} (M_n m_n)^{1/n} = 1 \ .$$

We are thus able to conclude that (3.5) is uniformly convergent.

In order now to show that $F(x,z) \to Pe^{qx}$ is a function of grade q we observe the identity,

$$\sum_{n=0}^{\infty} P^{(n)}(x)\,(D-x)^n/n! = P(D) \ ,$$

and write (3.5) in the form,

$$F(x,z) \to Pe^{qx} = P(D) \to I(x,q) \ .$$

Making use of (3.6) we achieve the further simplification,

$$P(D) \to I(x,q) = P(0)I(x,q) + e^{qx}Q(D+x) \to Y(q) \ ,$$

where we write $Q(z) = [P(z) - P(0)]/z$.

Expanding $Q(D+x)$ as a series in D and operating upon $Y(q)$ we get,

$$Q(D+x) \to Y(q) = Q(x)Y(q) + Q'(x)Y'(q) + \cdots + Q^{(m)}(x)Y^{(m)}(q)/m! + \cdots \ .$$

Let us now operate upon this equation with z^n and discuss the result.

$$S_n(x) = z^n \to \{Q(D+x) \to Y(q)\} = Q^{(n)}(x)Y(q) + Q^{(n+1)}(x)Y'(q) + Q^{(n+2)}(x)Y''(q)/2! + \cdots + Q^{(n+m)}(x)Y^{(m)}(q)/m! + \cdots \ .$$

Since $Y(z)$ is analytic throughout the interior of the circle $\varrho = R$, $Y^{(m)}(q)$ is dominated by $M_m a^m m!$, $q \leqq a < R$, $\lim\limits_{m=0} [M_m]^{1/m} = 1$, $M_m > 0$. Moreover, since $Q(x)$ is an entire function of genus zero, $Q^{(n)}(x)$ is dominated by a sequence of positive values γ_n which has the property that $\lim\limits_{n=\infty} A^n\gamma_n = 0$ for all finite values of A. We can also assume without loss of generality that $\gamma_n \geqq \gamma_{n+1}$. Hence we attain the inequality,

$$|S_n(x)| \leq \sum_{m=0}^{\infty} M_m \gamma_{n+m} a^m \ .$$

Making use of the limiting property of the sequence $\{\gamma_n\}$ we see that this series converges and hence furnishes a Weierstrass majorant for the function $S_n(x)$ for every value of n and for x in any finite interval.

Furthermore, from the assumption $\gamma_n \geqq \gamma_{n+1} > 0$, we obtain,

$$|S_n(x)| \leq \gamma_n \sum_{m=0}^{\infty} M_m (\gamma_{n+m}/\gamma_n) a^m \ .$$

But from the limiting property of the sequence $\{\gamma_n\}$ there exists a positive quantity C, independent of n, such that $\gamma_n < C/A^n$, $A > a$. Hence we can write,

$$|S_n(x)| \leq A^n \gamma_n \sum_{m=0}^{\infty} M_m (a/A)^m \ .$$

From the limitations upon M_m this series is seen to converge and the majorant thus obtained establishes the fact that

$$S_0(x) = Q(D+x) \to Y(q)$$

is a function of genus zero in the variable x.

Combining these results we can write (3.5) in the form,

$$F(x,z) \to Pe^{qr} = P(0)I(x,q) + e^{qx}S(x) \quad , \qquad (3.7)$$

where $S(x)$ is a function of genus zero.

We now need the lemma:

Lemma. The grade of $I(x,q)$ does not exceed q.

The proof is immediately attained by the use of the Schwarz inequality, (see section 9, chapter 3)

$$\int_0^q |u^2(t)|\,dt \int_0^q |v^2(t)|\,dt \geqq \left| \int_0^q u(t)\; v(t)\; dt \right|^2 .$$

Forming the nth derivative of $I(x,q)$,

$$I^{(n)}(x,q) = \int_0^q e^{xt} t^n Y(t)\,dt \ ,$$

we specialize $u(t) = e^{xt}Y(t)$, $v(t) = t^n$, and thus obtain,

$$|I^{(n)}(x,q)| \leq \{ \int_0^q |e^{2xt}Y^2(t)|dt\}^{\frac{1}{2}}\, q^n (2n+1)^{-1/2} \ .$$

Since the integral exists by hypothesis we obtain at once the desired inequality, $\lim_{n\to\infty} [|I^{(n)}(x,q)|]^{1/n} \leqq q$.

Employing this lemma and making use of theorems 3 and 4 we are able to conclude that the grade of the function defined by (3.7) is q.

Corollary 1. If $F(x,z)$ is the operator,

$$F(x,z) = e^{-xz} \int_a^z e^{xt}Y(t)\,dt \ ,$$

then the function $F(x,z) \to f(x)$, where $f(x)$ is of grade q, is of grade not larger than the larger of the two numbers $|a|$ and q.

The proof is immediate if we write

$$F(x,z) = e^{-xz} \{ \int_a^0 e^{xt}Y(t)\,dt + \int_0^z e^{xt}Y(t)\; dt\}$$

and note that

$$e^{-xz} \int_a^0 e^{xt}Y(t)\,dt \to f(x) = f(0) \int_a^0 e^{xt}Y(t)\,dt \ .$$

The result of the lemma proved above combined with theorem 6 is a statement of the corollary.

Corollary 2. If $F(x,z)$ is defined to be the operator of theorem 6 *and if $P(x)$ is a function of genus zero, then $F(x,z) \to P(x)$ is a function of genus zero.*

4. Asymptotic Expansions. Perhaps the most characteristic difficulty which one encounters in a practical application of formal operators to the solution of linear equations is the frequent intrusion of divergent series into the calculation. We have already encountered such a difficulty in section 11 of the preceding chapter and many others will appear in the ensuing pages.

It is impossible to make an exhaustive investigation here into the fascinating but profound problems presented by divergent series. We shall find it necessary, however, to be familiar with certain aspects of what are generally referred to as *semi-convergent* or *asymptotic* series.

An asymptotic expansion of a function $f(x)$ is a series of the form,

$$a_0 + a_1/x + a_2/x^2 + \cdots + a_n/x^n + \cdots ,$$

which, although divergent, satisfies the condition that

$$\lim_{|x|\to\infty} x^n|f(x) - S_n(x)| = 0 , \quad (n \text{ fixed}) , \tag{4.1}$$

where $S_n(x)$ is the sum $a_0 + a_1/x + a_2/x^2 + \cdots + a_n/x^n$.

If this condition, which was first stated by H. Poincaré in 1886,* is satisfied we say that the series is the *asymptotic expansion* of $f(x)$ and we represent it by the symbol,

$$f(x) \backsim a_0 + a_1/x + a_2/x^2 + \cdots a_n/x^n + \cdots . \tag{4.2}$$

The first example of an asymptotic expansion encountered by mathematicians, and certainly one of the most interesting, is that associated with the gamma function,

$$e^x x^{\frac{1}{2}-x}\Gamma(x)/(2\pi)^{\frac{1}{2}} \backsim 1 + 1/12x + 1/288x^2 - 139/51840x^3 - \cdots .\dagger$$

The characteristic property of asymptotic series which brought them into prominence in astronomical investigations long before they were regarded with even moderate favor by mathematicians, was the fact that for large values of x they converged in general with great rapidity to the value of the function. The error was noticed to be a function both of the magnitude of x and of the number of terms used. Hence for values of n which did not carry one too far into the divergent expansion and for correspondingly large values of x, the error

*Sur les intégrales irrégulières des équations linéaires. *Acta Mathematica*, vol. 8 (1886), pp. 295-344; in particular, pp. 295-303.

†This expansion to $O(1/x^8)$ is given in Davis: *Tables ol the Higher Mathematical Functions*, vol. 1, p. 180.

was in many practical applications extremely small. It has become almost a rule in the computation of tables of functions to seek first the asymptotic series.

One of the most important clews to the nature of asymptotic series is the *Borel theorem* on *continuation*, which we state as follows:

If $a_1, a_2, \cdots, a_n$ are singularities of the function,

$$u(x) = \int_0^\infty e^{-t}\varphi(xt)\,dt \;, \tag{4.3}$$

where $\varphi(z) = \sum_{n=0}^{\infty} A_n z^n/n!$ and $\lim_{n\to\infty} |A_n|^{1/n}$ exists, then $u(x)$ is analytic throughout the interior of the polygon containing the origin and formed by drawing through a_i perpendiculars to the lines joining the singular points to the origin.

The proof of this theorem will not be given here since it is to be found in numerous places.*

The application of the theorem to the summation of divergent series is at once apparent from the following consideration. If the expansion of $\varphi(xt)$ is placed in (4.3) and account taken of the integral identity, $\int_0^\infty e^{-t}t^n\,dt = n!$, the following series is obtained:

$$u(x) = \sum_{n=0}^{\infty} A_n x^n \;. \tag{4.4}$$

It is clear that this expansion converges for $|x| < a$, where a is the smallest of the moduli of the singular points, $x = a_1, a_2$, etc., but diverges outside of (and possibly on) the circle of radius a. Hence, in general, equation (4.3) furnishes a functional equivalent for series (4.4) in regions where it is divergent.

For example the series,

$$u(x) = 1 + x + x^2 + x^3 + \cdots \;,$$

is divergent on and exterior to the unit circle. However, its Borel equivalent,

$$\begin{aligned} u(x) &= \int_0^\infty e^{-t}(1 + tx + t^2x^2/2! + \cdots)\,dt \\ &= \int_0^\infty e^{(x-1)t}\,dt \;, \end{aligned}$$

converges in the half plane, $R(x) < 1$, where $R(x)$ designates the real part of x.

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*E. Borel: *Leçons sur les Series Divergentes.* Paris, (1901), chapter 4; Whittaker and Watson: *Modern Analysis*, 2nd ed., Cambridge, (1920), p. 141; Davis: *Tables of the Higher Mathematical Functions.* vol. 1 (1933), pp. 45-47.

Although the Borel theorem applies only to functions for which $\varphi(z)$ is an entire function, it may be extended to include the summation of totally divergent series. Here, however, we encounter a blemish in the fact that no simple way appears for defining in general the *region of summability.*

An example is found in the classical series,

$$u(x) = 1 - x + 2!x^2 - 3!x^3 + \cdots ,$$

which is totally divergent.

The equivalent Borel integral,

$$u(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\, dt ,$$

exists throughout the half plane, $R(x) > 0$.

In this book we shall be concerned mainly with asymptotic series which appear in expansions of a Laplace integral of the following general form,

$$F(x) = \int_L e^{-x\,g(t)}\, \vartheta(t)\, dt , \tag{4.5}$$

where L is a path in the complex plane. If the path is the positive real axis from 0 to ∞, the Borel theorem is usually effective either in determining the region of summability of the asymptotic series or, if the integral representation is assumed as given, in determining the form of the asymptotic representation of the integral.

In order to effect this equivalence, let us make the transformation $s = g(t) - g$, where we abbreviate $g = g(0)$. We shall then obtain (4.5) in the following form:

$$F(x) = e^{-gx} \int_0^\infty e^{-xs}\, \Theta(s)\, ds , \tag{4.6}$$

where we write, $\Theta(s) = \vartheta(g^{-1}[s+g])/g'(g^{-1}[s+g])$, in which $g^{-1}(x)$ means the inverse of the function $g(x)$.

If we now assume that $\Theta(s)$ is a function of the form,

$$\Theta(s) = s^\mu \varphi(s) , \quad \mu > -1 ,$$

where

$$\varphi(s) = \sum_{n=0}^\infty \varphi_n s^n$$

is any function for which the integral exists, then (4.6) has the formal expansion,

$$e^{gx} F(x) \backsim [\Gamma(\mu+1)/x^{\mu+1}][\varphi_0 + (\mu+1)\varphi_1/x + (\mu+1)(\mu+2)\varphi_2/x^2 + \cdots] .$$

A still more general expansion may be obtained by writing, $\theta(s) = s^\mu \log s\, \phi(s)$. Noting the integral,

$$\int_0^\infty e^{-xs} s^\mu \log s\, ds = [\Gamma(\mu+1)/x^{\mu+1}][\Psi(\mu+1) - \log x],$$

where $\Psi(\mu+1) = \Gamma'(\mu+1)/\Gamma(\mu+1)$, we may write,

$$e^{gx} F(x) \backsim [\Gamma(\mu+1)/x^{\mu+1}][\phi_0 + \sum_{n=1}^{\infty} (\mu+1)(\mu+2)\cdots (\mu+n)\phi_n \{\Psi(\mu+n+1) - \log x\}/x^n]$$

Other generalizations are obtained by taking the nth derivative with respect to μ of the Borel integral.

Returning now to equation (4.5), we may broaden our inquiry by seeking conditions under which the path may be so chosen that the integral has an asymptotic expansion.

This problem was apparently first suggested by G. F. B. Riemann (1826-1866) and the solution indicated in a posthumous paper.* The details of its development are due to a notable contribution made by P. Debye in 1909, who applied it to the study of the asymptotic development of Bessel functions.†

In equation (4.5) let the path L be so chosen that the following conditions are satisfied. First, $R\{g(t)\}$ shall change as rapidly as possible along L; second, as t approaches infinity along L we shall have the limit, $\lim_{t=\infty} R\{g(t)\} = \infty$, where $R\{g(t)\}$ means the real part of $g(t)$.

In order to see the significance of these conditions let us write $t = u + i\,v$, and thus obtain,

$$g(t) = R(u,v) + i\,I(u,v) .$$

As is well known both $R(u,v)$ and $I(u,v)$ satisfy Laplace's equation,

$$\frac{\partial^2 R}{\partial u^2} + \frac{\partial^2 R}{\partial v^2} = 0 , \qquad (4.7)$$

*Sullo svolgimento del quoziente di due serie ipergeometriche in frazione continua infinita. A fragment edited by H. A. Schwarz from Riemann's: *Gesammelte Werke*, 2nd ed. (1892), pp. 424-430.

†Näherungsformeln für die Zylinderfunktionen für grosse Werte des Arguments und unbeschränkt veränderliche Werte des Index. *Mathematische Annalen*, vol. 67 (1909), pp. 535-558.

and the Cauchy-Riemann conditions,

$$\frac{\partial R}{\partial u}=\frac{\partial I}{\partial v}\ , \quad \frac{\partial R}{\partial v}=-\frac{\partial I}{\partial u}\ . \tag{4.8}$$

In order to determine the path along which $R\{g(t)\}$ shall change as rapidly as possible, we set the derivatives of $R(u,v)$ equal to zero,

$$\frac{\partial R}{\partial u}=\frac{\partial R}{\partial v}=0\ . \tag{4.9}$$

Then from the Cauchy-Riemann equations we shall have, $\partial I/\partial u = -\partial I/\partial v = 0$, and hence derive,

$$dI=\frac{\partial I}{\partial u}\,du+\frac{\partial I}{\partial v}\,dv=0\ .$$

Thus we find that the desired path L is the one for which,

$$I(u,v)=k\ , \tag{4.10}$$

where k is to be determined from the solution of equations (4.9). If we now consider the equation $R = R(u,v)$, we note that the points u_0, v_0, R_0 derived from (4.9) are not points of maxima or minima in view of equation (4.7), but are *saddle points* or *passes* on the surface. Hence the curve (4.10) is a curve on the surface which passes through one of the saddle points and along which $R\,g(t)$ makes its most rapid change.

Returning now to equation (4.5), let us designate by $t = t_0$ a saddle point and let us effect the transformation: $s = g(t) - G$, where we write $G = g(t_0)$. Then obviously since $g'(t_0) = 0$, we shall have $s(t_0) = s'(t_0) = 0$.

It is then possible for us to write,

$$s=(t-t)^2\{g_0+g_1(t-t_0)+g_2(t-t_0)^2+\cdots\}\ . \tag{4.11}$$

Employing the transformation $s = T^2$, we shall have $dt/ds = \frac{1}{2}T^{-1}dt/dT$ from which we proceed to evaluate dt/dT in terms of T. To accomplish this we write,

$$dt/dT=\sum_{n=0}^{\infty}a_nT^n\ , \tag{4.12}$$

and then evalute the coefficients by means of Cauchy's theorem,

$$a_n=(1/2\pi i)\int\frac{dt}{dT}\frac{dT}{T^{n+1}}\ ,$$

where the integration is around the zeros of the T-plane.

In the t-plane this is equivalent to,

$$a_n = (1/2\pi i) \int [s(t)]^{-\frac{1}{2}(n+1)}\, dt \ , \qquad (4.13)$$

the values of a_n thus being determined from the coefficients of equation (4.11). The following explicit values will be found useful in the application of this theory:

$$a_0 = g_0^{-\frac{1}{2}} \ , \quad a_1 = g_0^{-1}(-g_1/g_0) \ , \quad a_2 = g_0^{-3/2}(-3g_2/2g_0 + 3\cdot 5\, g_1^2/8g_0^2) \ ,$$

$$a_3 = g_0^{-2}(-2g_3/g_0 + 6g_1g_2/g_0^2 - 4g_1^3/g_0^3) \ , \qquad (4.14)$$

$$a_4 = g_0^{-5/2}\Big[-5g_4/2g_0 + \frac{5\cdot 7}{8}(2g_1g_3/g_0^2 + g_2^2/g_0^2) - \frac{5\cdot 7\cdot 9}{16} g_1^2g_2/g_0^3 + \frac{5\cdot 7\cdot 9\cdot 11}{24\cdot 16} g_1^4/g_0^4\Big] \ , \text{ etc.}$$

If the two values of T, $T_1 = + s^{\frac{1}{2}}$, $T_2 = - s^{\frac{1}{2}}$ are substituted in (4.12), two expressions for dt/ds are obtained as follows:

$$dt_1/ds = \tfrac{1}{2}\, s^{-\frac{1}{2}} \sum_{n=0}^{\infty} a_n\, s^{\frac{1}{2}n} \ , \quad dt_2/ds = \tfrac{1}{2}\, s^{-\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^{n+1} a_n s^{\frac{1}{2}n} \ .$$

Returning to the original integral we are now able to write it in the form,

$$F(x) = e^{-Gx} \Big\{ \int_{L_1} e^{-xs}\, \vartheta(t_1) \frac{dt}{ds}\, ds + \int_{L_2} e^{-xs}\, \vartheta(t_2) \frac{dt}{ds}\, ds \Big\} \ .$$

where L_1 and L_2 are the two branches of the path L in the s-plane, these branches being separated by the saddle point. Since s is real over the path $L_1 + L_2$, it is seen that the integral can, in general, be expanded by the Borel theorem.

As an example let us consider the expansion of the integral,

$$F(x) = \int_L e^{-x \cosh t}\, dt \ .$$

Setting $g(t) = \cosh t$, we obtain the definitive equation,

$$g'(t) = \sinh t = 0 \ ,$$

from which we get $t_0 = n\pi i$, $n = 0, \pm 1, \pm 2, \cdots$. Selecting the value $t_0 = 0$, we then have,

$$s = \cosh t - 1 = t^2(1/2! + t^2/4! + t^4/6! + \cdots) \ .$$

We then compute the values,

$$a_0 = 2^{\frac{1}{2}} \ , \quad a_2 = -2^{\frac{1}{2}}/4 \ , \quad a_4 = 2^{\frac{1}{2}}\cdot 3/32, \cdots \ , \quad a_1 = a_3 = \cdots = 0 \ .$$

The path of integration is seen to be the axis of reals and from the condition $\lim_{t=\infty} R[g(t)] = \infty$, it is clearly the circuit from infinity about the point $t_0 = 0$. Hence we get,

$$F(x) = e^{-x} \int_0^\infty e^{-xs} \{ (dt_1/ds) - (dt_2/ds) \} \, ds \ ,$$

$$= e^{-x} \int_0^\infty e^{-xs} s^{-\frac{1}{2}} (a_0 + a_2 s + a_4 s^2 + \cdots) \, ds \ .$$

When the Borel integral is employed, this becomes,

$$F(x) \backsim e^{-x} \{\Gamma(\tfrac{1}{2})/x^{\frac{1}{2}}\} \{a_0 + a_2/2x + (1 \cdot 3) a_4/2^2 x^2 + \cdots \} \ ,$$

$$\backsim e^{-x} (2\pi/x)^{\frac{1}{2}} [1 - 1/(8x) + 9/(128x^2) + \cdots] \ .$$

If more terms of this series are desired, we return to equation (4.13), which in the present instance becomes,

$$a_n = (1/2\pi i) \int_L (\cosh t - 1)^{-\frac{1}{2}(n+1)} \, dt \ ,$$

$$= (1/2\pi i) \int_L 2^{\frac{1}{2}} (1 + \tfrac{1}{2} s)^{-\frac{1}{2}} \frac{ds}{s^{m+1}} \ ,$$

where we have written $n = 2m$. Since the path is about $s = 0$, it is clear that the values of a_{2m} are merely the coefficients of s^m in the development in power series about $s = 0$ of the function $2^{\frac{1}{2}} (1 + \frac{1}{2} s)^{-\frac{1}{2}}$.

The reader will notice that $F(x)$ is twice the Bessel function $K_0(x)$.

PROBLEMS

1. Derive the expansion

$$\int_x^\infty e^{-t^2} \, dt \backsim e^{-x^2} \left(\frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1 \cdot 3}{2^3 x^5} - \frac{1 \cdot 3 \cdot 5}{2^4 x^7} + \cdots \right) .$$

2. Determine the asymptotic expansion of the gamma function by considering the integral

$$F(x) = \int_C e^{-s} x^x \, ds = x^{x+1} e^{-x} \int_{C'} e^{-x(t-1-\log t)} \, dt \ .$$

Show that the saddle point is the point $t = 1$, and that the path of integration is the axis of reals consisting of the two branches $(0,1)$ and $(1,\infty)$.

[See G. N. Watson: An Expansion Related to Stirling's Formula, derived by the Method of Steepest Descents. *Quarterly Journal of Math.*, vol. 48 (1920), pp. 1-18. Also H. T. Davis: *Tables of the Higher Mathematical Functions*, vol. 1, (1933), pp. 56-57.]

3. Prove that

$$\psi(x) \backsim \log x - \frac{1}{2x} - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_n}{2n \cdot x^{2n}} ,$$

where B_n is the nth Bernoulli number and $\psi(x)$ is defined by

$$\psi(x) = \log x + \int_0^{\infty} e^{-tx} \left\{ \frac{e^t}{1-e^t} + \frac{1}{t} \right\} dt .$$

4. Show that

$$I(x) = \int_0^{\infty} \frac{e^{-t}}{x+t} dt \backsim \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots .$$

If $S_n(x)$ is the sum of n terms of the series, prove that if $x > 2n$, then

$$|I(x) - S_n(x)| < 1/(2^{n+1} n^2) \quad ;$$

also prove that

$$\lim_{x \to \infty} |x^n \{I(x) - S_n(x)\}| = 0 .$$

5. Find the asymptotic form of the solution of the equation

$$u(x) = -(1/2x) + \tfrac{1}{2} \int_x^{\infty} e^{(x-t)} u(t) dt .$$

6. Given the equation

$$\Delta u(x) = \log x ,$$

show that the asymptotic form of the solution to a first approximation is given by

$$u(x) \backsim k + (x - \tfrac{1}{2}) \log x - x + \frac{2}{x} \sum_{n=1}^{\infty} \frac{1}{(2\pi n)^2} .$$

If $k = \frac{1}{2} \log 2\pi$, show that $u(x) = \log \Gamma(x)$.

7. Derive the following expansions

$$\int_x^{\infty} \frac{\cos t}{(t)^{\frac{1}{2}}} dt = \frac{1}{(x)^{\frac{1}{2}}} (-A \sin x + B \cos x) ,$$

$$\int_x^{\infty} \frac{\sin t}{(t)^{\frac{1}{2}}} dt = \frac{1}{(x)^{\frac{1}{2}}} (A \cos x + B \sin x) ,$$

where we have

$$A \backsim 1 - \frac{1 \cdot 3}{(2x)^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{(2x)^4} - \cdots$$

$$B \backsim \frac{1}{2x} - \frac{1 \cdot 3 \cdot 5}{(2x)^3} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{(2x)^5} - \cdots .$$

Hint: Consider the integral

$$\int_x^{\infty} [e^{it}/(t)^{\frac{1}{2}}] \, dt ,$$

and make the transformation $t = x(s+1)$.

5. The Summability of Differential Operators with Constant Coefficients. In theorem 6 of section 3 we considered the case where $F(z)$, a differential operator with constant coefficients, operated upon a function of finite grade. It is clear, however, that this restriction excludes from consideration a large class of problems of which the equation which defines the psi function, $\psi(x)$, may be cited as typical:

$$u(x+1) - u(x) = 1/x \ , \tag{5.1}$$

or in terms of a differential operator,

$$(e^z - 1) \to u(x) = 1/x \ .$$

Since $1/x$ is a function of infinite grade it is certainly to be anticipated that, in general, an operation upon it will yield another function of infinite grade. As is well known, the principal solution of equation (5.1), namely $\psi(x)$, appears as the following divergent series:

$$\psi(x) \backsim \log x - 1/2x - \sum_{n=1}^{\infty} (-1)^{n-1} B_n / 2n \cdot x^{2n} \ , \tag{5.2}$$

where B_n are the Bernoulli numbers (see section 12, chapter 2). This series, however, is summable by the method of Borel and we are thus able to obtain the following integral equivalent of (5.2):

$$\psi(x) = \log x + \int_0^\infty e^{-tx} \{e^t/(1-e^t) + 1/t\}\, dt \ , \qquad R(x) > 0 \ .$$

The situation that is here revealed is covered in the following theorem:

Theorem 8. If $f(x)$ is of the form

$$f(x) = g(x) + h(x) \ ,$$

where $g(x)$ is a function of finite grade L and where $h(x)$ is of the form

$$h(x) = h_1/x + h_2/x^2 + h_3/x^3 + \cdots \ ,$$

then the summable equivalent of the function,

$$u(x) = F(z) \to f(x) \ ,$$

in which $F(z)$ is a differential operator with constant coefficients, exists and has the form

$$u(x) = \int_0^\infty e^{-xt} Q(t) \cdot F(-t)\, dt + F(z) \to g(x) \ ,$$

where

$$Q(t) = h_1 + h_2 t + h_3 t^2/2! + h_4 t^3/3! + \cdots \ , \tag{5.3}$$

provided, (a) $L < \varrho$, *where* ϱ *is the radius of convergence of* $F(z)$*;* (b) *positive values* k, A, *and* m *exist such that* $|Q(t)\cdot F(-t)| < Ae^{mt}$, *for* $0 < k \leq t \leq \infty$*; and* (c), $Q(t)\cdot F(-t)$ *is of limited variation in the interval* $0 \leq t \leq k$.

Proof: It will be clear that the series $\sum_{i=1}^{\infty} b_i L^i$ forms a majorant for the series

$$U(x) = F(z) \to g(x) = \{b_0 + b_1 z + b_2 z^2 + \cdots\} \to g(x) ,$$

since by hypothesis L is the grade of $g(x)$ and is smaller than the radius of convergence of the series expansion of $F(z)$. Since $U(x)$ is thus uniformly convergent we may form the derivative

$$U^{(n)}(x) = F(z) \to z^n \to g(x) . \tag{5.4}$$

From theorem 1, section 3 we know that $z^n \to g(x)$ is of grade L and thus that the series $L^n \sum_{i=1}^{\infty} b_i L^i$ forms a majorant for $U^{(n)}(x)$. It thus appears that $U(x)$ is at most of degree L.

It will be observed that this argument is merely a special case of the one employed in the proof of theorem 6, the annulus in this instance being merely the circle about the origin.

We now consider the equation

$$V(x) = F(z) \to h(x) .$$

Applying the explicit expansion of the resolvent to $h(x)$ we get

$$\begin{aligned} V(x) &= h_0\{b_0/x - b_1/x^2 + 2!b_2/x^3 - 3!b_3/x^4 + \cdots\} \\ &+ h_2\{b_0/x^2 - 2!b_1/x^3 + 3!b_2/x^4 - 4!b_3/x^5 + \cdots\} \\ &+ h_3\{b_0/x^3 - 3!b_1/2!x^4 + 4!b_2/2!x^5 - 5!b_3/2!x^6 + \cdots\} \\ &+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}$$

In general this series will be divergent, but it is usually summable by the method of Borel. This makes use of the identity

$$\int_0^\infty e^{-s} s^n ds = n!$$

from which we obtain $V(x)$ in the form

$$V(x) = (1/x) \int_0^\infty [e^{-s}\{h_1 + h_2(s/x) + h_3(s/x)^2/2! + \cdots\}\cdot F(-s/x)]ds .$$

Making the substitution $s = tx$, we obtain

$$V(x) = \int_0^\infty e^{-xt} Q(t) \cdot F(-t)\, dt \ ,$$

where $Q(t)$ is defined by (5.3).

The convergence of this result is easily established under hypotheses (b) and (c) stated in the theorem for we shall then have

$$|V(x)| < M/x + A \int_0^\infty e^{(m-x)t} dt = M/x + A/(x-m) \quad \text{for } R(x) > m \ .$$

The case where $Q(t) \cdot F(-t)$ has a pole of unit order in the interval $(0, \infty)$ is easily disposed of as follows:

Consider the function

$$G(t) = Q(t)\, F(-t) - R/(a-t) \ ,$$

where R is the residue of $Q(t) \cdot F(-t)$ at the point $t = a$. It is clear that $G(t)$ is regular at $t = a$ so we may consider the function

$$V(x) = \int_0^\infty e^{-tx} G(t)\, dt + \int_0^\infty [e^{-tx} R/(a-t)]\, dt \ . \qquad (5.5)$$

The second integral is divergent, but is seen to correspond formally for the case $m = 1$ to the solution of the equation,

$$(a - d/dx)^m \to W(x) = R/x \ ,$$

which has for its particular integral the function

$$W(x) = (-1)^m R \int_k^x \{e^{a(x-t)} (x-t)^{m-1}/t(m-1)!\}\, dt \ .$$

More generally, if on the positive real axis $a_1, a_2, \cdots, a_m$ are poles of unit order of the function $Q(t)/F(-t)$, then the solution of the original equation will be

$$V(x) = \int_0^\infty e^{-xt} G(t)\, dt - \sum_{i=1}^m R_i \int_{k_i}^x \{e^{a_i(x-t)}/t\} dt \ , \qquad (5.6)$$

where the R_i are the residues of $Q(t) \cdot F(-t)$ at the points a_i and

$$G(t) = Q(t) \cdot F(-t) - \sum_{i=1}^m R_i/(a_i - t) \ .$$

The case of poles of multiplicity m is treated in similar fashion by writing $G(t)$ in the form,

$$G(t) = Q(t) \cdot F(-t) - \sum_{m=1}^m A_m/(a-t)^m$$

where the A_m are the Laurent coefficients in the expansion of $Q(t)/F(-t)$. One then adds to the integral,

$$\int_0^\infty e^{-tx} G(t)\,dt \ ,$$

m functions obtained from $W(x)$ by letting m assume the values 1, 2, 3, $\cdots$, m and replacing R by A_m.

We shall consider a few simple examples:

Example 1. Let us assume the functions,

$$F(z) = 1 + z + z^2 + \cdots = 1/(1-z) \ ,$$

and

$$h(x) = 1/x - 1/x^2 + 1/x^3 - \cdots \ .$$

We then compute, $Q(t) = e^{-t}$, and hence obtain,

$$F(z) \to h(x) = 1/x - 2/x^2 + 5/x^3 - 16/x^4 + 65/x^5 - \cdots$$

$$= \int_0^\infty \{e^{-(x+1)t}/(1+t)\}\,dt \ .$$

Example 2. In the preceding example let us replace $h(x)$ by the following totally divergent series,

$$h(x) = 1/x - 1/x^2 + 2!/x^3 - 3!/x^4 + \cdots \ .$$

We then compute, $Q(t) = 1/(1+t)$, and hence get,

$$F(z) \to h(x) = 1/x - 2!/x^2 + 3!/x^3 - 4!/x^4 + \cdots \ ,$$

$$= \int_0^\infty \{e^{-xt}/(1+t)^2\}\,dt \ .$$

Example 3. Let us assume, $F(z) = 1/(e^z - 1)$, $Q(t) = 1$, $h(x) = 1/x$. We note that $F(-t)$ has a unit pole at $t = 0$, so we compute the residue at this point and hence construct the function,

$$G(t) = 1/(e^{-t}-1) + 1/t = e^t/(1-e^t) + 1/t \ .$$

Now employing formula (5.6) for the case $m = 1$, we readily obtain,

$$F(z) \to h(x) = \{1/z - 1/2 + B_1 z/2! - B_2 z^3/4! + B_3 z^5/6! - \cdots\} \to 1/x \ ,$$

$$= c + \log x + (1/x)\{-\tfrac{1}{2} + \sum_{p=1}^{\infty} (-1)^p B_p/2p x^{2p-1}\}$$

$$= \int_0^\infty e^{-xt}\{e^t/(1-e^t) + 1/t\}\,dt + \int_k^x dt/t, \ , \ R(x) > 1$$

$$F(z) \to h(x) = c + \log x + \int_0^\infty e^{-xt}\{e^t/(1-e^t) + 1/t\}\, dt \ ,$$
$$= c + \psi(x) \ ,$$

where B_n are the Bernoulli numbers.

The theorem which we have just proved may be generalized in a significant manner by introducing the function $f(x) = g(x)h(x)$ instead of $f(x) = g(x) + h(x)$. This theorem may be stated as follows:

Theorem 9. Under the following assumptions: (a) $f(x) = g(x)h(x)$, *where* $g(x)$ *is a function of finite grade* g *and* $h(x)$ *is of the form,* $h(x) = h_1/x + h_2/x^2 + h_3/x^3 \cdots$; (b) *the functions* $Q(t) = h_1 + h_2 t + h_3 t^2/2! + \cdots$ *and* $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ *are of finite grades* Q *and* F *respectively, then the operation* $F(z) \to f(x)$ *defines a series, in general divergent, which is summable by the method of Borel to the form,*

$$F(z) \to f(x) = \int_0^\infty e^{-xt} Q(t) \left[\sum_{n=0}^{\infty} g^{(n)}(x) F^{(n)}(-t)/n! \right] dt \ ,$$

where the real part of x, $R(x)$, *satisfies the inequality,*

$$Q + F < R \leq R(x) \leq R' < \infty \ .$$

Proof: The proof of this theorem is similar to that used in the preceding discussion. Employing the results of theorem 8, we first define the summable equivalent of $F^{(n)}(z) \to h(x)$, where $F^{(n)}(z)$ is the nth derivative of $F(z)$, to be,

$$V_n(x) = F^{(n)}(z) \to h(x) = \int_0^\infty e^{-xt} Q(t) F^{(n)}(-t)\, dt \ .$$

We next operate upon $f(x) = g(x)\, h(x)$ with $F(z)$ and for the interpretation of our result employ the generalized formula of Leibnitz, formula (2.2) of chapter 4, in which we write $u = h(x)$, $v = g(x)$. This yields the equation,

$$F(z) \to f(x) = \sum_{n=0}^{\infty} g^{(n)}(x)\ \{F^{(n)}(z) \to h(x)\}/n! \ ,$$
$$= \sum_{n=0}^{\infty} g^{(n)}(x) V_n(x)/n! \ . \qquad (5.7)$$

In order to discuss the convergence of this result we notice from theorem 1 and hypothesis (b) that the function $F^{(n)}(-t)$ is of grade F. Hence $F^{(n)}(-t)$ is bounded by the following inequality:

$$| F^{(n)}(-t) | < S_n(t)e^{Ft} \ , \quad S_n(t) = \begin{cases} S_n \ , & 0 \leq t \leq t_0 \ , \\ S_n e^{\delta t} \ , & t > t_0 \ , \end{cases}$$

where S_n is a positive quantity independent of t such that $\lim_{n=\infty} (S_n)^{1/n} = F$ and δ is an arbitrarily small positive number independent of n.

There exists a similar majorant for $Q(t)$ where $T(t)$ is substituted for $S_n(t)$, T for S, and Q for F.

We are thus able to attain the inequalities,

$$| V_n(x) | < S_nT \{ \int_0^{t_0} e^{-tp}\, dt + \int_{t_0}^{\infty} e^{-t(p-2\delta)}\, dt \}$$

$$< S_nT\{ (1-e^{-t_0p})/p + e^{-t_0(p-2\delta)}/(p-2\delta) \} \leqq S_nT(1+e^{2\delta})/p \ ,$$

where we have $p = x - Q - F$, $R(x) > Q + F + 2\delta$.

Referring now to equation (5.7), we employ the inequality just written down and hypothesis (a), from which we have $| g^{(n)}(x) | \leq R_n g^n$, where $\lim_{n=\infty} (R_n)^{1/n} = 1$. We may then write,

$$| F(z) \to f(x) | \leq \sum_{n=0}^{\infty} R_n g^n S_n T(1+e^{2\delta})/n!p \ , \ R(x) > Q + F + 2\delta \ .$$

Hence the series for $F(z) \to f(x)$ converges uniformly for values of x in some region $Q + F < R \leq R(x) \leq R' < \infty$. Therefore, since each integral representation of $V_n(x)$ exists in the specified region, we may interchange integration and summation signs and thus attain the statement of the theorem.

6. *The Summability of Operators of Laplace Type.* In section 5 we have considered the summability problem for differential operators with constant coefficients. It will be useful in another place to extend these ideas so as to include operators of the type considered in theorem 7.

We shall thus prove the following theorem:

Theorem 10. Under the following assumptions: (a) $f(x) = g(x)h(x)$ *is a function of finite grade* g *and* $h(x)$ *is of the form* $h(x) = h_1/x + h_2/x^2 + h_3/x^3 + \cdots$; (b) $Q(t) = h_1 + h_2t + h_3t^2/2! + \cdots$ *exists and defines a function*

$$P(s) \int_s^{\infty} Q(t)\, dt$$

throughout the interval $0 \leq s \leq \infty$, (c) $Y(t)$ *is a function such that the following inequalities hold*:

$$| P(t)\, Y^{(n)}(-t) | \leqq \begin{cases} C_n \ , & 0 \leq t \leq t_0 \ , \\ C_n e^{mt}, & t_0 < t \leqq \infty \ , \end{cases}$$

where $\lim_{n=\infty} (C_n)^{1/n} = C$, $C_{n+1} > C_n$, $m > 0$, *then* $F(x, z) \to f(x)$, *where we write*

$$F(x, z) = e^{-xz} \int_0^z e^{xt}\, Y(t)\, dt \;, \tag{6.1}$$

defines a series, in general divergent, which is summable by the method of Borel to the form,

$$F(x, z) \to f(x) =$$

$$\int_0^\infty e^{-xt} Q(t) \{ \sum_{n-1}^{\infty} g^{(n)}(x) A_n(x,t)/n! - g(0) P(t) Y(-t) \}\, dt \;,$$

for x *in the region,* $m < R \leq R(x) \leq R' < \infty$, *where we employ the abbreviation,* $A_n(x, t) = [\{D^n - (-x)^n\}/(D + x) \to Y(z)]_{z=-t}$, $D = d/dz$.

Proof: By an argument which coincides with that of the previous theorem since it is unaffected by the parameter x in $F(x, z)$, we express the operation $F(x, z) \to f(x)$ in the form,

$$F(x, z) \to f(x) = \sum_{n=0}^{\infty} g^{(n)}(x)\, W_n(x)/n!$$

where we abbreviate

$$W_n(x) = \int_0^\infty e^{-xt} Q(t) F_z^{(n)}(x, -t) \cdot dt \;,$$

in which we write

$$F_z^{(n)}(x, -t) = \partial^n F(x, z)/\partial z^n|_{z=-t} \;.$$

Since we have $F_z^{(n)}(x, z) = e^{-xz}(d/dz - x)^{(n)} \to I(x, z)$, where $I(x, z)$ is defined by (3.4), we get from (3.6) and its consequences,

$$W_n(x) = \int_0^\infty e^{-xt} Q(t) A_n(x, t)\, dt$$

$$+ (-x)^n \int_0^\infty Q(t) I(x, -t)\, dt \;,$$

where $A_n(x, t)$ is defined above.

Substituting this value in $F(x, z) \to f(x)$ and noting that

$$g(0) = \sum_{n=0}^{\infty} g^{(n)}(x)\, (-x)^n/n!,$$

we get

$$F(x,z) \to f(x) = \sum_{n=1}^{\infty} \int_0^\infty e^{-xt} Q(t) A_n(x,t) g^{(n)}(x)\, dt/n! \tag{6.2}$$
$$+ g(0) \int_0^\infty Q(t) I(x, -t)\, dt \; .$$

Considerng the second integral we write,

$$\int_0^\infty Q(t) I(x, -t)\, dt = -\int_0^\infty Q(t)\, dt \int_0^t e^{-xs}\, Y(-s)\, ds$$
$$= -\int_0^\infty e^{-xs} P(s)\, Y(-s)\, ds \; .$$

The second equality sign is seen to be justified by hypotheses (b) and (c) of the theorem which permit the interchange of the order of integration by Dirichlet's formula and insure the existence of the infinite integral.

Referring finally to the first integral of (6.2) we are able to attain from hypothesis (a), i. e. that $|g^{(n)}(x)| \leqq R_n g^n \; \lim_{n=\infty} (R_n)^{1/n} = 1$, and from hypothesis (c), the inequality

$$\left| \int_0^\infty e^{-xt} Q(t) A_n(x,t) g^{(n)}(x)\, dt \right|$$
$$\leqq R_n g^n \int_0^\infty e^{-xt} \sum_{r=0}^{n-1} | Q(t)\, Y^{(n-1-r)}(-t) | \; |x|^r \, |dt|$$
$$\leqq R_n g^n C_{n-1} \{ (1 - |x|^n) / (1 - |x|) \} \{ 1 + e^{-t_0(x-m)} \} / |x - m|$$
$$\leqq R_n g^n C_{n-1} (1 + R')^n \{ 1 + e^{-t_0(x-m)} \} / |x - m| \; ,$$

where x is limited to the region $m < R \leqq R(x) \leqq R' < \infty$.

Hence within the stated region the integrals

$$I_n(x) = \int_0^\infty e^{-xt} Q(t) A_n(x,t)\, dt$$

all exist and the series

$$\sum_{n=1}^{\infty} I_n(x) g^{(n)}(x)/n!$$

converges uniformly. Therefore we can interchange summation and integral signs and thus attain the statement of the theorem.

CHAPTER VI

Differential Equations of Infinite Order With Constant Coefficients

1. Introduction. In the first chapter we have presented the history of methods devised to solve the differential equation of infinite order

$$a_0 u(x) + a_1 u'(x) + a_2 u''(x) + \cdots = f(x) \ , \qquad (1.1)$$

in which the coefficients are assumed to be constants. It is the purpose of this chapter to explain some of these methods in more detail, special attention being paid to conditions imposed upon the function $f(x)$.

It is clear that equation (1.1) is of rather general application since its theory is closely associated with the theory of functional equations of the following types:

(a) $u(x) + \int_x^{\infty} \sum_{i=1}^{n} e^{s_i(x-t)} u(t)\, dt = f(x) \ , \qquad s_i > 0 \ ,$

(b) $u(x) + \int_a^b K(t) u(x+ct)\, dt = f(x) \ ,$

(c) $u(x+b) + \lambda u(x) = f(x) \ ,$

(d) $A_0 u(x) + A_1 u(x+1) + A_2 u(x+2) + \cdots + A_n u(x+n) = f(x) \ .$

This relationship is formally exhibited by expanding $u(T)$ in a Taylor's series about x,

$$u(T) = u(x) + (T-x) u'(x) + (T-x)^2 u''(x)/2! + \cdots \ .$$

If we replace T by t and substitute in (a) we get a differential equation in which the coefficients are,

$$a_0 = 1 + \sum_{m=1}^{n} 1/s_m \ , \qquad a_i = \sum_{m=1}^{n} 1/s_m^{\,i+1} \ , \qquad i > 0 \ .$$

Similarly, if we set $T = x + ct$, equation (b) is seen to reduce to a differential equation of type (1.1) with the coefficients

$$a_0 = 1 + \int_a^b K(t)\, dt \ , \qquad a_i = \int_a^b [K(t)\, (ct)^i / i!]\, dt \ , \qquad i > 0 \ .$$

Letting $T = x + b$, equation (c) reduces to a differential equation with coefficients

$$a_0 = 1 + \lambda \ , \quad a_i = b^i/i! \ , \quad i > 0 \ .$$

As for the general difference equation with constant coefficients given by (d), we note from equation (12.1), chapter 2, the equivalence of $u(x + n)$ and $e^{nz} \to u(x)$ and hence we may write (d) in the form,

$$[A_0 + A_1 e^z + A_2 e^{2z} + A_3 e^{3z} + \cdots + A_n e^{nz}] \to u(x) = f(x) \ .$$

2. Expansion of the Resolvent Generatrix. In order to attain the formal solution of equation (1.1), we first write it in the form

$$F(z) \to u(x) = f(x) \ ,$$

where

$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots \ ,$$

is the generatrix function.

In order to determine the resolvent generatrix $G(z)$, as defined in section 10, chapter 4, we set Bourlet's operational product equal to 1,

$$G(z) \to F(z) = [G \cdot F](z) = 1 \ .$$

Since $F(z)$ is independent of x, this immediately yields the result

$$G(z) = 1/F(z) \ .$$

Simple as this result appears, it invokes certain questions which can best be raised by means of an elementary example. If the generatrix is $F(z) = (1—z)(2—z)$, then the resolvent is $G(z) = 1/(1—z)(2—z)$, which has the three expansions,

$$G(z) = \tfrac{1}{2} + 3z/4 + 7z^2/8 + 15z^3/16 + \cdots \ , \quad |z| < 1 \ ;$$

$$G_1(z) = —(1/2 + z/4 + z^2/8 + z^3/16 + \cdots$$
$$+ 1/z + 1/z^2 + 1/z^3 + 1/z^4 + \cdots) \ , \ 1 < |z| < 2 \ ;$$

$$G_2(z) = 1/z^2 + 3/z^3 + 7/z^4 + 15/z^5 + \cdots \ . \qquad 2 < |z| \ ;$$

A question of primary concern, of course, is to determine which of these three forms is a valid operator. Applying each to the function $f(x)$ and taking account of the identity,

$$1/z^{n+1} \to f(x) = \int_a^x (x—t)^n f(t)\,dt/n! \ ,$$

we obtain the results,

$$G(z) \to f(x) = \tfrac{1}{2}f(x) + 3f'(x)/4 + 7f''(x)/8 + \cdots ,$$

$$G_1(z) \to f(x) = -\tfrac{1}{2}f(x) - f'(x)/4 - f''(x)/8 - \cdots$$

$$- \int_a^x e^{(x-t)} f(t)\,dt ,$$

$$G_2(z) \to f(x) = \int_a^x [e^{2(x-t)} - e^{(x-t)}]\, f(t)\,dt .$$

For illustrative purposes we may now specialize, $f(x) = 1$, and thus obtain,

$$G(z) \to 1 = \tfrac{1}{2} , \quad G_1(z) \to 1 = \tfrac{1}{2} - e^{-a}e^{x} ,$$

$$G_2(z) \to 1 = \tfrac{1}{2} + \tfrac{1}{2}\, e^{-2a}e^{2x} - e^{-a}e^{x} .$$

We now note that all three functions are solutions of the equation,

$$F(z) \to u(x) = 1 ,$$

and that the difference of any two of them is a solution of the homogeneous equation,

$$F(z) \to u(x) = 0 .$$

This fact is very fundamental to the application of operational methods to the equation considered in this chapter and we may state it in general form in the following theorem:

Theorem 1. If $G_1(z)$ designates a Laurent expansion of $G(z)$ in an annulus formed by two concentric circles about the origin and if $G(z)$ is any other expansion about the origin, then the function

$$U(x) = \{G_1(z) - G(z)\} \to f(x) ,$$

where $f(x)$ is arbitrary to within the limits of the existence of the right hand member, is a solution of the homogeneous equation

$$F(z) \to u(x) = 0 .$$

Proof: In order to prove this theorem let us assume first that $F(z)$ is of the form

$$F(z) = (z-a)/\varphi(z) ,$$

where $\varphi(z)$ has no singularity within or upon the circle of radius $r = a$. The inverse operator, $G(z) = \varphi(z)/(z-a)$, then has the two expansions,

$$G(z) = -\{1 + (z/a) + (z/a)^2 + \cdots\}\, \varphi(z)/a ,$$

$$G_1(z) = \{1/z + a/z^2 + a^2/z^3 + \cdots\}\, \varphi(z) ,$$

the first expansion being valid within the circle of radius a and the second in the region exterior to it.

Without essentially impairing the generality of the discussion we may write

$$1/z \to f(x) = \int_0^x f(t)\,dt \ .$$

It will be convenient to replace $1/z^n$ by its equivalent generatrix,

$$Q_n(x,z) = \{1 - [1 + xz + x^2z^2/2! + \cdots \\ + x^{n-1}z^{n-1}/(n-1)!]\, e^{-xz}\}/z^n \ ,$$

which we have discussed in section 6 of chapter 2.

Letting $\varphi(z) = 1$, and replacing $1/z^n$ in the expansion of

$$G_1(z) \to f(x)$$

by its generatrix function we get:

$$G_1(z) \to f(x) = \{Q_1(x,z) + aQ_2(x,z) + a^2Q_3(x,z) \\ + \cdots\} \to f(x) \ .$$

It will be clear from this explicit expansion that

$$G_1(0) = \sum_{n=1}^{\infty} a^n x^n/n!a = (e^{ax} - 1)/a \ ,$$

$$G_1'(0) = -\sum_{n=1}^{\infty} a^{n+1}x^{n+1}/[(n-1)!(n+1)a^2] \\ = -\{e^{ax}(ax-1) + 1\}/a^2 \ ,$$

$$G_1''(0) = \{e^{ax}(a^2x^2 - 2ax + 2) - 2\}/a^3 \ ,$$

and in general,

$$G_1^{(n)}(0) = (-1)^n \{e^{ax}[a^nx^n - na^{n-1}x^{n-1} + n(n-1)\, a^{n-2}x^{n-2} - \cdots \\ \pm n!] - n!\}/a^{n+1} \ .$$

If we replace these values in the expansion

$$G_1(z) = G_1(0) + G_1'(0)z + G_1''(0)z^2/2! + \cdots \ ,$$

and then collect the coefficient of e^{ax}, we see that the operator reduces to the expression,

$$G_1(z) = e^{ax}e^{-xz}(1/a + z/a^2 + z^2/a^3 + \cdots) \\ - (1/a + z/a^2 + z^2/a^3 + \cdots) \ .$$

We now note the operational equivalence, $z^ne^{-xz} = e^{-xz} \to z^n$, which may be established from Bourlet's formula [See (3.1), chap-

ter 4] by setting $X = e^{-xz}$ and $F = z^n$. For example, $z^{2m}e^{-xz} \to \sin x = \{z^{2m} - xz^{2m+1} + x^2z^{2m+2}/2! - \cdots\} \to \sin x = (-1)^m (\sin x \cos x - \sin x \cos x) = 0$. Hence we see that $z^n e^{-xz} \to f(x) = f^{(n)}(0)$.

Taking account of this fact, we get:

$$G_1(z) \to f(x) = e^{ax}\{f(0)/a + f'(0)/a^2 + f''(0)/a^3 + \cdots\}$$
$$- \{f(x)/a + f'(x)/a^2 + f''(x)/a^3 + \cdots\} ,$$
$$= e^{ax}\{f(0)/a + f'(0)/a^2 + f''(0)/a^3 + \cdots\} + G(z) \to f(x) .$$

Similarly for $\varphi(z) = z$, we obtain

$$G_1(z) \to f(x) = e^{ax}\{f(0) + f'(0)/a + f''(0)/a^2 + \cdots\}$$
$$- \{f'(x)/a + f''(x)/a^2 + f'''(x)/a^3 + \cdots\} ,$$
$$= e^{ax}\{f(0) + f'(0)/a + f''(0)/a^2 + \cdots\} + G(z) \to f(x) \quad ;$$

and more generally, for $\varphi(z) = z^n$,

$$G_1(z) \to f(x) = a^{n-1}e^{ax}\{f(0) + f'(0)/a + f''(0)/a^2 + \cdots\}$$
$$+ G(z) \to f(x) .$$

Assuming that $\varphi(z)$ can be expressed in the form $\varphi_0 + \varphi_1 z + \varphi_2 z^2 + \cdots$, we then derive by addition the result

$$G_1(z) \to f(x) = \{\varphi(a)/a\}e^{ax}\{f(0) + f'(0)/a + f''(0)/a^2 + \cdots\}$$
$$+ G(z) \to f(x) .$$

Since $U(x) = Ce^{ax}$ is obviously a solution of the equation

$$F(z) \to u(x) = 0 ,$$

the truth of the theorem is demonstrated for the special case assumed above.

Let us next assume that $F(z)$ is of the form

$$F(z) = (z-a_1)(z-a_2)\cdots(z-a_n)/\varphi(z) .$$

where $a_1, a_2, \cdots, a_n$ are points within an annulus formed by two concentric circles r and R, (the latter having the larger radius), about the origin, and where $\varphi(z)$ has no singularity within or upon R.

Then the generatrix may be written

$$G(z) = \varphi(z)\{1/[P'(a_1)(z-a_1)] + 1/[P'(a_2)(z-a_2)] + \cdots$$
$$+ 1/[P'(a_n)(z-a_n)]\} , \tag{2.1}$$

where we employ the abbreviation

$$P(z) = (z-a_1)(z-a_2)\cdots(z-a_n) .$$

Let us designate the expansion of the resolvent within the circle r by $G(z)$ and the expansion in the region exterior to R by $G_1(z)$. We then have from the result of the case of one pole, the expansion

$$G_1(z) \to f(x) = \sum_{i=1}^{n} \{\varphi(a_i)/a_i P'(a_i)\} e^{a_i x} \{f(0) + f'(0)/a_i + f''(0)/a_i^2 + \cdots\} + G(z) \to f(x) \;, \qquad (2.2)$$

$$= \sum_{i=1}^{n} \{1/a_i F'(a_i)\} e^{a_i x} \{f(0) + f'(0)/a_i + f''(0)/a_i^2 + \cdots\} + G(z) \to f(x) \;.$$

Hence the difference

$$U(x) = \{G_1(z) - G(z)\} \to f(x) \;, \qquad (2.3)$$

is a solution of the homogeneous equation.

We should make specific note of one exceptional case, i. e. where $f(x) = e^{\lambda x}$, the value of λ coinciding with one of the zeros a_i, let us say for simplicity of notation the value a. We shall then have

$$G_1(z) \to e^{\lambda x} = \lim_{\lambda = a} e^{\lambda x} G_1(\lambda)$$

$$= \sum_{i=1}^{n}{}' e^{a_i x}/\{F'(a_i)(a_i - a)\} + \lim_{\lambda = a} [e^{ax}/\{F'(a)(a - \lambda)\} + e^{\lambda x}/F(\lambda)] \;,$$

where Σ' means that a has been omitted from the sum.

Taking the limit we thus obtain

$$G_1(z) \to e^{\lambda x} = \lim_{\lambda = a} e^{\lambda x} G_1(\lambda)$$

$$= \sum_{i=1}^{n}{}' e^{a_i x}/\{F'(a_i)(a_i - a)\} + \{e^{ax}/F'(a)\}\{x - F''(a)/2F'(a)\} \;. \qquad (2.4)$$

Example: Let us evaluate,

$$1/F(z) \to e^{5x} \;,$$

where $F(z) = (z-2)(z-5) = z^2 - 7z + 10$.

We have $F'(5) = 3$, $F''(5) = 2$, $F'(2) = -3$. Hence we get

$$1/F(z) \to e^{5x} = e^{2x}/9 + (e^{5x}/3)(x - 1/3) \;.$$

The case of multiple poles is treated by a simple device . If the resolvent is

$$G(z) = \varphi(z)/(z-a)^r \;,$$

we may write it in the form

$$G(z) = (\partial^{r-1}/\partial a^{r-1})\, \varphi(z)/(z-a)\,(r-1)!\ .$$

Hence we have

$$G(z) \to f(x) = [1/(r-1)!]\,(\partial^{r-1}/\partial a^{r-1})\{[\varphi(a)/a]e^{ax}[f(0) + f'(0)/a + f''(0)/a^2 + \cdots]\}$$
$$+ (-1)^r[\varphi(z)/a^r]\{1 + rz/a + r(r+1)z^2/a^2\cdot 2! + r(r+1)(r+2)z^3/a^3\cdot 3! + \cdots\} \to f(x)\ .$$

The difference between the left hand member and the second term of the right hand side is again seen to be a solution of the homogenous equation. The general proof is then easily constructed from this fact.

It remains for us to discuss the values of the solution and its derivatives at the point $x = 0$. We obtain the following theorem:

Theorem 2. If $G_1(z)$ denotes the Laurent expansion of the function

$$G(z) = \varphi(z)/(z-a_1)(z-a_2)\cdots(z-a_n)\ ,$$

in the region exterior to the poles $a_1, a_2, \cdots, a_n$, and if $\varphi(z)$ is a polynomial of degree $m < n$, then $u^{(r)}(x) = z^r G_1(z) \to f(x)$ vanishes at $x = 0$ for $r = 0, 1, 2, \cdots, n-m-1$.

Proof: Writing $G(z)$ in the form (2.1) we have from the results of the last theorem,

$$u^{(r)}(x) = z^r G_1(z) \to f(x) = \sum_{i=1}^{n} [\varphi(a_i)a_i^r e^{a_i x}/P'(a_i)]\,\{f(0)/a_i + f'(0)/a_i^2 + \cdots\} - \sum_{i=1}^{n} \varphi(z)/P'(a_i) \to \{f^{(r)}(x)/a_i + f^{(r+1)}(x)/a_i^2 + \cdots\}\ .$$

Recalling the algebraic identity

$$I(p) = \sum_{i=1}^{n} a_i^p/P'(a_i) = \begin{cases} 1\ , & p = n-1\ , \\ 0\ , & 0 \leq p < n-1\ , \end{cases}$$

we see that

$$u^{(r)}(0)=\sum_{i=1}^{n}\{\varphi_0+\varphi_1 a_i+\cdots+\varphi_m a_i^m\}\, a_i^r\,\{f(0)/a_i$$

$$+f'(0)/a_i^2+\cdots\}/P'(a_i)-\sum_{i=1}^{n}\ \sum_{j=0}^{m}\varphi_j\,\{f^{(r+j)}(0)/a_i$$

$$+f^{(r+j+1)}(0)/a_i^2+\cdots\}/P'(a_i)\ ,$$

$$=\{\sum_{j=0}^{m}\varphi_j I(r+j-1)f(0)$$

$$+\sum_{j=0}^{m}\varphi_j I(r+j-2)f'(0)+\cdots\}$$

$$-\{\sum_{j=0}^{m}I(-1)\ \varphi_j\ f^{(r+j)}(0)$$

$$+\sum_{j=0}^{m}I(-2)\ \varphi_j\ f^{(r+j+1)}(0)+\cdots\}\ ,$$

$$=\sum_{j=0}^{m}\varphi_j I(r+j-1)f(0)$$

$$+\sum_{j=0}^{m}\varphi_j I(r+j-2)\ f'(0)+\cdots$$

$$+\sum_{j=0}^{m}\varphi_j I(j)\ f^{(r-1)}(0)\ .$$

If $r+j-1\leqq n-2$, then $u^{(r)}(0)=0$. Since j does not exceed m we have $r\leqq n-m-1$, which is the statement of the theorem.

Corollary. If $f(x)$ is a function which vanishes together with its first q derivatives at $x=0$, then $u^{(r)}(0)=0$ for $r=n+q-m$.

In order to illustrate the preceding results we shall apply them to several examples:

Example 1. The fundamental equation satisfied by the polynomials representing the sums of powers of the natural numbers, $S_n(x)=1^n+2^n+\cdots+x^n$, is obviously

$$S_n(x)-S_n(x-1)=x^n\ ,$$

which may be written symbolically as

$$(1-e^{-z})\rightarrow S_n(x)=x^n\ .$$

The solution follows immediately:

$$S_n(x)=1/(1-e^{-z})\rightarrow x^n\ ,$$

$$=\{1/z+1/2+B_1z/2!-B_2z^3/4!+B_3z^5/6!-\cdots\}\rightarrow x^n\ ,$$

$$S_n(x) = x^{n+1}/(n+1) + x^n/2 + nB_1x^{n-1}/2!$$
$$- n(n-1)(n-2)B_2x^{n-3}/4!$$
$$+ n(n-1)(n-2)(n-3)(n-4)(n-5)B_3x^{n-5}/6! - \cdots ,$$

where the B_n are the Bernoulli numbers. The formula is at once recognized as the classical solution of Bernoulli.

Example 2. The following problem is taken from Bromwich [See *Bibliography,* Bromwich (1), p. 413], and will be used in the illustration of other methods.

We shall solve the equation,

$$\{z^2 + 2\nu z + (\nu^2 + n^2)\} \to u(x) = Fe^{-\nu x}\cos(nx + \omega) ,$$

subject to the condition that $u(x)$ shall vanish to as high an order as possible at the origin.

It is clear from the boundary condition that this problem is a special case of theorem 2. The generatrix is written in the form

$$F(z) = z^2 + 2\nu z + (\nu^2 + n^2) = (z - a_1)(z - a_2) ,$$

where we abbreviate, $a_1 = -\nu + ni$, $a_2 = -\nu - ni$.

By theorem 2 we must take the expansion of the resolvent generatrix in its outer annulus, which leads us to write,

$$1/F(z) = (1/2ni)[1/(z - a_1) - 1/(z - a_2)]$$
$$= (1/2ni)[(1/z + a_1/z^2 + \cdots) - (1/z + a_2/z + \cdots)] .$$

Designating by $f(x)$ the function $Fe^{-\nu x}\cos(nx + \omega)$, we get

$$1/F(z) \to f(x)$$
$$= (1/2ni)\{\int_0^x [1 + a_1(x-t) + a_2^2(x-t)^2/2! + \cdots] f(t)dt$$
$$- \int_0^x [1 + a_2(x-t) + a_2^2(x-t)^2/2! + \cdots] f(t)dt\}$$
$$= (1/2ni)\int_0^x [e^{a_1(x-t)} - e^{a_2(x-t)}] f(t)dt .$$

By simple integration this leads to the solution,

$$u(x) = [Fe^{-\nu x}/(2n^2)] [nx\sin(nx + \omega) - \sin nx \sin\omega] ,$$

which is seen to vanish together with its first derivative at $x = 0$.

Example 3. We shall illustrate the efficacy of formula (2.3) by applying it to the following elementary Sturmian problem, in which we seek to solve the differential equation,

$$u''(x) + \lambda^2 u(x) = 0 \ ,$$

subject to the boundary conditions,

$$u(0) = u(1) = 0 \ .$$

The resolvent generatrix is obviously,

$$1/F(z) = (1/2\lambda i)\,[1/(z-\lambda i) - 1/(z+\lambda i)] \ ,$$

from which we obtain by means of (2.3) the solution,

$$\begin{aligned} u(x) = \{G_1(z) - G(z)\} \to f(x) = (-1/\lambda^2)\ [\cos \lambda x \ \{f(0) \\ - f''(0)/\lambda^2 + f^{(4)}(0)/\lambda^4 - \cdots\} - i \sin \lambda x \ \{f'(0)/\lambda \\ - f^{(3)}(0)/\lambda^3 + \cdots\}] \ . \end{aligned}$$

The boundary condition, $u(0) = 0$, leads to the equation,

$$f(0) - f''(0)/\lambda^2 + f^{(4)}(0)/\lambda^4 - f^{(6)}(0)/\lambda^6 + \cdots = 0 \ . \tag{2.5}$$

Similarly the assumption that $u(1) = 0$ gives us the condition,

$$\sin \lambda \ \{f'(0)/\lambda - f^{(3)}(0)/\lambda^3 + f^{(5)}(0)/\lambda^5 - \cdots\} = 0 \ .$$

If we now define characteristic numbers by the equation,

$$\sin \lambda = 0 \ ,$$

then a solution of the original system will be obtained by operating upon any function which satisfied (2.5). We might choose, for example, $f(x) = x$ and thus obtain,

$$\begin{aligned} u(x) &= \{G_1(z) - G(z)\} \to x \\ &= \{(1/z^2 - \lambda^2/z^4 + \lambda^4/z^6 - \cdots) - 1/\lambda^2 + z^2/\lambda^4 \\ &\qquad - z^4/\lambda^6 + \cdots\} \to x \\ &= (1/\lambda) \int_0^x \sin \lambda (x-t)\, t \, dt - x/\lambda^2 = - \sin \lambda x/\lambda^3 \ . \end{aligned}$$

Let us now consider a special case of the homogeneous equation which will be of importance to us in another section. We shall consider the solution of the equation

$$F(z) \to u(x) = 0 \ , \tag{2.6}$$

where $u(x)$ is subject to the boundary conditions:

$$u(0) = u_0 \ , \ u'(0) = u_1 \ , \ u''(0) = u_2 \ , \cdots \ , \ u^{(n-1)}(0) = u_{n-1} \ , \tag{2.7}$$

and $F(z)$ is a polynomial of the nth degree,

$$F(z) = z^n - Az^{n-1} + Bz^{n-2} - Cz^{n-3} + \cdots \pm N$$
$$= (z-a_1)(z-a_2)(z-a_3)\cdots(z-a_n) \ .$$

Referring to theorem 1 and expansion (2.2) we see that the function

$$u(x) = \sum_{i=1}^{n} \frac{p(a_i)}{a_i F'(a_i)} e^{a_i x} \qquad (2.8)$$

is a solution of equation (2.6).

Let us now determine the function $p(a_i)$ in such a way that $u(x)$ shall satisfy the boundary conditions (2.7). For this purpose we write

$$p(t) = p_1 t + p_2 t^2 + \cdots + p_n t^n \ .$$

Hence making use of the abbreviation

$$I(p) = \sum_{i=1}^{n} a_i^p / F'(a_i) \ ,$$

we can write (2.8) in the form

$$u(x) = \sum_{i=1}^{n} I(i)\, p_i\, e^{a_i x} \ .$$

Moreover we have

$$u^{(r)}(x) = \sum_{r=1}^{n} I(i+r-1)\, p_i\, e^{a_i x} \ .$$

Letting $x = 0$, we obtain the following set of equations for the determination of the coefficients p_i:

$$u^{(r)}(0) = \sum_{i=1}^{n} I(i+r-1) p_i = u_r \ , \quad r = 0, 1, 2, \cdots, n-1 \ . \qquad (2.9)$$

The function $I(p)$ will be seen to have the following properties:

$$\begin{aligned} I(p) &= 0 \ , \quad 0 \leq p < n-1 \ , \\ I(p) &= 1 \ , \quad p = n-1 \ , \\ I(p) &= \sigma_r \ , \quad p = n+r-1 \ . \end{aligned} \qquad (2.10)$$

where we abbreviate $\sigma_1 = \Sigma a_i$, $\sigma_2 = \Sigma a_i a_j$, $\sigma_3 = \Sigma a_i a_j a_k$, etc., where the summations range independently over the integers from 1 to n.

Since the σ_r are symmetric functions they can be expressed in terms of the elementary symmetric functions and hence in terms of

the coefficients of $F(z)$. The calculations are somewhat involved, although entirely elementary, and we shall give only a few values as follows:*

$$\begin{aligned} \sigma_1 &= A \ , \\ \sigma_2 &= A^2 - B \ , \\ \sigma_3 &= A^3 - 2AB + C \ , \\ \sigma_4 &= A^4 - 3A^2B + 2AC + B^2 - D \ , \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{2.11}$$

Returning to (2.9) and making use of (2.10) we obtain the following system of equations for the determination of the coefficients p_i:

$$\begin{aligned} &p_n = u_0 \ , \\ &p_{n-1} + \sigma_1 p_n = u_1 \ , \\ &p_{n-2} + \sigma_1 p_{n-1} + \sigma_2 p_n = u_2 \ , \\ &p_{n-3} + \sigma_1 p_{n-2} + \sigma_2 p_{n-1} + \sigma_3 p_n = u_3 \ , \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Employing the explicit values given in (2.11), we have the following solutions for the cases, $n = 2, 3, 4$ and 5:

$$\begin{aligned} n = 2;\ p_1 &= -\sigma_1 u_0 + u_1 = -Au_0 + u_1 \ , \quad p_2 = u_0 \ ; \\ n = 3;\ p_1 &= (\sigma_1^2 - \sigma_2)u_0 - \sigma_1 u_1 + u_2 = Bu_0 - Au_1 + u_2 \ , \\ p_2 &= -\sigma_1 u_0 + u_1 = -Au_0 + u_1 \ , \ p_3 = u_0 \ ; \\ n = 4;\ p_1 &= (-\sigma_1^3 + 2\sigma_1\sigma_2 - \sigma_3)u_0 + (\sigma_1^2 - \sigma_2)u_1 - \sigma_1 u_2 + u_3 \ , \\ &= -Cu_0 + Bu_1 - Au_2 + u_3 \ , \\ p_2 &= Bu_0 - Au_1 + u_2 \ , \ p_3 = -Au_0 + u_1 \ , \ p_4 = u_0 \ ; \\ n = 5;\ p_1 &= (\sigma_1^4 + 2\sigma_1\sigma_3 - 3\sigma_1^2\sigma_2 + \sigma_2^2 - \sigma_4)u_0 \\ &\quad + (-\sigma_1^3 + 2\sigma_1\sigma_2 - \sigma_3)u_1 + (\sigma_1^2 - \sigma_2)u_2 - \sigma_1 u_1 + u_4 \ , \\ &= Du_0 - Cu_1 + Bu_2 - Au_3 + u_4 \ , \\ p_2 &= -Cu_0 + Bu_1 - Au_2 + u_3 \ , \\ p_3 &= Bu_0 - Au_1 + u_2 \ , \ p_4 = -Au_0 + u_1 \ , \ p_5 = u_0 \ . \end{aligned}$$

The reader may readily verify that these coefficients are identical with the coefficients of the polynomial part of the product

$$v(t)F(t)t \ ,$$

*See M. Bôcher: *Introduction to Higher Algebra.* New York (1921), chap. 18.

where we define,

$$v(t) = u_0/t + u_1/t^2 + u_2/t^3 + \cdots u_{n-1}/t^n \ .$$

The significance of this relationship is discussed in the next section.

As an example, let us consider the special case,

$$F(z) \to u(x) = 0 \ ,$$

where we specialize,

$$F(z) = (z-1)(z-2)(z-3)(z-4) = z^4 - 10z^3 + 35z^2 - 50z + 24 \ .$$

We then have,

$$F'(1) = -6 \ , \ F'(2) = 2 \ , \ F'(3) = -2 \ , \ F'(4) = 6 \ ,$$

and employing (2.8) we attain the solution

$$u(x) = (24u_0 - 26u_1 + 9u_2 - u_3)e^x/6 + (-12u_0 + 19u_1 - 8u_2 + u_3)e^{2x}/2 + (8u_0 - 14u_1 + 7u_2 - u_3)e^{3x}/2 + (-6u_0 + 11u_1 - 6u_2 + u_3)e^{4x}/6 \ .$$

One may easily verify that this solution satisfies the boundary conditions, $u^{(r)}(0) = u_r \ , \ r = 0, 1, 2, 3$.

Let us make the final observation that the solution (2.8) is equivalent to the development,

$$u(x) = [p(z)/F(z)] \to 1 \ , \tag{2.12}$$

where the operator is first expanded in its outer Laurent annulus. This observation then leads to the following rule, the significant character of which will appear in section 5 of the next chapter:

The solution of equation (2.6), *subject to the boundary values* (2.7), *may be attained by first solving the algebraic equation,*

$$F(z)\ U = p(z) \ , \tag{2.13}$$

The solution, $u(x)$, *is then explicitly found by interpreting the operation* $[p(z)/F(z)] \to 1$ *as equivalent to* (2.8).

3. *The Method of Cauchy-Bromwich.* We have already indicated in the first chapter the important part which T. J. I'A. Bromwich has played in laying a rationelle for the Heaviside calculus by an adaptation of a method of circuit integrals originally due to Cauchy.* We shall discuss this method for the case where the resolvent generatrix has the following form:

*This method has much in common with what is called the *method of Appell polynomials.* The reader will find it profitable at this point to consult section 13, chapter 8.

$$1/F(z) = [F_1/(z-a_1) + F_2/(z-a_2) + \cdots + F_n/(z-a_n)]\,\varphi(z) \ . \tag{3.1}$$

It is assumed that $\varphi(z)$ has no singularities within a circle about the origin which includes the largest of the a_i. It will be convenient to apply the method first to the homogeneous case of equation (1.1).

The method depends upon the success of finding a solution of the form

$$u(x) = (1/2\pi i) \int_C e^{xt} v(t)\,dt \ . \tag{3.2}$$

In another chapter we shall apply this Laplace transformation to discover the solution of a more general equation in which the coefficient of z^n is a polynomial. Obviously here we are dealing with a degenerate case of this method since (3.2) substituted in (1.1) leads to the equation:

$$(1/2\pi i) \int_C e^{xt} F(t)\, v(t)\,dt = 0 \ .$$

In order to interpret this result we see that $v(t)$ must be a function which erases the singularities of $F(t)$ within the contour C, but it may have simple poles at the points $a_1, a_2, \cdots, a_n$. That is to say, we may write,

$$v(t) = \varphi(t)\,[A_1/(t-a_1) + A_2/(t-a_2) + \cdots + A_n/(t-a_n)] \ . \tag{3.3}$$

Choosing C as a closed circuit containing all the points a_i, we are led to the solution

$$u(x) = \varphi(a_1) A_1 e^{a_1 x} + \varphi(a_2) A_2 e^{a_2 x} + \cdots + \varphi(a_n) A_n e^{a_n x} \ .$$

It will be seen that this furnishes a solution of equation (1.1) with the n arbitrary constants $A_1, A_2, \cdots, A_n$. In order to specify these we note the equations,

$$\begin{aligned}
&\varphi(a_1) A_1 + \varphi(a_2) A_2 + \cdots + \varphi(a_n) A_n = u_0 \ , \\
&\varphi(a_1) A_1 a_1 + \varphi(a_2) A_2 a_2 + \cdots + \varphi(a_n) A_n a_n = u_1 \ , \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\varphi(a_1) A_1 a_1^{n-1} + \varphi(a_2) A_2 a_2^{n-1} + \cdots + \varphi(a_n) A_n a_n^{n-1} = u_{n-1} \ .
\end{aligned} \tag{3.4}$$

where we employ the abbreviation, $u^{(p)}(0) = u_p$.

Bromwich finally observes that if (3.3) be expanded in the Laurent annulus exterior to all the values of the a_i, then the coefficients are precisely the boundary conditions which form the right hand member of the system (3.4), that is,

$$v(t) = u_0/t + u_1/t^2 + u_2/t^3 + \cdots + u_{n-1}/t^n + \Psi(t) \ ,$$

where $\Psi(t) = O(t^{-n-1})$. (3.5)

Since, moreover, in the present case the function,

$$p(z) = F(z)v(z) \ ,$$

is a polynomial, the value of $v(z)$ is readily computed by first evaluating $p(z)$ through the direct multiplication of (3.5) with $F(z)$ and the subsequent rejection of negative powers of z.

Example: Let us solve, $[z^2 + 2\nu z + (\nu^2 + n^2)] \rightarrow u(x) = 0$, subject to the boundary conditions $u(0) = u_0$, $u'(0) = u_1$.

Since $v(z) = u_0/z + u_1/z^2 + \cdots$, and $F(z) = z^2 + 2\nu z + (\nu^2 + n^2)$, we get, $p(z) = u_0 z + u_1 + 2\nu u_0$. Employing this polynomial we then obtain,

$$\begin{aligned} v(z) &= (u_0 z + u_1 + 2\nu u)/[(z-a_1)(z-a_2)] \\ &= A/(z-a_1) + B/(z-a_2), \end{aligned}$$

where we abbreviate, $a_1 = -\nu + ni$, $a_2 = -\nu - ni$, $A = (\nu u_0 + u_1 + ni\,u_0)/2ni$, $B = (\nu u_0 + u_1 - ni\,u_0)/(-2ni)$.

Integral (3.2) obviously yields the solution

$$\begin{aligned} u(x) &= Ae^{a_1 x} + Be^{a_2 x} \ , \\ &= (e^{\nu x}/n)[(\nu u_0 + u_1)\sin nx + nu_0 \cos nx] \ . \end{aligned}$$

We consider next the non-homogeneous case and for simplicity of argument we shall assume that $f(x)$ has the expansion,

$$f(x) = f_0 + f_1 x + f_2 x^2/2! + f_3 x^3/3! + \cdots \ .$$

We first note that we may write formally

$$f(x) = (1/2\pi i) \int_C e^{xt} V(t)\, dt \ ,$$

where

$$V(t) = f_0/t + f_1/t^2 + f_2/t^3 + \cdots \ , \tag{3.6}$$

and the path of integration is a circle about the origin.

If we assume that the solution may be written as the integral (3.2) we must have after substitution in (1.1)

$$(1/2\pi i) \int_C e^{xt} F(t) v(t)\, dt = f(x) \ , \tag{3.7}$$

and hence $F(t)v(t) = V(t)$, that is to say,

$$v(t) = V(t)/F(t) \ .$$

Bromwich limits himself to the case where the impressed force is

$$f(x) = Ae^{mt} ,$$

from which he obtains,

$$V(t) = A/(t-m) .$$

The path of integration, it should be noted, can be in this case about $t = m$ alone, or about this value and any number of zeros of $F(t)$.

Perhaps the most significant fact in the situation can be summarized in the following theorem:

Theorem 3. If in the integral

$$u(x) = (1/2\pi i) \int_C e^{xt}[V(t)/F(t)]dt \tag{3.8}$$

the function $V(t)$ is defined by (3.6) *and the path of integration is a circle about the origin sufficiently large to include the zeros of $F(t)$, then $u(x)$ is formally equivalent to the operational solution* (2.2),

$$u(x) = G_1(z) \to f(x) .$$

The proof of this theorem can be constructed by the reader from a consideration of the following special case.

Let us write

$$1/F(t) = 1/(t-a) = 1/t + a/t^2 + a^2/t^3 + \cdots ,$$

so that we have

$$V(t)/F(t) = f_0/t^2 + (f_1 + af_0)/t^3 + (f_2 + f_1a + f_0a^2)/t^4 + \cdots .$$

We then obtain from (3.8) the expansion

$$\begin{aligned} u(x) &= f_0x + (f_1 + af_0)x^2/2! + (f_2 + f_1a + f_0a^2)x^3/3! + \cdots \\ &= (f_0/a)(e^{ax} - 1) + (f_1/a^2)(e^{ax} - ax - 1) \\ &\qquad + (f_2/a^3)(e^{ax} - a^2x^2/2! - ax - 1) + \cdots \\ &= e^{ax}(f_0/a + f_1/a^2 + f_2/a^3 + \cdots) \\ &\qquad - (1/a)(f_0 + f_1x + f_2x^2/2! + \cdots) \\ &\qquad - (1/a^2)(f_1 + f_2x + f_3x^2/2! + \cdots) \\ &\qquad - (1/a^3)(f_2 + f_3x + f_4x^2/2! + \cdots) \\ &\qquad - \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &= e^{az}(f_0/a + f_1/a^2 + f_2/a^3 + \cdots) \\ &\qquad - (1/a)f(x) - (1/a^2)f'(x) - (1/a^3)f''(x) - \cdots \\ &= e^{ax}(f_0/a + f_1/a^2 + f_2/a^3 + \cdots) + \{1/F(z)\} \to f(x) . \end{aligned}$$

Example. As an illustration we shall apply this method to the problem solved in example 2 of the preceding section.

From the fact that

$$f(x) = Fe^{-\nu x} \cos(nx + \omega) = \tfrac{1}{2} F(e^{a_1 x + \omega i} + e^{a_2 x - \omega i}) ,$$

we get at once,

$$V(t) = A/(t-a_1) + B/(t-a_2) ,$$

where we abbreviate $A = \tfrac{1}{2} Fe^{\omega i}$, $B = \tfrac{1}{2} Fe^{-\omega i}$.

We also have

$$F(t) = a/(t-a_1) + b/(t-a_2) ,$$

where we write $a = -b = 1/2ni$.

From this we derive the expansion

$$V(t)/F(t) = Aa/(t-a)^2 + (aB + Ab)[a/(t-a) + b/(t-a)] + bB/(t-a)^2 ,$$

which when introduced into integral (3.8) yields the solution

$$u(x) = Aaxe^{a_1 x} + (a^2B + Aab)e^{a_1 x} + (abB + Ab^2)e^{a_2 x} + bBxe^{a_2 x}$$
$$= [Fe^{-\nu x}/(2n^2)][nx \sin(nx + \omega) - \sin nx \sin \omega] .$$

The solution is thus seen to be identical with the one previously obtained.

PROBLEMS

1. Discuss the solution of

$$(1 + \lambda \sin z) \rightarrow u(x) = f(x) ,$$

when (a) $f(x) = x$; (b) $f(x) = e^{ax}$; (c) $f(x) = 1/x$.

2. Show that

$$u(x) = -\tfrac{1}{2} f(x) - \tfrac{1}{4} \int_0^\infty e^{-\frac{1}{2}t} f(x+t)\,dt$$

is a solution of the equation

$$2\,u(x) = -f(x) + \int_x^\infty e^{(x-t)}\, u(t)\, dt .$$

Reduce the solution to the form

$$u(x) = \frac{(1-z)}{(1-2z)} \rightarrow f(x) .$$

3. Given $u(x+1) + u(x) = 2\,f(x)$, show that if $f(x)$ in the half plane, $R(x) > c$, has the form

$$f(x) = k/x + p(x)/x^2 ,$$

where k is a constant and $p(x)$ is bounded, then

$$u(x) = (1/2\pi i) \int_{a-i\infty}^{a+i\infty} f(t) g(x-t)\, dt \ , \quad R(x) > a > c \ ,$$

where we abbreviate

$$g\ (s) = \int_0^{\infty} [2\, e^{-ts}/(1+e^{-t})]\, dt = \Psi[\tfrac{1}{2}(s+1)] - \Psi(\tfrac{1}{2}s) \ .$$

(Hilb).

4. Prove that

$$\frac{1}{e^z - 1} = z^{-1} - \tfrac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{z + 2n\pi i} + \frac{1}{z - 2n\pi i} \right] .$$

Hence show that the solution of the equation

$$\Delta\, u(x) = f(x)$$

is given by

$$u(x) = \Pi(x) - \tfrac{1}{2} f(x) + \int^x f(t)\, dt + 2 \sum_{n=1}^{\infty} \int^x \cos 2n\pi (x-t)\, f(t)\, dt \ ,$$

where $\Pi(x)$ is an arbitrary periodic function of unit period and the lower limits of the integrals are arbitrary constants the imaginary parts of which are all equal. (Wedderburn).

5. Prove the following theorem [see *Bibliography*: Sheffer (3), p. 280]: If $u(x) = \sum_{n=0}^{\infty} u_n\, x^n/n!$ is a function of grade $q < R$, which satisfies the equation

$$\sum_{n=-\infty}^{\infty} a_n\, u^{(n)}(x) = f(x) \ ,$$

where $F(z) = \sum_{n=-\infty}^{\infty} a_n\, z^n$ converges for $0 < |z| < R$, then $f(x)$ may be expanded in the form

$$f(x) = \sum_{n=0}^{\infty} u_n\, P_n(x) \ , \quad \sup_{n=\infty} \lim |u_n|^{1/n} < R \ ,$$

where $P_n(x)$ are Appell polynomials generated by

$$e^{xz}\, F(z) = \sum_{n=-\infty}^{\infty} P_n(x)\, z^n \ .$$

Show also that the converse is true.

(For the definition and discussion of Appell polynomials the reader is referred to section 6, chapter 1 and section 13, chapter 8).

4. The Method of Carson. We have commented in the first chapter to the effect that the Heaviside calculus may be regarded from one point of view as a special case of the generatrix calculus. This connection is essentially the contribution of J. R. Carson, whose ideas we shall now reconstruct.

Let us first consider the equation

$$F(z) \to v(x) = 1 \ . \tag{4.1}$$

From (2.2) we then obtain

$$v(x) = \sum_{i=1}^{n} e^{a_i x}/[a_i F'(a_i)] + 1/F(0) \ .$$

We now note the following formal equivalence:

$$\int_0^\infty v(s)\, e^{-zs} ds = \sum_{i=1}^{n} 1/[a_i F'(a_i)\,(a_i - z)] + 1/[zF(0)] \ .$$

But this expansion is identically equal to $1/[zF(z)]$ so that we attain the result,

$$\int_0^\infty v(s)\, e^{-zs} ds = 1/[zF(z)] \ . \tag{4.2}$$

In this manner the solution of equation (4.1) is reduced to the inversion of this Laplace integral equation. For practical applications rather extensive tables have been prepared to facilitate the inversion and this mechanization of the method has given it a merited popularity. A brief table is appended at the end of this section.

Let us now see how to proceed in solving the more general equation,

$$F(z) \to u(x) = f(x) \ .$$

We shall show that in terms of $v(x)$ the solution is given by

$$\begin{aligned} u(x) &= \frac{d}{dx} \int_0^x f(t)\, v(x-t)\, dt \ , \\ &= f(x)\, v(0) + \int_0^x f(t)\, v'(x-t)\, dt \ . \end{aligned} \tag{4.3}$$

In order to prove this statement we shall identify (4.3) with (2.2). Referring to the latter solution we see that we can write,

$$\begin{aligned} u(x) = \{ &\sum_{i=1}^{n} e^{a_i x} f(0)/a_i F'(a_i) + \sum_{i=1}^{n} e^{a_i x} f'(0)/a_i^2 F'(a_i) \\ &+ \sum_{i=1}^{n} e^{a_i x} f''(0)/a_i^3 F'(a_i) + \cdots \} + f(x)/F(0) \\ &- \sum_{i=1}^{n} f'(x)/a_i^2 F'(a_i) - \sum_{i=1}^{n} f''(x)/a_i^3 F'(a_i) - \cdots \ , \end{aligned}$$

$$u(x) = \sum_{j=1}^{\infty} \sum_{i=1}^{n} \{e^{a_i x} f^{(j-1)}(0)/a_i{}^j F'(a_i) - f^{(j-1)}(x)/a_i{}^j F'(a_i)\}$$

$$+ f(x)/F(0) + \sum_{i=1}^{n} f(x)/a_i F'(a_i) \ ,$$

$$= \int_0^x \{\sum_{i=1}^{n} e^{a_i(x-t)} f(t)/F'(a_i)\}\, dt + f(x)/F(0)$$

$$+ \sum_{i=1}^{n} f(x)/a_i F'(a_i) \ .$$

This last expression is seen to be the equivalent of

$$f(x)v(0) + \int_0^x f(t)v'(x-t)\,dt \ ,$$

which establishes the validity of (4.3).

Example. It will be instructive to apply this method to the solution of the illustrative problem given in example 2 of section 2.

In this case we begin with the integral equation,

$$\int_0^\infty v(s)e^{-zs}ds = 1/\{z[z^2 + 2\nu z + (\nu^2 + n^2)]\} \ ,$$

$$= A/z + B/(z-a_1) + C/(z-a_2) \ ,$$

where we abbreviate

$$A = 1/(\nu^2 + n^2) \ , \ B = 1/[a_1(2ni)] \ , \ C = -1/[a_2(2ni)] \ .$$

Noting the integral

$$\int_0^\infty e^{-ax}dx = 1/a \ ,$$

we at once derive,

$$v(s) = A + Be^{a_1 s} + Ce^{a_2 s} \ .$$

Since $v(0) = A + B + C = 0$, we are concerned only with the evaluation of the integral

$$\int_0^x v'(x-t)f(t)\,dt \ .$$

This leads immediately to the solution,

$$u(x) = [Fe^{-\nu x}/(2n^2)][nx \sin(nx + \omega) - \sin nx \sin \omega] \ .$$

It should be pointed out here that the Carson theory is less general than that of Bromwich from the fact that it is designed to solve

a single type of boundary value problem, namely, the one most interesting to electrical engineers where the solution vanishes to as high an order as possible at the origin. Within the scope of this problem, however, the Carson theory is unusually effective.

For the sake of applications, a brief table is given below for the inversion of the integral,

$$\int_0^\infty e^{-zs} v(s)\, ds = 1/[zF(z)] \ .$$

In applying the theory of this section to more complicated operators, the reader will find very useful the great work of D. Bierens de Haan (1822-1895): *Nouvelles Tables d'Intégrales Définies,* published in 1867.

TABLE OF LAPLACE TRANSFORMS

	$F(z)$	$1/[zF(z)]$	$v(s)$
(1)	$(z+\lambda)/z$	$1/(z+\lambda)$	$e^{-\lambda s}$
(2)	$z^n/n!$	$n!/z^{n+1}$	s^n
(3)	$(z+\lambda)^{n+1}/z$	$n!/(z+\lambda)^{n+1}$	$s^n e^{-\lambda s}$
(4)	$(z^2+\lambda^2)/\lambda z$	$\lambda/(z^2+\lambda^2)$	$\sin \lambda s$
(5)	$(z^2+\lambda^2)/z^2$	$z/(z^2+\lambda^2)$	$\cos \lambda s$
(6)	$[(z+\mu)^2 + \lambda^2]/\lambda z$	$\lambda/[(z+\mu)^2+\lambda^2]$	$e^{-\mu s} \sin \lambda s$
(7)	$[(z+\mu)^2 + \lambda^2]/[z(z+\mu)]$	$(z+\mu)/[(z+\mu)^2+\lambda^2]$	$e^{-\mu s} \cos \lambda s$
(8)	$(z^2+\lambda^2)/\lambda^2$	$\lambda^2/[z(z^2+\lambda^2)]$	$1 - \cos \lambda s$
(9)	$(z^2+\lambda^2)/(z-\lambda)^2$	$(z-\lambda)^2/[z(z^2+\lambda^2)]$	$1 - 2 \sin \lambda s$
(10)	$(z+a)$	$1/[z(z+a)]$	$(1-e^{-as})/a$
(11)	$(z+a)^2$	$1/[z(z+a)^2]$	$[1-e^{-as}(1+as)]/a^2$
(12)	$(z+a)^3$	$1/[z(z+a)^3]$	$[1-e^{-as}(1+as + a^2s^2/2!)]/a^3$
(13)	$(z+a)^n$	$1/[z(z+a)^n]$	$[1-e^{-as}p_{n-1}(s)]/a^n$, $p_{n-1}(s) = 1 + as + a^2s^2/2! + a^3s^3/3! + \cdots + a^{n-1}s^{n-1}/(n-1)!$
(14)	$(z+a)(z+b)$	$1/[z(z+a)(z+b)]$	$1/ab + (e^{-bs}/b - e^{-as}/a)/(b-a)$
(15)	$(z+a)(z+b)/z$	$1/[(z+a)(z+b)]$	$(e^{-bs} - e^{-as})/(a-b)$
(16)	$(z+a)(z+b)/z^2$	$z/[(z+a)(z+b)]$	$(be^{-bs} - ae^{-as})/(b-a)$

	$F(z)$	$1/[zF(z)]$	$v(s)$
(17)	$z(z+a)(z+b)$	$1/[z^2(z+a)(z+b)]$	$(ab-a-b)/a^2b^2 + s/ab + (e^{-bs}/b^2 - e^{-as}/a^2)/(a-b)$
(18)	$(z+a)(z+b) \times (z+c)$	$1/[z(z+a)(z+b) \times (z+c)]$	$1/abc - e^{-as}/[a(a-b) \times (a-c)] + e^{-bs}/[b(a-b) \times (b-c)] - e^{-cs}/[c(a-c) \times (b-c)]$
(19)	$(z+a)(z+b) \times (z+c)/z$	$1/[(z+a)(z+b) \times (z+c)]$	$[(b-c)e^{-as} - (a-c)e^{-bs} + (a-b)e^{-cs}]/[(a-b) \times (a-c)(b-c)]$
(20)	$(z^2-a^2) \times (z+c)/z$	$1/[(z^2-a^2)(z+c)]$	$[\cosh as - (c/a)\sinh as - e^{-cs}]/(a^2-c^2)$
(21)	$(z^2+a^2) \times (z+c)/z$	$1/[(z^2+a^2)(z+c)]$	$[e^{-cs} + (c/a)\sin as - \cos as]/(a^2+c^2)$
(22)	$(z^2+\lambda^2)^{\frac{1}{2}}/z$	$(z^2+\lambda^2)^{-\frac{1}{2}}$	$J_0(\lambda s)$
(23)	$e^{\lambda(z^2+1)^{1/2}} \times (z^2+1)^{1/2}/z$	$e^{-\lambda(z^2+1)^{1/2}}(z^2+1)^{-\frac{1}{2}}$	$J_0[(s^2-\lambda^2)^{\frac{1}{2}}]$ for $s > \lambda$, 0 for $s \leqq \lambda$
(24)	$(a^n/z)(z^2+a^2)^{\frac{1}{2}} /[(z^2+a^2)^{\frac{1}{2}} - z]^n$	$[(z^2+a^2)^{\frac{1}{2}} - z]^n /[a^n(z^2+a^2)^{\frac{1}{2}}]$	$J_n(as)$
(25)	$e^{(az)^{1/2}}/z$	$e^{-(az)^{1/2}}$	$\frac{1}{2}(a/\pi)^{\frac{1}{2}}s^{-3/2}e^{-a/4s}$
(26)	$e^{(az)^{1/2}}$	$e^{-(az)^{1/2}}/z$	$1 - I[(a/4s)^{\frac{1}{2}}]$ where $I(x) = (2/\pi^{\frac{1}{2}})\int_0^x e^{-t^2}dt$
(27)	z^μ	$z^{-(\mu+1)}$	$s^\mu/\Gamma(\mu+1)$, $\mu > -1$
(28)	$z^{\frac{1}{2}}$	$z^{-3/2}$	$2s^{\frac{1}{2}}/\pi^{\frac{1}{2}}$
(29)	$z^{-\frac{1}{2}}$	$z^{-\frac{1}{2}}$	$1/(\pi s)^{\frac{1}{2}}$
(30)	$e^{a/z}$	$(e^{-a/z})/z$	$J_0[2(as)^{\frac{1}{2}}]$
(31)	$(z+\lambda)^{\frac{1}{2}}$	$1/[z(z+\lambda)^{\frac{1}{2}}]$	$\int_0^s [e^{-\lambda t}/(\pi t)^{\frac{1}{2}}]dt$
(32)	$(z+\lambda)^{\frac{1}{2}}/z$	$(z+\lambda)^{-\frac{1}{2}}$	$e^{-\lambda s}/(\pi s)^{\frac{1}{2}}$
(33)	$(z+\lambda)^{-\frac{1}{2}}$	$(z+\lambda)^{\frac{1}{2}}/z$	$e^{-\lambda s}/(\pi s)^{\frac{1}{2}} + \lambda \int_0^s [e^{-\lambda t}/(\pi t)^{\frac{1}{2}}]dt$

	$F(z)$	$1/[zF(z)]$	$v(s)$
(34)	$(1+2a/z)^{\frac{1}{2}}$	$(1+2a/z)^{-\frac{1}{2}}/z$	$e^{-as}I_0(as)$, where $I_0(x) = J_0(ix)$
(35)	$1 + (a/z)^{\frac{1}{2}}$	$1/\{z[1+(a/z)^{\frac{1}{2}}]\}$	$e^{as}[1 - (a/\pi)^{\frac{1}{2}} \int_0^s \{e^{-at}/t^{\frac{1}{2}}\}dt]$
(36)	$[(z+a)^2 - b^2]^{\frac{1}{2}}/z$	$[(z+a)^2 - b^2]^{-\frac{1}{2}}$	$e^{-as}I_0(bs)$
(37)	$z^{-\frac{1}{2}}(z+\lambda)$	$1/[z^{\frac{1}{2}}(z+\lambda)^{\frac{1}{2}}]$	$v(s) \sim (\lambda^2\pi s)^{-\frac{1}{2}} [1 + 1/(2\lambda x) + 1 \cdot 3/(2\lambda s)^2 + 1 \cdot 3 \cdot 5 / (2\lambda s)^3 + \cdots]$
(38)	$z^{-3/2}(z+\lambda)$	$z^{\frac{1}{2}}/(z+\lambda)$	$v(s) \sim -(\pi s)^{-\frac{1}{2}}[1/(2\lambda s) + 1 \cdot 3/(2\lambda s)^2 + 1 \cdot 3 \cdot 5/(2\lambda s)^3 + \cdots]$
(39)	$z^{a-1}/[\Gamma(a) \times (\Psi(a) - \log z)]$	$z^{-a}\Gamma(a)[\Psi(a) - \log z]$	$\log s\, s^{a-1}$
(40)	$-1/[(\pi z)^{\frac{1}{2}}(\log z + 2 \log 2 + C)]$, where C is Euler's constant.	$-(\pi/z)^{\frac{1}{2}}(\log z + 2 \log 2 + C)$	$\log s\, s^{-\frac{1}{2}}$

5. *Inversion as a Problem of the Fourier Transform.* Unquestionably the most important single method for the inversion of the operators which appear in the problems of mathematical physics is that founded upon the theorems of Fourier series and the Fourier integral. The method when it is warily applied exhibits extraordinary power and the existence of a large literature of both theoretical and formal results makes its application a comparatively simple one. We shall begin by stating without proof the fundamental theorem on Fourier series:

If $f(x)$ is a single valued function which has at most a finite number of infinite discontinuities and is of limited variation in the interval $-a \leq x \leq a$ when such discontinuities have been excluded, and for other values of x is defined by the equation,

$$f(x + 2a) = f(x) \quad ,$$

and if, moreover, the integral

$$\int_{-a}^{a} |f(x)| \, dx$$

exists, then $f(x)$ can be represented by means of the Fourier series,

$$f(x) = \tfrac{1}{2} A_0 + A_1 \cos(\pi x/a) + A_2 \cos(2\pi x/a) + \cdots$$
$$+ B_1 \sin(\pi x/a) + B_2 \sin(2\pi x/a) + \cdots \,, \qquad (5.1)$$

where the coefficients are determined from the integrals

$$A_n = \frac{1}{a} \int_{-a}^{a} f(t) \cos(n\pi t/a)\,dt \ , \quad B_n = \frac{1}{a} \int_{-a}^{a} f(t) \sin(n\pi t/a)\,ds \ .$$

The Fourier series gives the value,

$$\lim_{\varepsilon=0} \tfrac{1}{2}[f(x+\varepsilon) + f(x-\varepsilon)] \ ,$$

this limit existing except where the function is infinite.

In order to apply this theorem to operational analysis, let us write,

$$\cos px = \tfrac{1}{2}[e^{pxi} + e^{-pxi}] \ , \ \sin px = -\tfrac{1}{2}i[e^{pxi} - e^{-pxi}] \ ,$$

and note the operational identity: $F(z) \rightarrow e^{ax} = F(a)e^{ax}$.

Hence we get,

$$F(z) \rightarrow \cos px = \tfrac{1}{2}[F(pi)e^{pxi} + F(-pi)e^{-pxi}] \ ,$$
$$F(z) \rightarrow \sin px = -\tfrac{1}{2}i[F(pi)e^{pxi} - F(-pi)e^{-pxi}] \ .$$

If these operations are substituted in equation (5.1), with proper specialization of p, the following formal expansion is obtained:

$$F(z) \rightarrow f(x) = \tfrac{1}{2} A_0 F(0)$$
$$+ \tfrac{1}{2} \sum_{m=1}^{\infty} A_m[F(m\pi i/a)e^{m\pi x i/a} + F(-m\pi i/a)e^{-m\pi x i/a}]$$
$$- \tfrac{1}{2}i \sum_{m=1}^{\infty} B_m[F(m\pi i/a)e^{m\pi x i/a} - F(-m\pi i/a)e^{-m\pi x i/a}] \qquad (5.2)$$

If $F(z)$ is an even function, that is to say, if $F(z) = F(-z)$, then this expansion takes the simpler form,

$$F(z) \rightarrow f(x) = \tfrac{1}{2} A_0 F(0) + \sum_{m=1}^{\infty} A_m F(m\pi i/a) \cos(m\pi x/a)$$
$$+ \sum_{m=1}^{\infty} B_m F(m\pi i/a) \sin(m\pi x/a) \ . \qquad (5.3)$$

Similarly, if $F(z)$ is an odd function, that is to say, if $F(z) = -F(-z)$, then (5.2) becomes

$$F(z) \to f(x) = \tfrac{1}{2} A_0 F(0) + i \sum_{m=1}^{\infty} A_m F(m\pi i/a) \sin(m\pi x/a)$$
$$- i \sum_{m=1}^{\infty} B_m F(m\pi i/a) \cos(m\pi x/a) \ . \tag{5.4}$$

All these expansions, (5.2), (5.3) and (5.4), are valid representations of the operator $F(z) \to f(x)$ provided the series converge uniformly in the interval $(-a,a)$.

If the conditions which we have stated above for expansibility in a Fourier series do not hold, then equation (5.1) must be replaced by the *Fourier integral.* A sufficient condition for this representation is found in the following theorem:

If $f(x)$ is a single valued function which has at most a finite number of infinite discontinuities and is of limited variation in any finite interval from which such discontinuities have been excluded, and if the integral,

$$\int_{-\infty}^{\infty} |f(x)|\, dx$$

exists, then $f(x)$ may be represented by the integral,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} f(t) \cos s(x-t)\, dt \ . \tag{5.5}$$

The integral gives the value,

$$\lim_{\varepsilon=0} \tfrac{1}{2}[f(x+\varepsilon) + f(x-\varepsilon)] \ ,$$

this limit existing except where the function is infinite.

If the cosine in (5.5) is written in exponential form and a change of variable made in one of the terms, it will be found that the integral can be written in the following form:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} e^{\pm s(x-t)i} f(t)\, dt \ .$$

Hence the operation $F(z) \to f(x)$ can be represented formally by means of the integral,

$$F(z) \to f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} F(\pm si)\, e^{\pm s(x-t)i} f(t)\, dt \ , \tag{5.6}$$

provided the integral exists.

Choosing the plus sign in this representation for convenience, we now resolve $F(si)$ into its real and imaginary parts,

$$F(si) = G(s) + iH(s) \ ,$$

and thus obtain the expansion:

$$F(z) \to f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} [G(s) \cos s(x-t) - H(s) \sin s(x-t)]\, f(t)\, dt$$
$$+ \frac{i}{2\pi} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} [H(s) \cos s(x-t) + G(s) \sin s(x-t)] f(t)\, dt\,. \tag{5.7}$$

Since in general $F(z) \to f(x)$ is real, the second integral will vanish provided the path of integration does not pass over singularities of the integrand. In this case equation (5.7) becomes

$$F(z) \to f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} [G(s) \cos s(x-t) - H(s) \sin s(x-t)] f(t)\, dt\,. \tag{5.8}$$

A problem of peculiar interest that meets us on the threshold of the theory is the determination of the form of the Fourier integral when $f(x)$ is the unit function of the Heaviside calculus, that is,

$$f(x) = 1\,, \quad x \geqq 0\,,$$
$$f(x) = 0\,, \quad x < 0\,.$$

It is obvious that the criterion of the theorem quoted above is not met since the integral of $f(x)$ over the infinite range does not converge. Several devices have been formulated to overcome this difficulty. N. Wiener [see *Bibliography*: Weiner (4)] has employed a method for separating a function into *high-and-low frequency ranges,* the purpose of which is to segregate "those difficulties which come from the singularity of $f(x)$ at the origin from those which arrive from its behaviour at infinity." T. C. Fry [see *Bibliography*: Fry (2)] has used a "fortuitously chosen set of definitions" of a certain Stieltjes integral which, together with the Cesàro limit of a class of divergent integrals, is employed to interpret Fourier integrals that "in any ordinary sense, are without meaning." G. Giorgi [see *Bibliography*: Giorgi (4)] makes use of the idea of *improper functions* to the class of which the so called *impulsive functions* belong. He says in part:

"By the symbol

$$Fu(t)$$

and by the name of *impulsive unitary function* I mean a function of t which is everywhere $= 0$ for all values of t, except in an infinitesi-

mal interval containing the point $t = 0$, in which interval the function becomes infinitely great with such values that

$$\int_{-\infty}^{\infty} Fu(t)\,dt = 1 \ .$$

This $Fu(t)$ may be considered as a limiting form of a rectangle or of a function of a kind $A\, e^{-h^2t^2}$, or of others, according to the degree of continuity or of vicinity required. Following these ideas, we may, by differentiating $Fu(t)$, define impulsive functions of the second, third order and so on.

"At first sight it may appear that the use of these impulsive functions is strange and illegitimate, as it involves the consideration of actual infinitesimals and infinities. But in fact it is very useful because it simplifies the formulae very greatly and removes the exceptions; in fact not one of the most rigorous writers on dynamics has refrained from introducing the impulsions. As regards the theoretical standpoint, it is to be remarked that actual infinitesimals and infinities may be introduced with perfect rigour as a class of non-archimedean numbers, involving special postulates; or, what amounts to the same thing, we may say that all formulae containing improper functions are formulae wherein the sign of *lim*, is understood, so that $Fu(t)$ and similar symbols may be regarded as a kind of short-hand notation."

In order to include the unit function of the Heaviside calculus mentioned above, we shall adopt the device offered by the concept of impulsive functions and thus introduce a function, $Q(a,t)$, which has the following properties:

$$\text{I.} \quad \lim_{a=0} Q(a,t) = 1 \ , \qquad t \geqq 0 \ ,$$

$$\text{II.} \quad Q(a,t) = 0 \ , \qquad t < 0 \ ,$$

$$\text{III.} \quad \lim_{t=\infty} Q(a,t) = 0 \ , \qquad a > 0 \ ,$$

the order of vanishing being sufficient to assure the convergence of the integrals,

$$\int_0^{\infty} \cos s(x-t)\ Q(a,t)\,dt \quad \text{and} \quad \int_0^{\infty} \sin s(x-t)\ Q(a,t)\,dt \ .$$

It is sufficient to select for this function, $Q(a,t) = e^{-at}$, $t \geqq 0$. If this is then substituted in equation (5.8), there results,

$$F_a(z) \rightarrow f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \int_0^{\infty} [G(s) \cos s(x-t)$$
$$- H(s) \sin s(x-t)\, e^{-at}\, dt$$

$$F_a(z) \to f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(s) \left[\frac{s \sin xs}{a^2 + s^2} + \frac{a \cos xs}{a^2 + s^2}\right] ds$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} H(s) \left[\frac{s \cos xs}{a^2 + s^2} - \frac{a \sin xs}{a^2 + s^2}\right] ds \; .$$

Taking the limit, as $a \to 0$, we obtain the following representation of the operation $F(z) \to f(x)$, where $f(x)$ is the unit function described above:

$$\lim_{a \to 0} F_a(z) \to f(x) = F(z) \to f(x)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(s) \frac{\sin xs}{s} ds + \frac{1}{2\pi} \int_{-\infty}^{\infty} H(s) \frac{\cos xs}{s} ds \; . \tag{5.9}$$

If we note from the assumptions made regarding the unit function over the negative range that $F(z) \to f(-x) = 0$, that is to say,

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} G(s) \frac{\sin xs}{s} ds + \frac{1}{2\pi} \int_{-\infty}^{\infty} H(s) \frac{\cos xs}{s} ds = 0 \; ,$$

then we can write (5.9) as follows:

$$F(z) \to f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} G(s) \frac{\sin xs}{s} ds = \frac{1}{\pi} \int_{-\infty}^{\infty} H(s) \frac{\cos xs}{s} ds \; . \tag{5.10}$$

In case $G(s) = G(-s)$ and $H(s) = -H(-s)$, this equation can be written:

$$F(z) \to f(x) = \frac{2}{\pi} \int_{0}^{\infty} G(s) \frac{\sin xs}{s} ds = \frac{2}{\pi} \int_{0}^{\infty} H(s) \frac{\cos xs}{s} ds \; . \tag{5.11}$$

A few examples will illustrate the application of these formulas.

Example 1. Let us first consider the operator $F(z) = z/(z + \lambda)$. Since

$$F(si) = s^2/(\lambda^2 + s^2) + \lambda\, si/(\lambda^2 + s^2) \; ,$$

and $G(s) = G(-s)$, $H(s) = -H(-s)$, we have

$$F(z) \to f(x) = \frac{2}{\pi} \int_{0}^{\infty} G(s) \frac{\sin xs}{s} ds = \frac{2}{\pi} \int_{0}^{\infty} \frac{s \sin xs}{\lambda^2 + s^2} ds = e^{-\lambda x} \; .$$

The connection between this method and the method of Carson is immediately ascertained by referring to the first formula in the table of section 4.

Example 2. As a second illustration consider the operator,

$$F(z) = \lambda z/[(z+\mu)^2 + \lambda^2] ,$$

from which we derive,

$$G(s) = 2s^2\mu\lambda/[(s^2+\mu^2-\lambda^2)^2 + 4\lambda^2\mu^2] ,$$

$$H(s) = \lambda s[(\lambda^2+\mu^2) - s^2]/[(s^2+\mu^2-\lambda^2)^2 + 4\lambda^2\mu^2] .$$

Since $G(s) = G(-s)$ and $H(s) = -H(-s)$, we then obtain from (5.11)

$$F(z) \to f(x) = \frac{2}{\pi} \int_0^\infty \frac{2\, s\, \mu\, \lambda \sin xs}{(s^2+\mu^2-\lambda^2)^2 + 4\lambda^2\mu^2}\, ds = e^{-x\mu} \sin \lambda x \;.$$

This answer will be found to coincide with that obtained by the Carson method as recorded in (6) of the table in section 4.

Example 3. As a somewhat more complicated example, let us consider the fractional operator,

$$F(z) = z^a , \quad a < 1 ,$$

from which we derive, $G(s) = \cos \tfrac{1}{2}\, a\, \pi\, s^a$, $H(s) = \sin \tfrac{1}{2}\, a\, \pi\, s^a$.

Since both $G(s) \sin xs/s$ and $H(s) \cos xs/s$ have branch points at the origin, the path of integration in (5.7) must be deformed so as to pass below the origin. One then obtains,

$$\frac{1}{2\pi} \int_{-\infty}^\infty G(s) \frac{\sin xs}{s}\, ds$$
$$= \sin \tfrac{1}{2}\, a\, \pi \cos \tfrac{1}{2}\, a\, \pi\, \Gamma(a)\, (1 + e^{-\pi a i})/(2\, \pi\, x^a) ,$$

$$\frac{1}{2\pi} \int_{-\infty}^\infty H(s) \frac{\sin xs}{s}$$
$$= \sin^2 \tfrac{1}{2}\, a\, \pi\, \Gamma(a)\, (1 + e^{-\pi a i})/(2\, \pi\, x^a) ,$$

$$\frac{1}{2\pi} \int_{-\infty}^\infty G(s) \frac{\cos xs}{s}\, ds$$
$$= \cos^2 \tfrac{1}{2}\, a\, \pi\, \Gamma(a)\, (1 - e^{-\pi a i})/(2\, \pi\, x^a) ,$$

$$\frac{1}{2\pi} \int_{-\infty}^\infty H(s) \frac{\cos xs}{s}\, ds$$
$$= \sin \tfrac{1}{2}\, a\, \pi \cos \tfrac{1}{2}\, a\, \pi\, \Gamma(a)\, (1 - e^{-\pi a i})/(2\, \pi\, x^a) .$$

Substituting these values in the formula,

$$F(z) \to f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(s) \frac{\sin xs}{s} ds + \frac{1}{2\pi} \int_{-\infty}^{\infty} H(s) \frac{\cos xs}{s} ds$$

$$+ \frac{i}{2\pi} \int_{-\infty}^{\infty} H(s) \frac{\sin xs}{s} ds - \frac{i}{2\pi} \int_{-\infty}^{\infty} G(s) \frac{\cos xs}{s} ds ,$$

we obtain,

$$z^a \to f(x) = \Gamma(a) \sin a\pi/(\pi x^a) .$$

By means of the identity, $\Gamma(a) = \pi/[\sin a\pi\, \Gamma(1-a)]$, this may be simplified to

$$z^a \to f(x) = 1/[\Gamma(1-a) x^a] ,$$

which agrees with the result stated in (27) of the table given in section 4.

One of the most useful features of the Fourier theory of operational process is found in the ease with which inverses may be constructed. This inversion is obtained through the use of what is called the *Fourier transform*, which we write as follows:

$$f(x) = \sqrt{2/\pi} \int_0^{\infty} \cos xs\, g(s) ds ,$$

$$g(s) = \sqrt{2/\pi} \int_0^{\infty} \cos sx\, f(x) dx ; \qquad (5.12a)$$

$$f(x) = \sqrt{2/\pi} \int_0^{\infty} \sin xs\, g(s) ds ,$$

$$g(s) = \sqrt{2/\pi} \int_0^{\infty} \sin sx\, f(x) dx . \qquad (5.12b)$$

These inversions are valid under the general conditions stated in the Fourier integral theorem previously quoted. They may be used, in particular, to reconstruct operators from functions derived from their application to the unit function of the Heaviside calculus.

Thus, if we designate by $J(x)$ the function $F(z) \to f(x)$, we shall have from (5.11),

$$J(x) = \frac{2}{\pi} \int_0^{\infty} \frac{G(s)}{s} \sin xs\, ds = \frac{2}{\pi} \int_0^{\infty} \frac{H(s)}{s} \cos xs\, ds .$$

From this we obtain by means of (5.12a) and (5.12b) the inversions,

$$G(s) = s \int_0^{\infty} J(t) \sin st\, dt , \qquad H(s) = s \int_0^{\infty} J(t) \cos st\, dt .$$

The operator $F(z)$ is then constructed from the equation,

$$F(z) = G(-iz) + iH(-iz) .$$

For example, if $J(t) = e^{-at}$, we obtain

$$G(s) = s^2/(a^2 + s^2) \ , \quad H(s) = as/(a^2 + s^2) \ ,$$

and the corresponding operator is,

$$F(z) = -z^2/(a^2 - z^2) - i(iz)a/(a^2 - z^2) = z/(z + a) \ .$$

As an aid to application the following table has been appended which gives a few of the Fourier transforms frequently encountered. An exhaustive table to which reference has been made in the first chapter was prepared in 1931 by G. A. Campbell and R. M. Foster.* Numerous complicated examples are also to be found in the work of D. Bierens de Haan: *Nouvelles Tables d'Intégrales Définies,* referred to in the preceding section.

6. The Solution of Fractional Equations. In preceding sections we have applied the method of operators to differential equations in which z appeared to integral powers. Many of the most interesting applications of the operational calculus, however, will be found associated with problems in which z enters with a fractional index. It is desirable, therefore, to consider a rather general formulization of the operational solution of equations of the following forms:

$$L(u) = A_0 u + A_1 u^{(1/n)} + A_2 u^{(2/n)} + \cdots + A_{n-1} u^{(1-1/n)} = f(x) \ , \qquad (6.1)$$

$$M(u) = B_0 u + B_1 u^{-(1/n)} + B_2 u^{-(2/n)} + \cdots + B_{n-1} u^{-(1-1/n)} = g(x) \ , \qquad (6.2)$$

where $A_0, A_1, \cdots, A_{n-1}$ and $B_0, B_1, \cdots, B_{n-1}$ are constants.

Let us designate the generatrix functions of these equations by $F(z)$ and $G(z)$ respectively, that is to say,

$$F(z) = A_0 + A_1 z^{1/n} + A_2 z^{2/n} + \cdots A_{n-1} z^{1-1/n} \ ,$$
$$G(z) = B_0 + B_1 z^{-1/n} + B_2 z^{-2/n} + \cdots B_{n-1} z^{-1+1/n} \ .$$

To begin with let us consider $F(z)$ for which we shall first find a rationalizing factor. In order to accomplish this we multiply $F(z)$ successively by $z^{1/n}$, $z^{2/n}$, etc., thus obtaining the system of equations,

$$\begin{aligned}
F(z) &= A_0 + A_1 z^{1/n} + A_2 z^{2/n} + \cdots + A_{n-1} z^{1-1/n} \ , \\
z^{1/n} F(z) &= z A_{n-1} + A_0 z^{1/n} + A_1 z^{2/n} + \cdots + A_{n-2} z^{1-1/n} \ , \\
z^{2/n} F(z) &= z A_{n-2} + z A_{n-1} z^{1/n} + A_0 z^{2/n} + \cdots + A_{n-3} z^{1-1/n} \ , \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
z^{1-1/n} F(z) &= z A_1 + z A_2 z^{1/n} + z A_3 z^{2/n} + \cdots + A_0 z^{1-1/n} \ .
\end{aligned}$$

*Fourier Integrals for Practical Applications. Bell Telephone System. *Technical Publications* (1931), Monograph B-584. 177 p.

$$f(x) = \int_0^\infty \sin xt \, g(t) \, dt$$

	$g(t)$	$f(x)$
(1a)	t^{p-1}	$x^{-p}\Gamma(p) \sin \frac{1}{2}p\pi \;, \; (p^2 < 1)$.
(2a)	$1/t$	$\frac{1}{2}\pi \;, \; x > 0$ $0 \;, \; x = 0$ $-\frac{1}{2}\pi \;, \; x < 0$
(3a)	$t/(a^2 + t^2)$	$\frac{1}{2}\pi \, e^{-ax}$
(4a)	$\sin xt/(a^2 + t^2)$	$(\pi/4a)(1 - e^{-2ax})$
(5a)	$t/[(t^2 + a^2)^2 + b^2]$	$(\pi/2b) \, e^{-x\lambda} \sin \mu x$*
(6a)	$\dfrac{t(t^2 + a^2)}{(t^2 + a^2)^2 + b^2}$	$\frac{1}{2}\pi e^{-x\lambda} \cos \mu x$*
(7a)	e^{-at}	$x/(a^2 + x^2)$
(8a)	e^{-at^2}	$\sum_{n=0}^{\infty} (-1)^n \dfrac{(n+1)!}{(2n+2)!} \left(\dfrac{x^2}{a}\right)^n \left(\dfrac{x}{a}\right)$
(9a)	$(e^{at} + 1)^{-1}$	$1/2x - \pi/(2a \sinh x\pi/a)$
(10a)	$(e^{at} - 1)^{-1}$	$\dfrac{(\pi \cosh x\pi/a)}{(2a \sinh x\pi/a)} - \dfrac{1}{2x}$
(11a)	$1/\sinh at$	$\dfrac{\pi \sinh x\pi/2a}{2a \cosh x\pi/2a}$
(12a)	$t^{2m+1} e^{-\frac{1}{2}t}$	$(-1)^m \sqrt{\frac{1}{2}\pi} \, e^{\frac{1}{2}x^2} h_{2m+1}(x)$†
(13a)	$J_n(at)/t$	$n^{-1} \sin\{n \operatorname{arc\,sin}(x/a)\} \;, \; x \leqq a$, $\dfrac{a^n \sin \frac{1}{2}n\pi}{n\{x + (x^2 - a^2)^{\frac{1}{2}}\}^n} \;, \; x \geqq a$, $R(n) > -1$.
(14a)	$J_n(at)$	$\dfrac{\sin\{n \operatorname{arc\,sin}(x/a)\}}{(a^2 - x^2)^{\frac{1}{2}}} \;, \; x \leqq a$, ∞ or $0 \;, \; x = a$, $\dfrac{a^n \cos \frac{1}{2}n\pi}{(x^2 - a^2)^{\frac{1}{2}}\{x + (x^2 - a^2)^{1/2}\}^n}$ $x > a \;, \; R(n) > -2$,
(15a)	$J_0(at)$	$(x^2 - a^2)^{-\frac{1}{2}} \;, \; x > a$. $\infty \;, \; x = a$, $0 \;, \; x < a$,
(16a)	$t/(a^2 + t^2)^4$	$\dfrac{\pi}{96a^4}(3 + 3ax + a^2x^2) \, x e^{-ax}$
(17a)	$[t/(t^2 - 1)]^{1/2}$	$(\pi/4x)^{1/2}(\cos x + \sin x)$
(18a)	$\operatorname{arc\,cot}(t/a)$	$(\pi/2x)(1 - e^{-ax})$
(19a)	$t^{p-1} e^{-at}$	$\dfrac{\Gamma(p)}{(a^2 + x^2)^{\frac{1}{2}p}} \sin(p \operatorname{arc\,tan} x/a)$
(20a)	$\dfrac{e^t - e^{-t}}{e^t + e^{-t} + 2 \cos a}$	$\pi \dfrac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}} \;, \; a \leqq \pi$

*$2\lambda^2 = (a^4 + b^2)^{\frac{1}{2}} + a^2 \;, \quad 2\mu^2 = (a^4 + b^2)^{\frac{1}{2}} - a^2$.
†$h_n(x) = (-1)^n \phi_n(x)/\phi_0(x)$, where $\phi_0(x) = e^{-\frac{1}{2}x^2}$, $\phi_n(x) = d^n \phi_0(x)/dx^n$.
See A. Adamoff: *Ann. de St. Pétersbourg*, vol. 5 (1906), pp. 127-143.

$$f(x) = \int_0^{\infty} \cos xt\, g(t)\, dt$$

	$g(t)$	$f(x)$
(1b)	t^{p-1}	$x^{-p}\Gamma(p)\cos \frac{1}{2}p\pi$, $(p^2 < 1)$.
(2b)	$a/(a^2 + t^2)$	$\frac{1}{2}\pi e^{-ax}$
(3b)	$\cos xt/(a^2 + t^2)$	$(\pi/2a)(1 + e^{-2ax})$
(4b)	$1/[(t^2 + a^2)^2 + b^2]$	$\dfrac{\pi\, e^{-x\lambda}}{2b(a^4+b^2)^{1/2}}(\mu \cos \mu x^{*} + \lambda \sin \mu x)$.
(5b)	$\dfrac{t^2 + a^2}{(t^2 + a^2)^2 + b^2}$	$\frac{1}{2}\pi \dfrac{e^{-x\lambda}}{(a^4+b^2)^{1/2}}(\lambda \cos \mu x^{*} - \mu \sin \mu x)$
(6b)	e^{-at}	$a/(a^2 + x^2)$
(7b)	e^{-at^2}	$\frac{1}{2}\sqrt{\pi/a}\, e^{-x^2/4a}$
(8b)	$1/\cosh at$	$\pi/(2a \cosh x\pi/2a)$
(9b)	$(1 - e^{-t})^{-1}$	$\sum_{n=0}^{\infty} n/(n^2 + x^2)$
(10b)	$t^{2m} e^{-\frac{1}{2}t}$	$(-1)^m \sqrt{\frac{1}{2}\pi}\, e^{\frac{1}{2}x^2} h_{2m}(x)$ †
(11b)	$1/(\lambda^2 - t^2)^{\frac{1}{2}}$	$\pi J_0(x\lambda)$, $x > 0$.
(12b)	$J_n(at)/t$	$n^{-1}\cos\{n \text{ arc sin } (x/a)\}$ $x \leqq a$, $\dfrac{a^n \cos \frac{1}{2}n\pi}{n\{x + (x^2 - a^2)^{\frac{1}{2}}\}^n}$ $x \geqq a$, $R(n) > 0$.
(13b)	$J_n(at)$	$\dfrac{\cos\{n \text{ arc sin}(x/a)\}}{(a^2 - x^2)^{\frac{1}{2}}}$ $x < a$, ∞ or 0 , $x = a$, $\dfrac{a^n \sin \frac{1}{2}n\pi}{(x^2 - a^2)^{1/2}\{x + (x^2 - a^2)^{1/2}\}^n}$ $x > a$, $R(x) > -1$.
(14b)	$J_0(at)$	$(x^2 - a^2)^{-\frac{1}{2}}$, $x < a$, ∞ , $x = a$, 0 , $x > a$,
(15b)	$t^2/(a^2 + t^2)^2$	$\dfrac{\pi}{4a}(1 - ax)e^{-ax}$
(16b)	$[t/(t^2 - 1)]^{1/2}$	$(\pi/4x)^{1/2}(\cos x - \sin x)$
(17b)	$\log(1 + a^2/t^2)$	$(\pi/x)(1 - e^{-ax})$
(18b)	$t^{p-1} e^{-ax}$	$\dfrac{\Gamma(p)}{(a^2 + x^2)^{\frac{1}{2}p}} \cos(p \text{ arc tan } x/a)$
(19b)	$\dfrac{e^t + e^{-t}}{e^t + e^{-t} + 2\cos a}$	$-\pi \cot a \dfrac{e^{ax} - e^{-ax}}{e^{\pi x} - e^{-\pi x}}$, $a \leqq \pi$

*$2\lambda^2 = (a^4 + b^2)^{\frac{1}{2}} + a^2$, $2\mu^2 = (a^4 + b^2)^{\frac{1}{2}} - a^2$.

†$h_n(x) = (-1)^n \phi_n(x)/\phi_0(x)$, where $\phi_0(x) = e^{-\frac{1}{2}x^2}$, $\phi_n(x) = d_n\, \phi_0(x)/dx^n$.

Eliminating $z^{1/n}, z^{2/n}, \cdots, z^{1-1/n}$ from the right hand side of these equations we obtain

$$\begin{vmatrix} A_0 - F(z) , & A_1 , & A_2 , \cdots , A_{n-1} \\ zA_{n-1} - z^{1/n}F(z) , & A_0 , & A_1 , \cdots , A_{n-2} \\ zA_{n-2} - z^{2/n}F(z) , & zA_{n-1} , & A_0 , \cdots , A_{n-3} \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ zA_1 - z^{1-1/n}F(z) , & zA_2 , & zA_3 , \cdots , A_0 \end{vmatrix} = 0 .$$

From this it follows that

$$F(z)\Delta(z) = A(z) ,$$

where we abbreviate,

$$\Delta(z) = \begin{vmatrix} 1 , & A_1 , & A_2 , \cdots , A_{n-1} \\ z^{1/n} , & A_0 , & A_1 , \cdots , A_{n-2} \\ z^{2/n} , & zA_{n-1} , & A_0 , \cdots , A_{n-3} \\ \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot \\ z^{1-1/n} , & zA_2 , & zA_3 , \cdots , A_0 \end{vmatrix} \tag{6.3}$$

and $A(z)$ is the same determinant with the first column replaced by $A_0, zA_{n-1}, zA_{n-2}, \cdots, zA_1$.

Since $A(z)$ is a polynomial in z of degree less than or equal to n—1 it is clear that $\Delta(z)$ is a rationalizing factor and equation (6.1) can be formally replaced by the ordinary linear equation

$$A(z) \to u(x) = \Delta(z) \to f(x) . \tag{6.4}$$

To this equation there corresponds an integral of the form

$$u(x) = \int_c^x W(x,t) \; [\Delta(d/dt) \to f(t)] \; dt , \tag{6.5}$$

where $W(x,t)$ is the Cauchy function associated with (6.4) [See section 2, chapter 11], provided $W(x,t)[\Delta(d/dt) \to f(t)]$ does not become infinite to a higher order than $\mu < 1$ at $t = c$.

It will be noticed, however, that equation (6.5) is not a complete solution of the original equation since we must adjoin to it such solutions as exist for the homogeneous equation,

$$F(z) \to u(x) = 0 . \tag{6.7}$$

Let us assume that a solution exists of the form,

$$u(x) = \Delta(z) \to v(x) , \tag{6.8}$$

where $v(x)$ must be further specified. Substituting this in (6.7) and noting the commutability of the operators $F(z)$ and $\Delta(z)$, we readily obtain,

$$
\begin{aligned}
F(z) \to u(x) &= F(z) \to [\Delta(z) \to v(x)] \ , \\
&= [\Delta(z) \to F(z)] \to v(x) \ , \\
&= A(z) \to v(x) = 0 \ .
\end{aligned}
$$

Hence if $v(x)$ is any solution of the ordinary differential equation

$$A(z) \to v(x) = 0 \ ,$$

then (6.8) *is a formal solution of* (6.7). *Since, moreover, there will be in general n—1 such functions $v(x)$, we see that in general equation* (6.7) *will have n—1 formal solutions. Some of these solutions, however, may be extraneous.*

It is of some interest to record the explicit solutions for the first three cases, namely, where $n = 2$, 3 and 4.

Case 1. We shall have for the equation,

$$(A_0 + A_1 z^{1/2}) \to u(x) = 0 \ ,$$

the solution,

$$u(x) = (A_0 - A_1 z^{\frac{1}{2}}) \to v(x) \ ,$$

where $v(x)$ is a solution of the differential equation,

$$A_0^2 v(x) - A_1^2 v'(x) = 0 \ ,$$

that is to say, $v(x) = C\, e^{kx}$, where $k = A_0^2/A_1^2$.

Case 2. For the equation,

$$(A_0 + A_1 z^{1/3} + A_2 z^{2/3}) \to u(x) = 0 \ ,$$

we have the solutions,

$$
\begin{aligned}
u(x) = [A_0^2 - A_0 A_1 z^{1/3} + (A_1^2 - A_0 A_2) z^{2/3} \\
- A_1 A_2 z + A_2^2 z^{4/3}] \to v_i(x) \ ,
\end{aligned}
$$

where the $v_i(x)$ are solutions of the differential equation,

$$A_0^3 v(x) + (A_1^3 - 3A_0 A_1 A_2) v'(x) + A_2^3 v''(x) = 0 \ .$$

Case 3. Considering the equation,

$$(A_0 + A_1 z^{1/4} + A_2 z^{2/4} + A_3 z^{3/4}) \to u(x) = 0 \ ,$$

we find the solutions,

$$
\begin{aligned}
u(x) = [&A_0^3 - A_0^2 A_1 z^{1/4} + (A_0 A_1^2 - A_0^2 A_2) z^{2/4} \\
&+ (A_1^3 + 2A_0 A_1 A_2 - A_0^2 A_3) z^{3/4} + (A_1^2 A_2 - 2A_0 A_1 A_3 \\
&- A_0 A_2^2) z + (A_1^2 A_3 + 2A_0 A_2 A_3 - A_1 A_2^2) z^{5/4} \\
&+ (A_2^3 - 2A_1 A_2 A_3 + A_0 A_3^2) z^{6/4} + (A_1 A_3^2 \\
&- A_2^2 A_3) z^{7/4} + A_2 A_3^2 z^2 - A_3^3 z^{9/4}] \to v_i(x) \ ,
\end{aligned}
$$

where the functions $v_i(x)$ are solutions of the differential equation

$$\begin{aligned} A_0^4 v(x) &+ (4A_0A_1^2A_2 - 2A_0^2A_2^2 - 4A_0^2A_1A_3 + A_1^4)v'(x) \\ &+ (4A_0A_2A_3^2 - 4A_1A_2^2A_3 + 2A_1^2A_3^2 + A_2^4)v''(x) \\ &- A_3^4 v^{(3)}(x) = 0 \ . \end{aligned}$$

In the application of these formulas it is frequently necessary to use the following derivative:

If

$$v(x) = \int_c^x \frac{g(x,s)u(s)}{(x-s)^a} ds \ ,$$

where $u(s)$ is continuous in the interval $c \leqq s \leqq x$ and $g(x,s)$ in the triangle $c \leqq s \leqq x \leqq x_0$ and where $0 < a < 1$, we then have

$$\begin{aligned} v'(x) = \frac{g(x,c)u(c)}{(x-c)^a} &+ \int_c^x \frac{D_x g(x,s) + D_s g(x,s)}{(x-s)^a} u(s) ds \\ &+ \int_c^x \frac{g(x,s)}{(x-s)^a} u'(s) ds \ . \qquad (6.9) \end{aligned}$$

This formula can be established by the method of "partie finie" due simultaneously to R. d'Adhémar and J. Hadamard in 1904.* A generalization of their method is contained in the following theorem from which (6.9) is derived as a special case:

Theorem 4. If $F(x)$ is defined by the integral

$$F(x) = \int_{B(x)}^{A(x)} f[u(x,s)]h(x,s) ds \ ,$$

where $A(x)$ and $B(x)$ are continuous functions of x with continuous first derivatives such that either $\lim_{s=A} f[u(x,s)]$ or $\lim_{s=B} f[u(x,s)]$ or both are infinite, although $f(x,s)$ is finite and continuous elsewhere in the interval (A,B) and where $u(x,s)$ and $h(x,s)$ are continuous functions having first and second derivatives with respect to both variables, and if $T(\varepsilon,x)$ is defined by the following expression:

$$\begin{aligned} T(\varepsilon,x) = &\left[\frac{dA}{dx} + \imath(x,A-\varepsilon)\right] f(x, A-\varepsilon)h(x, A-\varepsilon) \\ &- \left[\frac{dB}{dx} + \imath(x,B+\varepsilon)\right] f(x, B+\varepsilon)h(x, B+\varepsilon) \ , \end{aligned}$$

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*R.d'Adhémar: Thesis: *Journal de Math.*, (1904); see also *Exercices et leçons d'analyse*, Paris, (1908), p. 150; J. Hadamard: *Congres de Mathém.*, (1904).

where $\imath(x,s) = \dfrac{\partial u}{\partial x}\Big/\dfrac{\partial u}{\partial s}$, *then the derivative of* $F(x)$ *will be*

$$\frac{dF}{dx} = \int_B^A [D_x h - \imath\, D_s h - h D_s \imath] f\, ds + \lim_{\varepsilon=0} T(\varepsilon,x) \quad , (6.10)$$

provided $\lim\limits_{\varepsilon=0} T(\varepsilon,x)$ *and the integral both exist.*

Proof: Since both f and h are regular in s in the open interval $B < s < A$, we may apply the ordinary rule for differentiation under the sign of integration to the integral

$$F_\varepsilon(x) = \int_{B+\varepsilon}^{A-\varepsilon} f[u(x,s)]h(x,s)\, ds \ ,$$

where ε is an arbitrarily chosen positive quantity, and thus obtain

$$\frac{dF_\varepsilon}{dx} = \int_{B+\varepsilon}^{A-\varepsilon} \left[\frac{\partial f}{\partial x} h + \frac{\partial h}{\partial x} f\right] ds + \frac{dA}{dx} f(x,A-\varepsilon)h(x,A-\varepsilon)$$
$$- \frac{dB}{dx} f(x,B+\varepsilon)h(x,B+\varepsilon) \ .$$

Since $\dfrac{\partial f}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial f}{\partial s} = \dfrac{df}{du}\dfrac{\partial u}{\partial s}$, we shall have

$$\frac{\partial f}{\partial x} = \imath(x,s)\,\frac{\partial f}{\partial s} \text{ where } \imath(x,s) = \frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial s} \ .$$

Substituting above and integrating by parts we shall have

$$\frac{dF_\varepsilon}{dx} = \int_{B+\varepsilon}^{A-\varepsilon} [D_x h - \imath D_s h - h D_s \imath]\, f\, ds + T(\varepsilon,x) \ ,$$

where $T(\varepsilon,x)$ is the function already defined.

If the integral and $T(\varepsilon,x)$ approach finite limits as ε approaches zero, then $\dfrac{dF}{dx}$ will be the limit of the right hand member of this equation.

Equation (6.9) follows as a special case by letting

$$f(u) = \frac{1}{u^a}, \quad u(x,s) = (x-s) \ , \quad h(x,s) = g(x,s)u(s) \ .$$

We then have $\imath(x,s) = -1$ and

$$T(\varepsilon, x) = (1-1)g(x, x-\varepsilon)u(x-\varepsilon)/\varepsilon^a$$
$$+ g(x, c+\varepsilon)u(c+\varepsilon)/(x-c-\varepsilon)^a \ ,$$

which when substituted in (6.10) yield the desired derivative.

Let us now consider two simple examples.

Example 1. We shall solve,

$$\lambda u(x) + {}_0D_x^{\frac{1}{2}} u(x) = f(x) \ .$$

Rationalizing we obtain the ordinary equation

$$du/dx - \lambda^2 u = {}_0D_x^{\frac{1}{2}} f(x) - \lambda f(x) \ ,$$

the solution of which is

$$u(x) = e^{\lambda^2 x} \int_0^x \{ ({}_0D^{\frac{1}{2}}{}_t - \lambda) \ f(t) \} \, e^{-\lambda^2 t} \, dt \ .$$

To this we must add the solution of the homogeneous equation,

$$v(x) + {}_0D_x^{\frac{1}{2}} v(x) = 0 \ ,$$

which we readily find to be,

$$v(x) = C(\lambda - {}_0D_x^{\frac{1}{2}}) \ e^{\lambda^2 x} \ .$$

Let us specialize these results by setting $\lambda = -1$, $f(x) = -1$. We then obtain,

$$\begin{aligned} u(x) &= C'\{e^x + (\pi x)^{\frac{1}{2}} + (e^x \pi^{-\frac{1}{2}}) \int_0^x e^{-t} t^{-\frac{1}{2}} \, dt\} \\ &\qquad\qquad + e^x \int_0^x e^{-t}\{-1 - (\pi t)^{-\frac{1}{2}}\} dt \\ &= C'\{e^x + (\pi x)^{-\frac{1}{2}} + 2e^x (x/\pi)^{\frac{1}{2}} (1 - x/3 + x^2/5 \cdot 2! \\ &\qquad - x^3/7 \cdot 3! + \cdots)\} + 1 - e^x - 2e^x (x/\pi)^{\frac{1}{2}}) \ (1 - x/3 \\ &\qquad\qquad + x^2/5 \cdot 2! - x^3/7 \cdot 3! + \cdots) \ , \\ &= C'\{e^x + (\pi x)^{-\frac{1}{2}} + (\pi x)^{-\frac{1}{2}}[(2x) + (2x)^2/1 \cdot 3 \\ &\qquad + (2x)^3/1 \cdot 3 \cdot 5 + \cdots]\} + 1 - e^x - (\pi x)^{-\frac{1}{2}} \ [(2x) \\ &\qquad\qquad + (2x)^2/1 \cdot 3 + (2x)^3/1 \cdot 3 \cdot 5 + \cdots] \ . \end{aligned}$$

Example 2. Let us consider the equation,

$$u(x) + {}_0D_x^{1/3} \, u(x) + {}_0D_x^{2/3} \, u(x) = 0 \ .$$

We obtain at once from *case 2* above the solutions,

$$u_i(x) = (1 - z^{1/3} - z + z^{4/3}) \rightarrow v_i(x) \ ,$$

where the $v_i(x)$ are the solutions of the differential equation,

$$v(x) - 2v'(x) + v''(x) = 0 \ ,$$

that is to say, $v_1(x) = e^x$, $v_2(x) = xe^x$.

If we introduce the first of these functions into the solution, we reach the conclusion that

$$u_1(x) = -\frac{1}{3\Gamma(2/3)}\, x^{-1/3}$$

is a solution of the original equation. This is obviously not correct.

A further examination of the expression $(1 - z^{1/3} - z + z^{4/3})$ shows that it may be factored into the product $(1-z)(1-z^{1/3})$. Eliminating the extraneous factor $(1-z)$, we conclude that the original equation has only one solution,

$$u(x) = (1 - z^{1/3}) \to v(x) \ ,$$

where $v(x)$ is a solution of the equation,

$$v(x) - v'(x) = 0 \ ,$$

that is to say

$$\begin{aligned} u(x) &= (1 - z^{1/3}) \to e^x \ , \\ &= e^x - [e^x \int_0^x t^{-1/3} e^{-t}\, dt + x^{-1/3}]/\Gamma(2/3) \ . \end{aligned}$$

Returning now to the second solution, $u_2(x)$, obtained from $v_2(x)$, we find by direct computation that $u_2(x)$ is identical, except for a constant factor, with the value of $u(x)$ just written down.

From this example we learn that *extraneous solutions* may appear in the formal solution of fractional differential equations just as they appear in the analogous solution of algebraic equations with fractional index.

Integral Equations of Fractional Index

We now proceed to a discussion of integral equations of fractional index, that is to say, the integral equation,

$$G(z) \to u(x) = g(x) \ , \tag{6.11}$$

where $G(z)$ is the generatrix of the operator defined by (6.2).

As in the preceding case of differential equations of fractional index we first obtain a rationalizing factor, $\Delta'(z)$, which by an argument essentially the same as that used above we find to be,

$$\Delta'(z) = \begin{vmatrix} 1\ , & B_1\ , & B_2\ , & \cdots, & B_{n-1} \\ z^{-1/n}\ , & B_0\ , & B_1\ , & \cdots, & B_{n-2} \\ z^{-2/n}\ , & z^{-1}B_{n-1}\ , & B_0\ , & \cdots, & B_{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z^{-1+1/n}\ , & z^{-1}B_2\ , & z^{-1}B_3\ , & \cdots, & B_0 \end{vmatrix} \ . \tag{6.12}$$

Hence we may replace equation (6.11) formally by the rationalized equation,

$$B(z) \to u(x) = \Delta'(z) \to g(x) \ , \qquad (6.13)$$

where $B(z)$ is the determinant $\Delta'(z)$ in which the elements of the first column have been replaced by B_0, $z^{-1} B_{n-1}$, $z^{-1} B_{n-2}, \cdots, z^{-1} B_1$.

It is obvious that the solution of (6.13) will be unique since the only continuous solution of the integral equation,

$$B(z) \to u(x) = 0 \ ,$$

is in general $u(x) = 0$. The only exception to be noted here is the case where the integral z^{-1} has an infinite limit.

As an example let us consider the inversion of the generalized *integral equation of Abel*, a special case of which first appeared in the tautochrone problem. This equation may be written in the form,

$$g(x) = \int_c^x (x-t)^{m+a} u(t)\,dt \ , \ -1 < a < 1 \ , \ m = 0, 1, 2, \cdots \qquad (6.14)$$

In terms of fractional operators this may be written,

$$g(x) = \Gamma(m+a+1) \ {}_cD_x^{-(m+a+1)} u(x) \ .$$

Operating on both sides with the symbol ${}_cD_x^{m+a+1}$ we have

$$u(x) = {}_cD_x^{m+a+1} g(x) / \Gamma(m+a+1) \ ,$$

$$= \frac{d^{m+2}}{dx^{m+2}} \int_c^x \frac{(x-t)^{-a}}{\Gamma(m+a+1)\Gamma(1-a)} g(t)\,dt \ . \qquad (6.15)$$

Recalling elementary properties of the gamma function and making use of (6.9) we can write (6.15) in the form

$$u(x) = \frac{1}{\Gamma(m+a+1)\Gamma(1-a)} \Big\{ (-1)^{m+1} \frac{\Gamma(m+a+1)}{\Gamma(a)} \frac{g(c)}{(x-c)^{m+a+1}} + \cdots$$

$$+ (-1)^m \frac{\Gamma(m+a)}{\Gamma(a)} \frac{g'(c)}{(x-c)^{m+a}} + \cdots + \frac{\Gamma(a) g^{(m+1)}(c)}{\Gamma(a)(x-c)^a}$$

$$+ \Gamma(1-a) \, {}_cD_x^{-(1-a)} g^{(m+2)}(x) \Big\} \ . \qquad (6.16)$$

But it is clear that, if the value of $u(x)$ given by equation (6.16) is substituted in (6.14) the integral will not be convergent unless

$$g(c) = g'(c) = \cdots = g^{(m)}(c) = 0 \ .$$

Under these conditions $u(x)$ defined by (6.16) will be the desired solution. These results can be summarized as follows:

Theorem 5. If in equation (6.14), $g(x)$ *exists together with its first* $m+2$ *derivatives in the interval* $c \leqq x \leqq x_0$, *and if* $g(c) = g'(c) = \cdots = g^{(m)}(c) = 0$, *then the solution of equation* (6.14) *exists in the open interval* $c < x \leqq x_0$. *The solution is given by formula* (6.16).

The efficacy of the methods which we have stated may be further illustrated by means of the following examples:

Example 1. Let us consider the equation

$$ {}_cD_x^{-\frac{1}{2}}u + \lambda u = f(x) \;, \tag{6.17}$$

whose rational equivalent is the equation,

$$\lambda^2 \frac{d}{dx} u - u = \lambda f'(x) - {}_cD_x^{\frac{1}{2}} f(x) \;. \tag{6.18}$$

The solution of this equation is

$$u(x) = Ce^{x/\lambda^2} + e^{x/\lambda^2} \int_c^x \left[\frac{1}{\lambda} f'(t) - \frac{1}{\lambda^2}\, {}_cD_t^{\frac{1}{2}} f(t) \right] e^{-t/\lambda^2} dt \;. \tag{6.19}$$

In order to determine C consider equation (6.17) in the form

$$\frac{1}{\sqrt{\pi}} \int_c^x \frac{u(t)}{(x-t)^{\frac{1}{2}}} dt + \lambda u(x) = f(x) \;.$$

It is clear that for $x = c$ we have $\lambda u(c) = f(c)$. Then from (6.19) we get

$$u(c) = C\, e^{c/\lambda^2} = \frac{1}{\lambda} f(c)$$

and (6.19) can be written in the form

$$u(x) = \frac{1}{\lambda} f(c) e^{(x-c)/\lambda^2} + \int_c^x \left[\frac{1}{\lambda} f'(t) - {}_cD_t^{\frac{1}{2}} f(t) \right] e^{(x-t)/\lambda^2} dt \;. \tag{6.20}$$

As a specialization we may write

$${}_0D_x^{-\frac{1}{2}} u + u = 1 \;.$$

Hence replacing c by 0, λ by 1 and noting that ${}_0D_t^{-\frac{1}{2}} \to 1 = (\pi t)^{-\frac{1}{2}}$, we have from (6.20) the solution

$$u(x) = e^x - e^x \int_0^x e^{-t} \frac{1}{\sqrt{\pi t}} dt \;.$$

This solution can be expanded in two ways:

$$u_1 = e^x - \frac{1}{\sqrt{\pi x}} \left[(2x) + \frac{(2x)^2}{1\cdot 3} + \frac{(2x)^3}{1\cdot 3\cdot 5} + \cdots \right] ,$$

and

$$u_2 = \frac{1}{\sqrt{\pi x}} \left[1 - \frac{1}{2x} + \frac{1\cdot 3}{(2x)^2} - \frac{1\cdot 3\cdot 5}{(2x)^3} + \cdots \right] .$$

The second, although everywhere divergent, is the asymptotic expansion of the first.

Example 2. Let us consider the equation,

$$u(x) + \int_0^x \{u(t)/(x-t)^{1/3}\}\, dt = f(x) \ , \qquad (6.21)$$

which can be written in the form,

$$\{1 + \Gamma(2/3) z^{-2/3}\} \rightarrow u(x) = f(x) \ .$$

Obviously we have

$$B(z) = \begin{vmatrix} 1 & 0 & \Gamma(2/3) \\ z^{-1}\Gamma(2/3) & 1 & 0 \\ 0 & z^{-1}\Gamma(2/3) & 1 \end{vmatrix} = 1 + z^{-2}\Gamma^3(2/3) \ ,$$

and similarly

$$\Delta'(z) = 1 - \Gamma(2/3) z^{-2/3} + \Gamma^2(2/3) z^{-4/3} \ .$$

Hence equation (6.21) is equivalent to the *rationalized* equation,

$$u(x) + \Gamma^3(2/3) \int_0^x (x-t) u(t)\, dt = F(x) \ ,$$

in which we abbreviate

$$F(x) = f(x) + \int_0^x \{-(x-t)^{-1/3} + \Gamma^2(2/3)\,(x-t)^{1/3}/\Gamma(4/3)\}\, f(t)\, dt \ .$$

The solution of this equation is

$$u(x) = F(x) + \lambda \int_0^x \sin \lambda (t-x) f(t)\, dt \ ,$$

where we abbreviate $\lambda = \Gamma^{3/2}(2/3)$. This function may be shown to be the unique solution of (6.21) also.

Example 3. Let us consider,

$$u(x) = \mu x^{1-a}/\Gamma(2-a) - \{\lambda/\Gamma(1-a)\} \int_0^x \{u(t)/(x-t)^a\}\, dt \ , \tag{6.22}$$

which we may write in the form,

$$(1 - Bz^{-b}) \rightarrow u(x) = g(x) \ , \tag{6.23}$$

where we use the abbreviations,

$$B = -\lambda \ , \quad g(x) = \mu x^b/\Gamma(1+b) \ , \quad b = 1-a \ .$$

Since the numerical value of b is not specified we cannot now employ the method of rationalization, but instead can invert the equation formally as follows:

$$\begin{aligned} u(x) &= 1/(1 - Bz^{-b}) \rightarrow g(x) \ , \\ &= (1 + Bz^{-b} + B^2z^{-2b} + B^3z^{-3b} + \cdots) \rightarrow g(x) \ . \end{aligned} \tag{6.24}$$

Noting the formula,

$$z^{-nb} \rightarrow x^b = {}_0D_x{}^{-nb}x^b = \Gamma(1+b)\,x^{b(1+n)}/\Gamma(1+b+nb) \ ,$$

we immediately obtain the following expansion for (6.24):

$$\begin{aligned} u(x) &= \mu x^b\{1/\Gamma(1+b) - \lambda x^b/\Gamma(1+2b) + \lambda^2x^{2b}/\Gamma(1+3b) - \cdots\} \\ &= -(\mu/\lambda)\{1 - \lambda x^b/\Gamma(1+b) + \lambda^2x^{2b}/\Gamma(1+2b) \\ &\quad - \lambda^3x^{3b}/\Gamma(1+3b) + \cdots\} + (\mu/\lambda) \ . \end{aligned} \tag{6.25}$$

The function which we have obtained as a solution has more than a passing interest on its own account, since it is intimately connected with a generalization of the exponential function which was first studied by G. Mittag-Leffler* and later by E. Lindelöf† and E. W. Barnes.‡

A résumé of some of the results will be stated here. We shall first define as the Mittag-Leffler function,

$$E_a(x) = \sum_{n=0}^{\infty} x^n/\Gamma(1+an) \ , \quad a \geqq 0 \ ,$$

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*Un généralisation de l'intégrale de Laplace-Abel. *Comptes Rendus*, vol. 136 (1903), pp. 537-539. Sur la nouvelle fonction $E_a(x)$. *Comptes Rendus*, vol. 137 (1903), pp. 554-558.

†Sur la détermination de la croissance des fonctions entières définies par un developpement de Taylor. *Bulletin des Sciences Mathématiques*, vol. 27 (2nd series) (1903), pp. 213-226; in particular pp. 224-225.

‡The Asymptotic Expansion of Integral Functions Defined by Taylor's Series. *Trans. of London Phil. Soc.*, vol. 206 (A), (1906), pp. 249-297; in particular, pp. 285-289. See also: On Functions Defined by Simple Types of Hypergeometric Series. *Cambridge Phil. Transactions*, vol. 20 (1907), pp. 253-279.

in terms of which the solution (6.25) may then be written:

$$u(x) = (\mu/\lambda) - (\mu/\lambda)E_b(-\lambda x^b) \ . \qquad (6.26)$$

For $a = 0$, this function becomes $E_0(x) = 1/(1-x)$, for $a = 1$, $E_1(x) = e^x$, for $a = 2$, $E_2(x) = \cosh\sqrt{x}$, for $a = .5$, $E_{.5}(x) = e^{x^2}(1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt)$, etc. It is connected with the Laplace integral through the equation,

$$\int_0^\infty e^{-t}E_a(t^a x)\,dt = 1/(1-x) \ .$$

The most interesting properties of this function and those which led to its investigation by Mittag-Leffler are associated with its asymptotic development in various sectors of the complex plane, these properties being summarized as follows:

For the case, $0 < a < 2$, we have the following representations:

$$\text{(a)} \quad E_a(x) \backsim -\sum_{n=1}^{\infty} 1/\{x^n \Gamma(1-an)\} \ ,$$

provided $\frac{1}{2}a\pi <$ amp $x < 2\pi - \frac{1}{2}a\pi$;

$$\text{(b)} \quad E_a(x) - \frac{1}{a}\exp(x^{1/a}) \backsim -\sum_{n=1}^{\infty} 1/[x^n\Gamma(1-an)] \ ,$$

provided $-\frac{1}{2}a\pi \leq$ amp $x \leq \frac{1}{2}a\pi$.

For the case, $a \geqq 2$, we have

$$\text{(c)} \quad E_a(x) \backsim \frac{1}{a}\sum_{\mu}\exp\{x^{1/a}e^{2\pi i\mu/a}\} - \sum_{m=1}^{\infty} 1/[x^m\Gamma(1-am)] \ ,$$

where μ takes all integral values (positive, negative, or zero) such that

$$2\pi\mu + \text{amp } x \leq \tfrac{1}{2}a\pi \ ,$$

amp x having any value between $+\pi$ and $-\pi$ inclusive.

These results, first announced by Mittag-Leffler, were proved also by Barnes, who availed himself of the following representations:

$$\text{(a)} \quad E_a(-x) = \frac{1}{2\pi i}\int \frac{\pi x^s}{\Gamma(1+as)\sin s\pi}\,ds \ ,$$

the path of integration being taken along the positive half of the real axis and enclosing the points $s = 0, 1, 2, \cdots, \infty$, but no other poles of the integrand.

$$\text{(b)} \quad E_a(x) - \frac{1}{a}\exp(x^{1/a}) = \frac{1}{2\pi i}\int \frac{\Gamma(-as)\sin\pi(1-a)s}{\sin\pi s}\,x^s\,ds \ ,$$

the contour being taken along the positive half of the real axis and enclosing the poles of $\Gamma(-as)$ and the points $s=0, 1, 2, \cdots, \infty$.

$$\text{(c)}\quad E_a(x) - \frac{1}{a}\sum_{\mu=-p}^{p} \exp\{x^{1/a} e^{2\pi i\mu/a}\}$$
$$= \frac{1}{2\pi i}\int \{\Gamma(-as)\sin(2p+1-a)\, s\, x^s/\sin \pi s\}\, ds ,$$

where the path of integration is the positive half of the real axis and encloses the poles of $\Gamma(-as)$ and the points $s=0, 1, 2, \cdots, \infty$. The integer p is chosen so as to satisfy either of the inequalities,

$$a/2 < 2p+1 < a , \quad \text{or} \quad a < 2p+1 < 3a/2 .$$

PROBLEMS

1. Show that the equation

$$u(x) + \int_0^x t(x-t)^{-\frac{1}{2}} u(t)\, dt = f(x)$$

is equivalent to

$$u(x) - \tfrac{1}{2}\pi \int_0^x \{t(x+t)\}\, u(t)\, dt = F(x)$$

where we abbreviate

$$F(x) = f(x) - \sqrt{\pi}\, x\, {}_0D_x^{-1/2} f(x) + \tfrac{1}{2}\sqrt{\pi}\, {}_0D_x^{-3/2} f(x) .$$

2. Prove that the equation

$$u(x) + \int_0^x \{t^2(x-t)^{-2/3}\}\, u(t)\, dt = f(x)$$

is equivalent to

$$u(x) + \frac{\Gamma^3(1/3)}{243}\int_0^x t^2(44x^4+40x^3t+75x^2t^2+40xt^3+44t^4)\, u(t)\, dt = F(x)$$

where we abbreviate

$$F(x) = f(x) + \Gamma(1/3)[x^2\, {}_0D_x^{-1/3} f(x) - \frac{2}{3}x\, {}_0D_x^{-4/3} f(x) + \frac{4}{9}\, {}_0D_x^{-7/3} f(x)]$$

$$+ \Gamma^2(2/3)[x^4\, {}_0D_x^{-2/3} f(x) - 2x^3\, {}_0D_x^{-5/3} f(x) + \frac{34}{9}x^2\, {}_0D_x^{-8/3} f(x)$$

$$- \frac{16}{3}x\, {}_0D_x^{-11/3} f(x) + \frac{352}{81}\, {}_0D_x^{-14/3} f(x)] .$$

3. Prove that if

$$L(u) \equiv (x-a)(x-b)\, u''(x) + (c+hx)\, u'(x) + k\, u(x) ,$$

and if p is so chosen that $p(p-1) + ph + k = 0$, then

$$z^p \rightarrow L(u) \equiv (x-a)(x-b)\, u^{(p+2)} + [c - p(a+b) + (2p+h)x]\, u^{(p+1)} .$$

Hence employing the abbreviation

$$y(x) = u^{(p+1)}(x) \ ,$$

reduce the solution of the equation $L(u) = 0$ to an equation of first order. (Letnikoff).

4. Discuss Legendre's equation

$$(1 - x^2)\, u''(x) - 2x\, u'(x) + n(n+1)\, u(x) = 0$$

by the method given in problem 3.

7. Special Applications of the Fractional Calculus. In the next chapter we shall exhibit the efficacy of the fractional calculus in the solution of several types of problems which arise in the application of partial differential equations. It will be sufficient for our present purpose to discuss four problems from essentially different fields in which fractional equations appear.

Example 1. The first problem to be solved by integral equations was due to N. H. Abel, who by a curious coincidence also employed half-derivatives in attaining his inversion.* Abel considered the following question:

Suppose that a heavy bead is constrained to move under gravity on a curved wire situated in a vertical plane. If the time of descent from a height h to the lowest point of the curve is supposed to be a function $T(h)$ of the height, determine the equation of the curve, the initial velocity being zero.

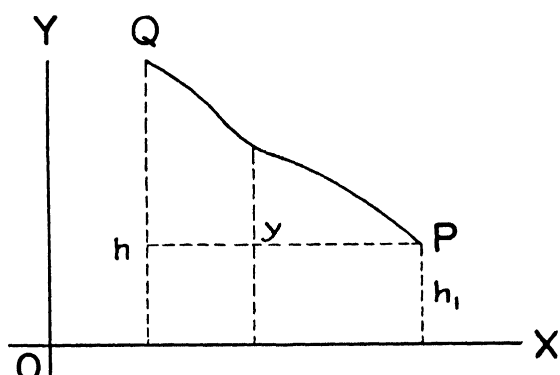

FIGURE 1

Let us suppose that the equation of the curve is $x = x(y)$. Then the element of arc will be $ds = u(y)\,dy$, where we abbreviate,

$$u(y) = [1 + (dx/dy)^2]^{\frac{1}{2}} \ . \qquad (7.1)$$

*Auflosung einer mechanischen Aufgabe. *Journal fur Mathematik*, vol. 1 (1826), pp. 153-157. Also *Oeuvres*. Christiania (1881), vol. 1, pp. 97-101.

From the energy formula

$$\tfrac{1}{2}\, m (ds/dt)^2 = mg(h - y)$$

we are led to the equation,

$$(2g)^{\frac{1}{2}}\, T(h) = \int_{h_1}^{h} (h - y)^{-\frac{1}{2}} u(y)\, dy \ . \qquad (7.2)$$

Employing fractional derivatives this may be written,

$$(2g)^{\frac{1}{2}}\, T(h) = \Gamma(\tfrac{1}{2})\ {}_{h_1}D_h^{-\frac{1}{2}}\, u(h) \ ,$$

from which we obtain by inversion,

$$\begin{aligned} u(h) &= [(2g)^{\frac{1}{2}}/\Gamma(\tfrac{1}{2})]\ {}_{h_1}D_h^{\frac{1}{2}}\, T(h) \ , \\ &= [(2g)^{\frac{1}{2}}/\pi]\ d/dh \int_{h_1}^{h} (h - y)^{-\frac{1}{2}}\, T(y)\ dy \ , \\ &= [(2g)^{\frac{1}{2}}/\pi]\, [T(h_1)\, (h - h_1)^{-\frac{1}{2}} + \int_{h_1}^{h} (h - y)^{-\frac{1}{2}}\, T'(y)\, dy] \ . \end{aligned}$$

When the function thus obtained is substituted in (7.1), the desired equation follows by a single integration,

$$x = \int_0^y [u^2(y) - 1]^{1/2}\, dy \ . \qquad (7.3)$$

As a specific example let us consider the tautochrone curve (the curve of equal descent) where we suppose that $T(h) = c$ and $h_1 = 0$. We then get, $u(h) = (c/\pi)\,(2g/h)^{\frac{1}{2}}$, which when substituted in (7.3) yields,

$$x = \int_0^y [(2a/y) - 1]^{\frac{1}{2}}\, dy \ ,$$

where we employ the abbreviation, $a = gc^2/\pi^2$.

Making the transformation, $y = 2a \sin^2 t$, we easily find

$$x = a(2t + \sin 2t) \ .$$

Employing the substitution: $2t = \vartheta$, the values of x and y take the form:

$$x = a(\vartheta + \sin\vartheta) \ ,$$

$$y = a(1 - \cos\vartheta) \ ,$$

which are recognized as the standard parametric representation of a cycloid generated by a circle of radius a.

Example 2. The problem of determining the shape of a weir notch so that the flow of water through it shall be a given function of the height has been solved by W. C. Brenke.*

In the figure the shaded area represents the cross section of a weir notch, which is symmetrical with respect to the T-axis. The quantity of water which flows through the notch per unit of time is given by the equation

$$Q = C \int_0^h (h-t)^{\frac{1}{2}} f(t)\,dt \ , \tag{7.4}$$

where C is a physical constant and the form of the notch is determined by $y = f(t)$, $t \geqq 0$.†

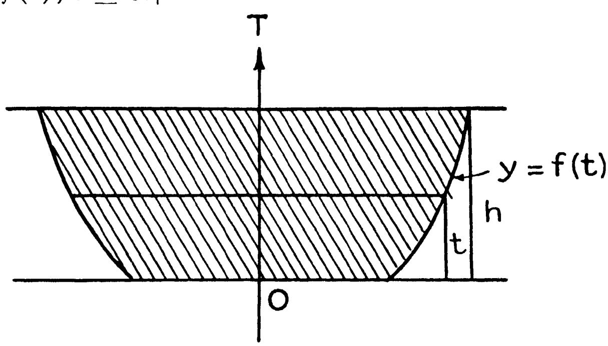

FIGURE 2

If we replace Q/C by $g(h)$ it is clear that we can write (7.4) in the form,

$$g(h) = \Gamma(3/2)\,{}_0D_h^{-3/2} f(h) \ ,$$

from which we obtain by inversion,

$$\begin{aligned} f(h) &= {}_0D_h^{3/2}\, g(h)/\Gamma(3/2) \ , \\ &= \frac{d^2}{dh^2} \int_0^h (h-t)^{-\frac{1}{2}}\, g(t)\,dt/\{\Gamma(3/2)\Gamma(1/2)\} \ , \\ &= (2/\pi)\,[-\tfrac{1}{2}\, g(0)\,h^{-3/2} + g'(0)\,h^{-1/2} \\ &\qquad + \int_0^h (h-t)^{-\frac{1}{2}}\, g''(t)\,dt] \ . \end{aligned}$$

As a special application we consider the case where $g(h) = kh^m$. We then obtain,

*An Application of Abel's Integral Equation. *American Math. Monthly*, vol. 29 (1922), pp. 58-60.

†For the dynamical considerations involved here the reader is referred to any standard work on hydraulics.

$$f(h) = \frac{2k}{\sqrt{\pi}} \frac{\Gamma(m+1)}{\Gamma(m-\frac{1}{2})} h^{m-3/2} , \quad m > \tfrac{1}{2} \; .$$

If we set $m = n$, where n is an integer, this formula reduces to $f(h) = (k/\pi)\,[2^n\, n!/1 \cdot 3 \cdot 5 \cdots (2n-3)\,]\,h^{n-3/2}$, and when $m = n + \frac{1}{2}$, it becomes $f(h) = k[1 \cdot 3 \cdot 5 \cdots (2n+1)/2^n \; (n-1)\,!]\,h^{n-1}$. From these results we have the interesting specializations that when the flow is directly proportional to the depth of the stream the form of the notch is $y = t^{-1/2}$, when $m = 3/2$ the notch is rectangular, $y =$ constant, and when $m = 5/2$ the notch is a parabola.

Example 3. The following problem, taken from J. Liouville,* is introduced as an example of the use of series in attaining the solution of a fractional equation. The problem does not strictly belong in this place since the coefficients are not constants, but the method of inversion employed is instructive.

Let AB and CD be two straight lines perpendicular to one another, the first terminating at the point A and extending to infinity on the side of B, the second infinite in both directions. We extend the line AB to the point P, where it cuts the other straight line. Midway between A and P a small mass M is placed which is attracted by the elements of AB and CD by a force represented by a function $u(r)$ of the distance. We then seek to find the function $u(r)$ such that the attraction of CD shall be twice the attraction of AB, whatever the distance $PA = 2y$.

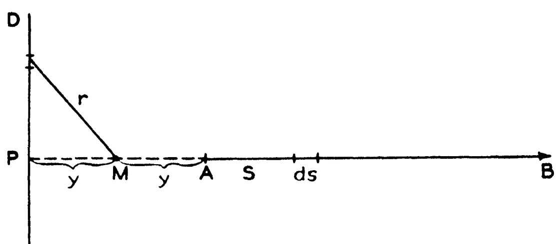

FIGURE 3

It is at once clear that the force F_1 exerted by the line AB on M may be written,

$$F_1 = \int_0^{\infty} u(y+s)\,ds = \int_y^{\infty} u(s)\,ds \; .$$

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*Mémoire sur quelques Questions de Géométrie et de Mécanique, et sur un nouveau genre de Calcul pour résoudre ces Questions. *Journal de l'École Polytechnique,* cahier 21 (1832), pp. 1-69.

Making the transformation: $s^2 = t,\ y^2 = z$, we may write

$$F_1 = \int_z^\infty [u(t^{\frac{1}{2}})/2t^{\frac{1}{2}}]dt = -\tfrac{1}{2}\,{}_\infty D_z^{-1} v(z) \quad ,$$

where we employ the abbreviation, $v(z) = u(z^{\frac{1}{2}})/z^{\frac{1}{2}}$.

Similarly the force F_2 exerted by the line CD may be written,

$$F_2 = 2y \int_0^\infty [u(r)/r]ds \quad ,$$

where we abbreviate, $r^2 = s^2 + y^2$.

Making the transformation: $s^2 + y^2 = t,\ y^2 = z$, this becomes

$$F_2 = 2z^{\frac{1}{2}} \int_z^\infty \frac{u(t^{\frac{1}{2}})}{t^{\frac{1}{2}}} \frac{dt}{2(t-z)^{\frac{1}{2}}}$$

$$= z^{\frac{1}{2}}\{-i\,\pi^{\frac{1}{2}}\,{}_\infty D_z^{-\frac{1}{2}} v(z)\} \; .$$

Now introducing the condition of the problem, namely that $F_2 = 2F_1$ we reach the equation,

$$ {}_\infty D_z^{-1} v(z) = i(\pi z)^{\frac{1}{2}}\,{}_\infty D_z^{-\frac{1}{2}} v(z). \tag{7.5}$$

From physical consideration we see that $v(z)$ approaches zero as z indefinitely increases. Hence it is not unreasonable to assume an expansion for $v(z)$ of the form,

$$v(z) = \Sigma\, A_n/z^{n+\nu} \; .$$

When this series is substituted in (7.5) and the coefficients of corresponding terms equated, we obtain the following equation for the determination of $n + \nu$:

$$1/(n+\nu-1) = \sqrt{\pi}\,\Gamma(n+\nu-\tfrac{1}{2})/\Gamma(n+\nu) \; .$$

This equation has the single solution $n + \nu = 3/2$, from which we immediately obtain $v(z) = A/z^{3/2}$ and hence the desired law of force,

$$u(y) = A/y^2 \; .$$

Example 4. The following example, taken from the field of biology, has been furnished the author by Dr. Kenneth S. Cole.*

A living nerve can be stimulated by passing a direct current through a short portion of it between two electrodes, provided the potential difference exceeds a certain critical value known as the *rheobase.* As the duration of the potential applied across the elec-

*For a more extensive description of the background of this problem the reader is referred to the following papers by Dr. Cole: *Electric Conductance of Biological Systems,* and *Electric Excitation in Nerves,* Cold Spring Harbor Symposia on Quantitative Biology, vol. 1 (1933).

trodes decreases, it is found that the intensity necessary for stimulation increases rapidly in a hyperbolic manner. The following analysis is designed to explain this phenomenon.

An idealized nerve fiber consists of a cylindrical core of electrolyte covered with a thin sheath or membrane. It is assumed that a local threshold change of the normal potential difference across the membrane will stimulate the fiber and cause an impulse to be propagated. The problem is then to express analytically the strength of stimulus which, when applied to the nerve bundle as a whole, will change the potential difference across the membrane of an individual fiber by a threshold amount in a given time.

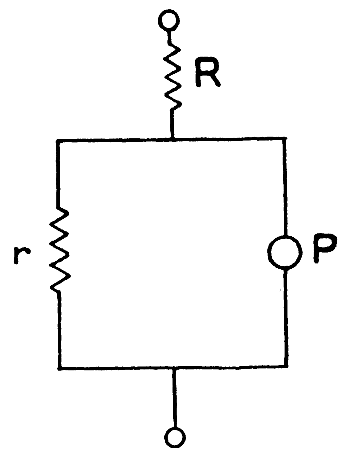

FIGURE 4

To begin with, experimental evidence points to the conclusion that the electrical behaviour of the nerve fiber may be simulated by the type of circuit illustrated in the figure, where r and R are constant resistances and the element P, called the *polarization element*, has an impedance* defined by the following equation:

$$Kp^{-a} \rightarrow I_p(t) = e_p(t) \ , \quad 0 < a < 1 \ , \tag{7.6}$$

The function $I_P(t)$ is the current in the element P, $e_P(t)$ the potential across it and p the operator d/dt. The positive constant K is determined experimentally. No combination of electrical circuits with ordinary resistances and capacities is known to lead to an impedance of the form postulated, but experimental evidence appears to indicate that such an impedance is essential to the description of the curious electrical behavor of biological materials in general and of nerve fibers in particular.

*For a definition of impedance see section 2, chapter 7.

Referring now to the figure, we compute the relationship between $I_p(t)$ and $e_p(t)$, when a constant e. m. f., E, is applied across the electrodes. This is easily found to be

$$e_P(t) = E - R\,[I_r(t) + I_P(t)]\ ,$$
$$= E - R\left[\frac{e_P(t)}{r} + I_P(t)\right]\ ,$$

where $I_r(t)$ and $I_P(t)$ are the instantaneous currents in r and P. Hence we get,

$$I_P(t) = \frac{E}{R} - \frac{R+r}{Rr}\,e_P(t)\ . \tag{7.7}$$

Making use of (7.6) to eliminate $I_p(t)$, we obtain

$$\left(\frac{1}{K}p^a + \frac{R+r}{Rr}\right) \to e_P(t) = E/R\ ,$$

or

$$(1 + \frac{R+r}{Rr}Kp^{-a}) \to e_P(t) = Kp^{-a} \to (E/R) = KEt^a/[R\,\Gamma(1+a)]\ .$$

This equation may be more simply written

$$(1 + \lambda p^{-a}) \to e_P(t) = g(t)\ ,$$

where we abbreviate,

$$\lambda = K(R+r)/Rr\ ,\ g(t) = KE\,t^a/[R\,\Gamma(1+a)]\ .$$

This equation has already been inverted in example 3, section 6 and the solution found to be

$$e_P(t) = \frac{KE}{R}t^a\left[\frac{1}{\Gamma(1+a)} - \frac{\lambda\,t^a}{\Gamma(1+2a)} + \cdots\right],$$
$$= \frac{Er}{R+r} - \frac{Er}{R+r}\left[1 - \frac{\lambda\,t^a}{\Gamma(1+a)} + \frac{\lambda^2 t^{2a}}{\Gamma(1+2a)} - \cdots\right],$$
$$= \frac{ER}{R+r} - \frac{ER}{R+r}E_a(-\lambda t^a)\ ,$$

where $E_a(x)$ is the Mittag-Leffler function discussed in the example referred to above.

From the asymptotic representation of $E_a(x)$ over the negative axis of reals and from the fact that λ is a positive constant, we know that

$$\lim_{t=\infty} e_P(t) = Er/(R+r)\ .$$

Moreover, since $E_a(-x)$ equals unity for $x = 0$ and decreases uniformly to zero as x increases, it is clear that $e_P(t)$ builds up uniformly from 0 to its limiting value $Er/(R + r)$.

Now let E_0 equal the *rheobase*, that is to say, the critical potential across the electrodes for which stimulation of the nerve will just take place. In terms of this potential, the maximum value of $e_p(t)$ is clearly $e_P(\infty) = E_0r/(R + r)$.

But if a higher potential, $E > E_0$, is put across the electrodes, then the critical value of $e_P(t)$ necessary to stimulate the nerve will be attained in a finite time determined from the equation,

$$\frac{E_0 r}{R+r} = \frac{Er}{R+r} - \frac{Er}{R+r} E_a(-\lambda t^a) \ ,$$

that is to say,

$$E_0 = E - EE_a(-\lambda t^a) \ . \tag{7.8}$$

In the study of the electrical stimulation of nerves a characteristic time, called the *chronaxie*, has been widely used. By the chronaxie is meant the duration of time, $t = \gamma$, which is necessary to build

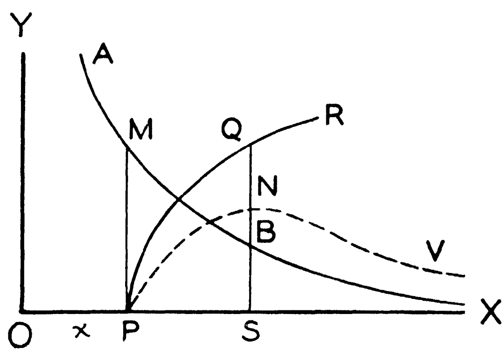

FIGURE 5

$e_P(t)$ up to stimulus strength, when the initial potential, E, is just twice the rheobase, $E = 2E_0$. From (7.8) we then obtain,

$$E_0 = 2E_0 - 2E_0E_a(-\lambda\gamma^a) \ ,$$

or

$$\tfrac{1}{2} = \sum_{n=1}^{\infty} (-1)^{n+1} x^n/\Gamma(1 + na) \ , \tag{7.9}$$

where we abbreviate, $x = \lambda\gamma^a$.

The inversion of this series is now assumed and since x is a function of a , $x = x(a)$, we may write ,

$$\lambda\gamma^a = x(a) \ , \quad \log\gamma = [\log x(a) - \log\lambda]/a \ .$$

PROBLEMS

Problem 1. (Liouville). We require the equation, $y = u(x)$, of a curve AMB which has the following property: Let us select any ordinate MP of the curve and with P as a vertex draw a parabola PQR with its axis along OX and its directrix midway between P and the origin O. We then construct a third curve PNV the ordinates of which are the product of the corresponding ordinates of the first two, that is, $NS = BS \times QS$. This proposed, we finally require that the area $XPNV$ under the new curve shall be a given function $f(x)$ of the abscissa of the point P. (See Figure 5).

Show that the function $u(x)$ is given by

$$u(x) = (\sqrt{2}/\pi) \int_0^\infty \frac{d^2}{dx^2}\left[\frac{f(x+t)}{(x+t)^{\frac{1}{2}}}\right]\frac{dt}{(t)^{\frac{1}{2}}} .$$

For the case $f(x) = a^2$, this yields the solution $u(x) = (\sqrt{2}/\pi)\ a^2/x^2$.

Problem 2. (Liouville). Suppose that a uniform distribution of masses symmetric with respect to the x axis is taken along a straight line y of infinite extent, and suppose that these masses exert an attraction upon a mass M situated on the x axis at a distance x from the line y. The total attraction $f(x)$ in the direction of x is known, but the law of attraction is unknown except that it depends upon the distance r of M from the given masses. The problem is to find this law of attraction, $u(r)$. (See Figure 6).

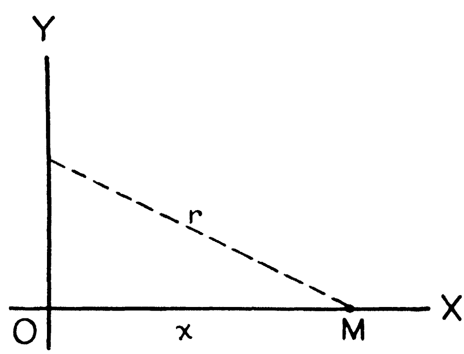

FIGURE 6

Show that $u(r)$ is determined from the equation,

$$\frac{u(z^{\frac{1}{2}})}{z^{\frac{1}{2}}} = \frac{1}{\pi}\frac{d}{dz}\int_z^\infty \frac{f(t^{\frac{1}{2}})}{t^{\frac{1}{2}}}\,\frac{dt}{(t-z)^{\frac{1}{2}}} .$$

For the case $f(x) = -\mu/x$, this leads to $u(r) = -\frac{1}{2}\mu/r^2$.

8. *Equations Involving the Logarithmic Operator.* Similar in kind although not so extensively found in applied problems as are equations involving fractional operators, equations in which the operator $\log z$ appears have attracted some attention. Their introduction and the theory of their inverse are due to V. Volterra who employed

them in his calculus of functions of composition which we have introduced in chapter 4.

We shall give a brief account of the application of the logarithmic operator by first considering the integral equation,

$$f(x) = \int_0^x [\log(x-t) + A]\, u(t)\,dt \ , \quad f(0) = 0 \ , \qquad (8.1)$$

which Volterra first discussed by other means.

Referring to the definitions of section 11, chapter 2, we see that we can write (8.1) symbolically in the form,

$$f(x) = (-z^{-1} \log z + az^{-1}) \rightarrow u(x) \ , \qquad (8.2)$$

where we abbreviate $a = A - C$ (Euler's constant).

In order to attain the inversion we first note from the corollary of section 3, chapter 4, that the operator $(-z^{-1} \log z + az^{-1})$ may be inverted by means of the Bourlet operational product. Hence, since x does not explicitly appear, we have the formal solution of (8.2) in the form,

$$u(x) = [z/(a - \log z)] \rightarrow f(x) \ .$$

In order to interpret this symbol we now appeal to the theory developed in section 11, chapter 2, in particular formula (11.14), where we find

$$z^{\nu+1} \int_0^\infty z^{-\mu-1} \vartheta(\mu)\,d\mu = z^\nu \int_0^\infty e^{(a-\log z)\mu} d\mu = z^\nu/(a - \log z)$$

$$= z^{\nu+1} \int_0^x e^{(t-x)z}\, I(x-t)\,dt \ , \qquad (8.3)$$

in which we write $\vartheta(\mu) = e^{a\mu}$ and

$$I(s) = \int_0^\infty \{e^{a\mu} s^\mu / \Gamma(\mu+1)\} d\mu \ .$$

Since we have

$$e^{(t-x)z} \rightarrow f(x) = f(t) \ ,$$

we immediately obtain the inversion of (8.2) out of (8.3) by setting $\nu = 1$, that is,

$$u(x) = z^2 \rightarrow \int_0^x f(t) I(x-t)\,dt \ .$$

It is of some interest in connection with this solution to note that the function $I(s)$ is asymptotic to e^{Es} in the infinite interval, where we use the abbreviation $E = e^a$. To see this we apply the Maclaurin integral test* for convergence to the series $e^{Es} = 1 + Es + (Es)^2/2! + (Es)^3/3! + \cdots$, and thus obtain the inequality

*T. J. Bromwich: *Infinite Series*, 2nd ed. (1926), p. 33.

$$e^{Es} - 1 \leq I(s) \leq e^{Es} \ .$$

Dividing by e^{Es}, we have

$$1 - e^{-Es} \leq e^{-Es} \, I(s) \leq 1 \ ,$$

which, for large positive values of s, establishes the desired property.

It should be pointed out further that the function $I(s)$ is also intimately related to the Mittag-Leffler function, $E_a(x)$ derived in the solution of problem 3, section 6. If we make the transformation, $\mu = at, \quad x = e^{aa}s^a$, then

$$I(e^{-a}x^{1/a})/a = \int_0^\infty \frac{x^t}{\Gamma(1+at)} \, dt$$

is seen to be the continuous counterpart of $E_a(x)$, in which the sign of integration replaces the sign of summation.

We shall next consider the integral equation,

$$f(x) = \int_0^x [\log^2(x-t) + A\log(x-t) + B]u(t) \, dt \ , \quad (8.4)$$

an equation also due to Volterra.

This equation can be written symbolically in the form

$$f(x) = z^{-1}(\log^2 z + \alpha \log z + \beta) \to u(x) \ ,$$

where we abbreviate,

$$\alpha = -A - 2\psi(1) \ , \ \beta = B + A\psi(1) + \psi^2(1) + \psi'(1) \ .$$

The function $\psi(x)$ is the psi function, which, together with its first four derivatives, has been extensively tabulated.* Numerically we have $\psi(1) = -C$ (Euler's constant) $= -.5772157$, $\psi'(1) = 1 + 1/2^2 + 1/3^2 + \cdots = 1.6449341$.

The solution then appears in the form

$$\begin{aligned} u(x) &= z/(\log^2 z + \alpha \log z + \beta) \to f(x) \\ &= z[\varphi_1/(\log z - \lambda_1) + \varphi_2/(\log z - \lambda_2)] \to f(x) \ , \end{aligned}$$

where $\varphi_1 = -\varphi_2 = 1/(\lambda_1 - \lambda_2)$, and where λ_1 and λ_2 are roots of the equation $\lambda^2 + a\lambda + \beta = 0$. The solution, when λ_1 and λ_2 are distinct, is thus seen to be attainable by means of the operator (8.3).

When $\lambda_1 = \lambda_2 = \lambda$, the preceding solution is replaced by

$$\begin{aligned} u(x) &= [z/(\log z - \lambda)^2] \to f(x) \ , \\ &= z^2 \to \int_0^x R(x-t) f(t) \, dt \ , \end{aligned}$$

where we abbreviate

$$R(s) = \int_0^\infty \{s^\mu \mu e^{\lambda\mu}/\Gamma(\mu+1)\} d\mu \ .$$

*See H. T. Davis: *Tables of the Higher Mathematical Functions*, vol. 1 (1933); vol. 2 (1935).

PROBLEMS

1. Show that the operator inverse to $X(\theta) = 1 - \theta$, $\theta = xz$, is the following:

$$Y(\theta) = 1 - \theta\, e^{-\theta}\, li(e^{\theta})$$

$$= 1 - \theta\, e^{-\theta}(C + \log \theta + \theta + \theta^2/(2 \cdot 2!) + \theta^3/(3 \cdot 3!) + \cdots) \ ,$$

where we abbreviate

$$li(e^{\theta}) = \int_{-\theta}^{\infty} (e^{-t}/t)\, dt \ ,$$

and C is Euler's constant.

2. Obtain the solution of the equation

$$(1 - \theta) \rightarrow u(x) = f(x)$$

in the form

$$u(x) = Y(\theta) \rightarrow f(x) = Ax + 1 - x \log x - \sum_{n=2}^{\infty} x^n f^{(n)}(0)/[(n-1) \cdot (n-1)!] \ .$$

9. *Convergent and Asymptotic Series.* In the preceding sections we have set up the formal machinery for the solution of differential equations of infinite order with constant coefficients. We shall now consider the nature of the convergence of the formal inverses that have been obtained. Leaving to section 10 the problem of the homogeneous equation, we shall consider the convergence of the formal operator

$$u(x) = \frac{1}{F(z)} \rightarrow f(z) \ .$$

The following theorem is pertinent:

Theorem 6. If the operator $1/F(z)$ has the expansion

$$\frac{1}{F(z)} = \sum_{m=-\infty}^{\infty} b_m z^m \ ,$$

which is convergent in the annulus $R' < |z| < R$, and if $f(x)$ is a function of finite grade q such that $R' \leqq q < R$, then

$$u(x) = \frac{1}{F(z)} \rightarrow f(x) \tag{9.1}$$

is a function of finite grade at most equal to q. If, moreover, $F(z)$ is expansible as a power series within the circle $|z| < R$, then $u(x)$ is a solution of the equation

$$F(z) \rightarrow u(x) = f(x) \ . \tag{9.2}$$

Proof: Since the conditions of theorem 6, chapter 5 are fulfilled, it follows that $[1/F(z)] \to f(x)$ exists and defines a function of grade at most equal to q. Since $u(x)$ as given by (9.1) is thus a function of grade at most equal to q, it follows by the theorem just cited and the conditions of the theorem that $F(z) \to u(x)$ exists.

Hence, since

$$F(z) \to \left[\frac{1}{F(z)} \to f(x) \right] = f(x) \ ,$$

from the properties of the operator, it follows that (9.1) is a solution of (9.2).

In order to establish the uniformity of the convergence of the function $F(z) \to u(x)$ to $f(x)$, we write

$$F(z) = \sum_{m=0}^{\infty} a_m z^m \ . \tag{9.3}$$

Representing the first n terms of this series by $F_n(z)$ and noting theorem 5, chapter 5, we obtain

$$\begin{aligned} | f(x) - F_n(z) \to u(x) | &\leqq M \sum_{m=n+1}^{\infty} | a_m | (q+\varepsilon)^m \sum_{k=-\infty}^{\infty} | b_k | (q+\varepsilon)^k \\ &\leqq M\, M' \sum_{m=n+1}^{\infty} (Q/R)^m \sum_{k=-\infty}^{\infty} | b_k | Q^k \\ &= M\, M'(Q/R)^{n+1} \frac{1}{1-Q/R} \sum_{k=-\infty}^{\infty} | b_k | Q^k \ , \end{aligned}$$

where M and M' are properly chosen constants independent of n and k and Q is subject to the inequality $R' < q < Q < R$.

Hence, since $Q < R$, and the series in the right hand member converges, it follows that $| f(x) - F_n(z) \to u(x) |$ converges uniformly to zero as $n \to \infty$.

The theorem which we have just established is not broad enough to include many important applications, as, for example, the inversion of the equation

$$u(x+1) - u(x) = 1/x \ ,$$

where the right-hand member is a function of infinite grade. By extending the domain of solutions to include divergent series which are summable by the method of Borel, it is possible to enlarge the field of application of operators.

The following theory indicates the nature of this extension:

Theorem 7. If in equation (9.2) *the function* $f(x)$ *has the expansion*

$$f(x) = h_1/x + h_2/x^2 + h_3/x^3 + \cdots + h_n/x^n + \cdots , \quad R < |x| ,$$

then a solution of (9.2) *exists of the form*

$$u(x) = \int_0^\infty e^{-xt} \frac{Q(t)}{F(-t)} dt , \tag{9.4}$$

where we write

$$Q(t) = h_1 + h_2 t + h_3 t^2/2! + h_4 t^3/3! + \cdots ,$$

provided positive values of k, A, and m exist such that*

(a) $|Q(t)/F(-t)|$ *and* $|F(-t)| < A e^{mt}$ for $0 < k \leqq t \leqq \infty$,

(b) $|Q(t)/F(-t)|$ *is of limited variation in the interval* $0 \leqq t \leqq k$.

The function $u(x)$, *in general, represents a solution of* (9.2) *asymptotically in the sense of Poincaré* (*see section* 4, *chapter* 5).

Proof: The conditions imposed by the theorem are those of theorem 8, chapter 5, and hence $u(x)$ determined formally is summable by the method of Borel. It is thus represented by the integral (9.4).

In general $u(x)$ is a function of infinite grade, since we have

$$u^{(n)}(x) = \int_0^\infty e^{-xt} t^n [Q(t)/F(-t)] dt$$

and hence

$$|u^{(n)}(x)| \leqq \int_0^\infty e^{-xt} t^n A e^{mt} dt$$

$$= A n!/|x-m|^{n+1} , \quad R(x) > m .$$

In order to establish the asymptotic character of the solution we assume $F(z)$ expansible in the series (9.3). Representing the first n terms by $F_n(z)$, we find

$$|f(x) - F_n(z) \to u(x)| \leqq \left| \int_0^\infty e^{-xt} \frac{Q(t)}{F(-t)} \sum_{m=n+1}^{\infty} a_m t^m dt \right|$$

$$\leqq \int_0^\infty e^{-xt} A^2 e^{2mt} t^{n+1} dt$$

$$= A^2 (n+1)!/|x-2m|^{n+2} , \quad R(x) > 2m .$$

From this inequality we see that for a fixed n, we have

$$\lim_{x=\infty} |f(x) - F_n(z) \to u(x)| = 0$$

which establishes the desired asymptotic convergence.

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*We should note that both k and m may assume the value 0.

10. The Homogeneous Case. In the foregoing theory we have touched rather lightly the problem of the homogeneous equation, namely, the problem of solving the equation

$$F(z) \to u(x) = 0 \ . \tag{10.1}$$

If λ_n is a zero of $F(z)$ of multiplicity m_n, then it is clear that the function

$$u_n(x) = e^{\lambda_n x} P_n(x) \ ,$$

where we write

$$P_n(x) = p_0^{(n)} + p_1^{(n)}x + p_2^{(n)}x^2 + \cdots + p_{m_n-1}^{(n)} x^{m_n-1} \ , \tag{10.2}$$

will be a particular solution of (10.1)

Two questions of interest present themselves: First, under what conditions will the sum

$$u(x) = \sum_{n=1}^{\infty} u_n(x) \tag{10.3}$$

converge? second, under what conditions will $u(x)$ furnish a solution of the equation (10.1)?

In order to answer the first of these questions we first find a majorant for the polynomial (10.2). If C_n represents the absolute value of the greatest of the coefficients and if the values of x lie within a circle of radius $R > 1$, then we shall have

$$|P_n(x)| < m_n C_n R^{m_n-1} \ . \tag{10.4}$$

In order to obtain a lower bound for $|P_n(x)|$ we note the following theorem due to H. Cartan:

Let $p_1, p_2, \cdots, p_n$ be any set of n distinct points in the plane and *let H be an arbitrary positive number. Then the points X of the plane for which one has the inequality*

$$Xp_1 \times Xp_2 \times Xp_3 \times \cdots \times Xp_n \leq H^n$$

*can be enclosed within the interior of circumferences in number at most equal to n, the sum of whose radii is equal to $2eH$, where e is the Naperian base.**

Now let R be a number such that all the zeros, $x_1, x_2, \cdots, x_p$, $p = m_n - 1$, *of* $P_n(x)$ lie within the circle $x = 2R$, and let x' be some

*Sur les systemes de fonctions holomorphes. *Annales de l'Ecole Normale Superieure*, vol. 45 (3rd ser.) (1928), pp. 255-346; in particular pp. 272 et seq. This theorem is a generalization of one originally given by A. Bloch: *Annales de l'Ecole Normale Superieure*, vol. 43 (3rd ser.) 1926), pp. 309-362; in particular, p. 321.

point on the circumference of the circle of radius R. The point x' may obviously be chosen so that $|P_n(x')|$ is at least as great as C_n. We shall then have

$$|P_n(x)| > C_n \prod_{i=1}^{p} \left| \frac{x - x_i}{x' - x_i} \right| > C_n (2R)^{-p} \prod_{i=1}^{p} | x - x_i | \ .$$

But by Cartan's theorem the last product is greater than $(R_n/2e)^p$, when x is exterior to a set of p circles the sum of whose radii is at most equal to R_n. Hence we get the inequality

$$|P_n(x)| > C_n (R_n/4eR)^{m_{n-1}} \ . \tag{10.5}$$

If we now assume that

$$\lim_{n \to \infty} (m_n/a_n) = 0 \ , \tag{10.6}$$

then it is clear that inequalities (10.4) and (10.5) limit the study of the convergence of (10.3) to that of the series

$$U(x) = \sum_{n=1}^{\infty} C_n \, e^{\lambda_n x} \ . \text{*}$$

The problem of the convergence of (10.3) has thus been reduced to that of a *Dirichlet series* and we shall postpone further discussion of this question to the next section.

We turn next to a consideration of the second question proposed above, namely, the validity of the solution. In this connection we shall prove the following theorem due to J. F. Ritt (see *Bibliography*):

Theorem 8. If $F(z)$ *is an entire function of genus zero and if* $u_1(x), u_2(x), \cdots, u_n(x), \cdots$ *are particular solutions of*

$$F(z) \rightarrow u(x) = 0 \ , \tag{10.7}$$

then the function

$$u(x) = \sum_{n=1}^{\infty} u_n(x) \tag{10.8}$$

is a solution of (10.7) *in any region within which the right-hand member converges.*

Proof: Let us consider the difference

$$|F(z) \rightarrow u(x) - F(z) \rightarrow U_m(x)| = |F(z) \rightarrow \sum_{n=m+1}^{\infty} u_n(x)| \ , \tag{10.9}$$

where $U_m(x)$ is the sum of the first m terms of (10.8).

*This conclusion and the foregoing analysis is due to G. Valiron (see *Bibliography*).

But by Cauchy's inequality [see (2.1), chapter 5], we have for every function $\varphi(x)$ which is analytic within a circle about $x = a$ of radius r, the following inequality

$$| z^p \to \varphi(x) | < p!\, M/r^p \,, \quad | a | < | x | < | a | + r \,,$$

where M is the maximum value of $\varphi(x)$ on the circumference of the circle.

Hence we shall have

$$| F(z) \to \varphi(x) | < M \sum_{p=0}^{\infty} \frac{p!\, |a_p|}{r^p} = M K \,,$$

provided the sum in the right-hand member converges to the value K.

But this implies (see section 2, chapter 5) that $(p!| a_p |)^{1/p} \to 0$ and hence that $F(z)$ is an entire function of genus zero.

Returning now to (10.9) we have from the uniform convergence of the series and from (10.10) the equivalence

$$F(z) \to \sum_{n=m+1}^{\infty} u_n(x) = \sum_{n=m+1}^{\infty} [F(z) \to u_n(x)] \,.$$

Since $U_m(x)$ and all the $u_n(x)$ are solutions of (10.7), it follows that $F(z) \to u(x)$ converges uniformly to zero, which establishes the theorem.

PROBLEMS

1. Solve the equation

$$(1 + z^2/2! + z^4/4! + \cdots) \to u(x) = 0 \,.$$

2. Since the function

$$u_m(x) = \frac{4}{\pi} \sum_{n=1}^{m} \frac{\cos(2n-1)\tfrac{1}{2}\pi x}{(2n-1)^2}$$

is a solution of the equation of problem 1, explain why

$$\lim_{m\to\infty} u_m(x) = \tfrac{1}{2}\pi - \tfrac{1}{2}\pi x \,, \quad 0 < x < 2 \,,$$

is not a solution.

3. Show that

$$f'(x) = \tfrac{1}{2} \int_0^{\infty} (e^{-t}/t)\,[f(x+t) - f(x-t)]\, dt$$

reduces to the linear differential equation

$$(z^3/3 + z^5/5 + \cdots) \to f(x) = 0 \,.$$

Hence solve the equation.

4. Solve the equation

$$u'(x) = a\,u(x) + b \sin z \to u(x) \,.$$

5. Prove the following theorem: (Pólya)

Given two series of numbers $\{a_n\}$ and $\{b_n\}$ such that sup lim $|a_n|^{1/n} = a$ (finite) and lim $|b_n|^{1/n} = 0$, construct the new set of numbers

$$c_n = \{a_0 b_n + {}_nC_1 a_1 b_{n-1} + {}_nC_2 a_2 b_{n-2} + \cdots + a_n b_0\} ,$$

where ${}_nC_r$ is the rth binomial coefficient. If not all the values of the set $\{b_n\}$ are zero, we shall have

$$\sup\lim |c_n|^{1/n} = a .$$

6. Employing the definitions of problem 5, discuss the function

$$g(x) = (b_0 - \frac{b_1}{1!} z + \frac{b_2}{2!} z^2 - \cdots) \rightarrow f(x) ,$$

where $f(x)$ has one singular point, $x = a$, on its circle of convergence and has the following development about $x = \infty$:

$$f(x) = a_0/x + a_1/x^2 + a_2/x^3 + \cdots .$$

Prove, in particular, that $g(x)$ has $x = a$ as a singular point. (Pólya).

7. M. Kalecki [*Econometrica*, vol. 3 (1935), pp. 327-344] has reduced his macrodynamic theory of business cycles to the solution of the following mixed difference and differential equation

$$u'(t) = au(t) - bu(t-\theta)$$

where θ is a constant.

Discuss the solution of this equation in general. Using Kalecki's values $a = .158$, $b = .279$, $\theta = 0.6$ discuss the solution of the equation.

This problem under the stimulus of Kalecki's application has been examined by R. Frisch and H. Holme [*Econometrica*, vol. 3 (1935, pp. 225-239]. Prior to this the equation in more general form was studied in a series of papers by F. Schürer (see *Bibliography*).

11. Dirichlet's Series. We have seen from the preceding section that the existence of solutions of (10.1) are fundamentally associated with the convergence properties of the Dirichlet's series

$$u(x) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n x} \tag{11.1}$$

where $\{\lambda_n\}$ and $\{c_n\}$ are sets of complex numbers. We may in particular assume that these numbers are real and that the set $\{\lambda_n\}$ is monotonically increasing with ∞ as its limiting value. For this case the convergence of the series (11.1) is a half plane bounded on the left by a *line of converengce*. The point (σ) where this line crosses the axis of reals is called the *abscissa of convergence* and it is defined analytically by the following limits:

$$\sigma = \limsup_{m=\infty} \log |\sum_{n=1}^{m-1} c_n| / \lambda_m ,$$

$$\sigma^* = \limsup_{m=\infty} \log \left| \sum_{n=m}^{\infty} c_n \right| / \lambda_m .$$

The first limit, σ, is to be used when the abscissa of convergence is positive and the second limit, σ^*, when the abscissa of convergence is negative.

The first of these limits is generally attributed to E. Cahen: Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analogues, *Annales de l'École Normale Supérieure*, vol. 11 (series 3) (1894), pp. 75-164. The second was given by S. Pincherle: Alcune spigolature nel campo delle funzioni determinanti, *Atti del IV Congresso dei Matematici*, vol. 2 (1908), pp. 44-48 and W. Schnee: Über die Koeffizientendarstellungs-formel in der Theorie der Dirichletschen Reihen. *Göttinger Nachrichten* (1910), pp. 1-42. Formulas which yield the abscissa of convergence for either sign have been designed by K. Knopp: Über die Abszisse der Grenzgeraden einer Dirichletschen Reihe, *Sitzungsberichte der Berliner Math. Gesellschaft,* Jahrg. 10 (1910), pp. 1-7 and T. Kojima: On the Convergence-Abscissa of General Dirichlet Series, *Tohoku Math. Journal,* vol. 6 (1914), pp. 134-139.

An excellent summary of the theory of Dirichlet series will be found in a monograph by G. H. Hardy and M. Riesz: *The General Theory of Dirichlet's Series*, Cambridge (1915), 78 p. For recent developments the reader is referred to E. Hille: Note on Dirichlet's Series with Complex Exponents, *Annals of Mathematics*, vol. 25 (2nd series) (1923-24), pp. 261-278, G. Pólya (see *Bibliography*) and G. Valiron (see *Bibliography*).

The half plane within which the series (11.1) is absolutely convergent is similarly defined by the abscissa of convergence computed from the following formulas:

$$\sigma = \limsup_{m=\infty} \log \sum_{n=1}^{m-1} |c_n| / \lambda_m ,$$

$$\sigma^* = \limsup_{m=\infty} \log \sum_{n=m}^{\infty} |c_n| / \lambda_m ,$$

where as before the first limit is to be used when the abscissa is positive and the second when the abscissa is negative.

If we now remove the restriction that the values of the set form a monotonically increasing series of positive numbers with ∞ as a limit and assume that they may be any set of complex numbers, we no longer find the region of convergence to be in general a half-plane. For convenience it will be assumed that series (11.1) converges at the origin, that is to say, the series $\sum_{n=0}^{\infty} c_n$ converges.*

It will first be shown that R, *the region of absolute convergence of* (11.1), *is convex, that is to say, if the series converges for two points x_1 and x_2 in R, then it converges for all values on the linear segment connecting x_1 and x_2.*

*This discussion is taken mainly from Hille: *loc. cit.*

To prove this, suppose that the series is absolutely convergent for the two points x_1 and x_2. Then any point on the linear segment (x_1x_2) can be represented by

$$x = tx_1 + (1-t)x_2 , \qquad 0 \leqq t \leqq 1 .$$

We then have

$$\sum_{n=0}^{\infty} |c_n e^{-\lambda_n x}| = \sum_{n=0}^{\infty} |c_n e^{-\lambda_n x_1 t} e_n^{-\lambda_n x_2(1-t)}| \leqq \sum_{n=0}^{\infty} |c_n| A_n{}^t B_n{}^{(1-t)} ,$$

where we use the abbreviations

$$A_n = |e^{-\lambda_n x_1}| , \quad B_n = |e^{-\lambda_n x_2}| .$$

But the series

$$\sum_{n=0}^{\infty} c_n A_n{}^t B_n{}^{(1-t)}$$

converges since (1) $A_n{}^t B_n{}^{(1-t)} \leqq tA_n + (1-t)B_n < A_n + B_n$, and (2) $\Sigma|c_n|A_n$ and $\Sigma|c_n|B_n$ are convergent by hypothesis.

For complex Dirichlet's series Hille defines a *maximal region of convergence* as follows:

Let us write

$$L_n = \log|c_n|$$

and let $\{L\}$ be the set of the limit points of L_n. Now take any point $L_0 \neq 0$ of the set $\{L\}$ and join it to the origin by a straight line segment. Through L_0 draw a line perpendicular to $(L_0 0)$, thus dividing the plane into two parts. Denote that half plane in which

$$R(x/L_0) \leqq 1 ,$$

where R denotes "the real part of", by Δ_0. When this construction has been repeated for all the limit points, a region will be defined common to all the half planes, $\Delta_0, \Delta_1, \cdots \Delta_m$ and this may be denoted by Δ. If any of the limit points is zero then the construction fails, but the division of the plane may be accomplished usually by a limiting process. Thus a set of half planes is constructed for some set having the limit point zero and the limit of this set of half planes is adjoined to the set Δ_1'.

It is not difficult to show that *any point exterior to the region Δ is a point of divergence for series* (11.1).

It may also be proved that if

$$\lim_{n\to\infty} |\lambda_n| / \log n = \infty ,$$

then the Dirichlet's series is absolutely convergent in the region Δ', where Δ' denotes the open region Δ minus its boundary set.

Hille also defines a *minimal region of convergence,* assuming only that the origin is a point of absolute convergence for the series (11.1). This definition furnishes sufficient conditions, independent of the magnitude of the set $\{\lambda_n\}$, for the absolute convergence of (11.1).

Under the condition that the series $\Sigma|c_n|$ converges, it is clear that

$$R_n = \sum_{m=n}^{\infty} |c_m|$$

converges to zero.

Now consider the sets of points $\{p_n\}$, where we define

$$p_n = \log R_n/\lambda_n \ .$$

This set has a set of limit points which we may designate by $\{P_n\}$. Employing this set of limit points exactly as we did the limit points $\{L_n\}$ above, we can construct a new set of half planes, $D_0, D_1, \cdots, D_k$, similar to the set of half planes $\Delta_0, \Delta_1, \cdots, \Delta_m$, except we now assume that

$$K(x/P_n) < 1 \ .$$

The region D, common to the set $\{D_n\}$, is included in the region Δ and series (11.1) *converges absolutely within it.*

It has long been known that the convergence of a Dirichlet's series also implies the convergence of factorial and generalized binomial (Newton) series. In order to establish this connection consider the following series:

$$v(x) = \sum_{n=1}^{\infty} c_n \frac{a_1 a_2 \cdots a_n}{(x+a_1)(x+a_2)\cdots(x+a_n)} \ , \qquad (11.2)$$

$$w(x) = \sum_{n=1}^{\infty} (-1)^n c_n (x-a_1)(x-a_2)\cdots(x-a_n)/(a_1 a_2 \cdots a_n) \ . \qquad (11.3)$$

Let us now employ the following abbreviations:

$$\lambda_n = \sum_{m=1}^{n} 1/a_m \ , \quad E_n(x) = c_n e^{-\lambda_n x} \ , \quad H_n(x) = \prod_{m=1}^{n} (1-x/a_m)\, e^{x/a_m} \ .$$

Using these abbreviations we can now write (11.1), (11.2) and (11.3) as follows:

$$u(x) = \sum_{n=1}^{\infty} E_n(x) \ ,$$

$$v(x) = \sum_{n=1}^{\infty} \frac{1}{H_n(-x)} E_n(x) \ ,$$

$$w(x) = \sum_{n=1}^{\infty} H_n(x) E_n(x) \ .$$

We now invoke Abel's lemma for complex series which states that if the series ΣA_n is convergent, and if the series $\Sigma(B_n - B_{n+1})$ is absolutely convergent, then the series $\Sigma A_n B_n$ is convergent.*

It will now be shown that if

$$\sum_{m=1}^{\infty} 1/|a_m|^2$$

converges, then the convergence of $\Sigma E_n(x)$ implies the convergence of $v(x)$ and $w(x)$.

In order to prove this, consider the difference

$$|H_n(x) - H_{n+1}(x)| = |\prod_{m-1}^{n} (1-x/a_m) e^{x/a_m}| \, |1 - (1-x/a_{n+1}) e^{x/a_{n+1}}|$$
$$= |H_n(x)| K_n |x|^2/|a_{n+1}|^2 \ ,$$

where K_n is uniformly bounded in every finite region of the x-plane. Moreover $H_n(x)$ is also similarly bounded and hence, since $\Sigma 1/|a_m|^2$ converges by hypothesis, the series

$$\sum_{n=1}^{\infty} \{H_n(x) - H_{n+1}(x)\}$$

converges absolutely.

Combining this with Abel's lemma we see that the convergence of $u(x)$ implies the convergence of $w(x)$. A similar conclusion may be drawn without essential change in the argument for the convergence of $v(x)$.†

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*See I. Bromwich: *The Theory of Infinite Series.* (2nd edition), London (1926), pp. 242-243.

†The argument here follows that given by Hille whose proof is based upon that given by S. Pincherle for $u(x)$ and $v(x)$: Sulle serie di fattoriali generalizzate. *Rendiconti del Circolo Matematico di Palermo,* vol. 37 (1914), pp. 379-390.

CHAPTER VII.

LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS OF INFINITE ORDER WITH CONSTANT COEFFICIENTS — THE HEAVISIDE CALCULUS.

1. Some General Dynamical Considerations. In the preceding chapter we have discussed a number of methods for the solution of a differential equation of infinite order with constant coefficients. These methods have a general validity beyond the applications which were made there and need only slight modification in order to be applied to systems of linear differential equations with constant coefficients. As is doubtless well known to the reader such systems are fundamental in many problems in dynamics. It is obviously beyond the scope of this work to enter fully into these dynamical considerations except through some special application. For this application we shall choose the theory of electrical conduction because it involves most of the elements to be found in dynamical problems of the closed cycle type and thus may serve as a prototype of oscillatory phenomena in general.

It would seem to be useful to set forth a few elementary concepts. Let us assume that y is the displacement of some dynamic variable from a position of equilibrium and that this displacement is a function of time,

$$y = y(t) \ .$$

Examples of such displacements are very numerous. Perhaps the simplest is the displacement of the bob of a pendulum from its position of equilibrium. Also y might represent the displacement of a bead on an elastic wire fastened at each end to rigid supports; it might represent the moving charge in a portion of an electric circuit or the displacement of a price index from its line of trend.

If $y(t)$ represents a *simple harmonic motion,* we can then write it in the form,

$$\begin{aligned} y(t) &= A\cos(2\pi t/T) + B\sin(2\pi t/T) \\ &= (A^2 + B^2)^{\frac{1}{2}} \sin\,(2\pi t/T + a) \ , \end{aligned}$$

where $a = \text{arc tan } A/B$. In this representation T is called the *period,* $(A^2 + B^2)^{\frac{1}{2}}$ the *amplitude* and a the *phase angle.* The reciprocal of T is called the *frequency* of the harmonic motion.

More generally a motion may be expressed as the sum of several simple harmonic terms

$$y(t) = A_0 + \sum_{n=1}^{N} A_n \cos(2\pi t/T_n) + \sum_{n=1}^{N} B_n \sin(2\pi t/T_n) \ . \qquad (1.1)$$

If $T_n = 1/n$ and the summation extends to infinity, we have the case of a Fourier series.

It should be noted that the total energy (E) of such a system is given by,

$$E = \tfrac{1}{2}[C_0 A_0^2 + \sum_{n=1}^{N} C_n (A_n^2 + B_n^2)] \ ,$$

where the C_n are the weighting factors determined from the physical conditions of the problem.

All dynamical systems, when not sustained by impressed forces, tend toward positions of equilibrium. Free energy is dissipated and disappears into the lowest energy frequencies of the system, that is to say, into molecular frequencies. Under these circumstances we say that a *damping factor* has been present. In many dynamical systems this damping factor is accurately represented by the exponential function, e^{-rt}, where r is positive and depends upon the physical properties of the system. It is obviously possible, therefore, to improve the description of the actual motion of a dynamical system by means of the damped harmonic series,

$$y(t) = A_0 + \sum_{n=1}^{N} e^{-r_n t}[A_n \cos(2\pi t/T_n) + B_n \sin(2\pi t/T_n)] \ .$$

2. The Problem of Electrical Networks. In the problem of the flow of current in an electrical network we are concerned with certain physical quantities called *resistance, inductance* and *capacity* which are represented customarily by the letters R, L and C. Resistance plays a rôle similar to that of friction, inductance to inertia, and capacity to the spring potential of ordinary material systems. In addition to these quantities, the description of an electrical system includes impressed forces designated as *e. m. f. s.* (electromotive forces).

The laws which govern the flow of electricity in a network may be stated as follows:

(*a*) The algebraic sum of the currents entering a branch point of the network is always zero.

(*b*) The total impressed force around any complete circuit in the network is equal to the potential drop due to resistance, inductive reaction and capacity reactance in the circuit.

(*c*) The potential drop in a given branch with resistance R, inductance L and capacity C is equal to $L\, d^2Q/dt^2 + R\, dQ/dt + Q/C$, where Q is the moving charge. The current is computed from the equation $I = dQ/dt$.

These laws may be illustrated by means of the following two typical circuits. Let us designate by $Z_i(p)$ the operator $L_i p^2 + R_i p + 1/C_i$, where $p = d/dt$. From (a) we have,

$$dQ_1/dt = dQ_2/dt + dQ_3/dt \ , \tag{2.1}$$

where the subscripts refer to the three branches of the circuit in figure 1.

By means of laws (b) and (c) we arrive at the equations

$$Z_1(p) \rightarrow Q_1 + Z_3(p) \rightarrow Q_3 = E_1 \ ,$$

$$Z_2(p) \rightarrow Q_2 - Z_3(p) \rightarrow Q_3 = E_2 \ .$$

Eliminating Q_3 by means of (2.1) we obtain

$$[Z_1(p) + Z_3(p)] \rightarrow Q_1 - Z_2(p) \rightarrow Q_2 = E_1 \ ,$$

$$-Z_3(p) \rightarrow Q_1 + [Z_2(p) + Z_3(p)] \rightarrow Q_2 = E_2 \ .$$

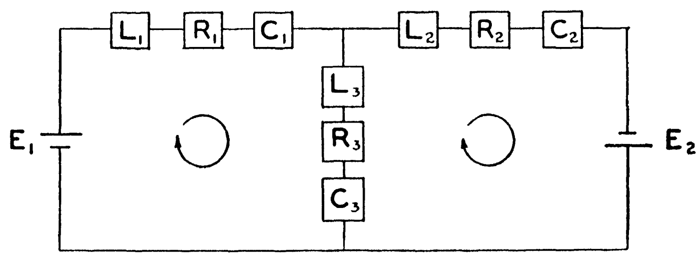

FIGURE 1

These equations may be written more specifically,

$$[(L_1 + L_3)p^2 + (R_1 + R_3)p + (1/C_1 + 1/C_3)] \rightarrow Q_1$$
$$-(L_3p^2 + R_3p + 1/C_3) \rightarrow Q_2 = E_1 \ ,$$

$$-(L_3 p^2 + R_3 p + 1/C_3) \rightarrow Q_1 + [(L_2 + L_3)p^2$$
$$+ (R_2 + R_3)p + (1/C_2 + 1/C_3)] \rightarrow Q_2 = E_2 \ .$$

In the second circuit diagramed in figure 2 we designate the mutual inductance by M. The equations of the two branches are written down by inspection as follows:

$$Z_1(p) \rightarrow Q_1 + M p^2 \rightarrow Q_2 = E_1 \ ,$$

$$M p^2 \rightarrow Q_1 + Z_2(p) \rightarrow Q_2 = E_2 \ .$$

or more specifically,

$$(L_1 p^2 + R_1 p + 1/C_1) \rightarrow Q_1 + M p^2 \rightarrow Q_2 = E_1 \ ,$$

$$M p^2 \rightarrow Q_1 + (L_2 p^2 + R_2 p + 1/C_2) \rightarrow Q_2 = E_2 \ .$$

Other more complicated circuits are similarly reduced to operators.

In terms of the simple circuit,

$$Z(p) \equiv (L\,p^2 + R\,p + 1/C) \to Q(t) = E(t) \;,$$

the function,

$$z(p) = Z(p)/p = L\,p + R + 1/(Cp) \;,$$

is referred to as the *impedance function.*

When an alternating e.m.f. is imposed upon the circuit, then the function,

$$z(\omega i) = R + [L\,\omega - 1/(C\omega)]\,i \;,$$

is called the *impedance of the alternating current* and the real and imaginary parts are called the *resistance* and *reactance* respectively.

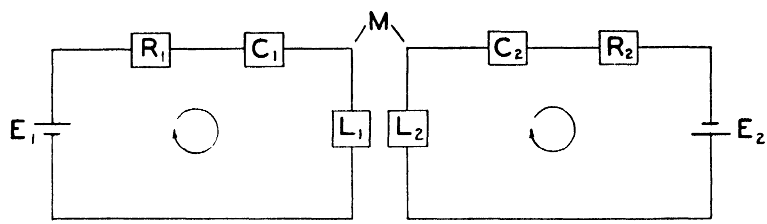

Figure 2.

The solution of the equation when unit *e.m.f.* has been impressed on the circuit at $t = 0$, namely,

$$z(p) \to I(t) = 1 \;, \qquad t > 0 \;,$$

is defined to be the *indicial admittance.* This solution is usually designated by $A(t)$.

PROBLEMS

1. If a condenser of capacity C_3 discharges into two circuits with impedances equal respectively to $Z_1(p)/p$ and $Z_2(p)/p$, show that the charge Q on the conductor is governed by the equation

$$\{Z_1(p)\,Z_2(p) + (1/C_3)[Z_1(p) + Z_2(p)]\} \to Q = 0 \;.$$

2. If in problem 1 the first circuit contains no capacity and the second no inductance, show that the equation reduces to

$$L_1\,R_2\,Q^{(3)} + \left(\frac{L_1}{C_2} + \frac{L_1}{C_3} + R_1\,R_2\right)Q'' + \left(\frac{R_1}{C_2} + \frac{R_2}{C_3} + \frac{R_1}{C_3}\right)\,Q' + \frac{1}{C_2C_3}\,Q = 0 \;.$$

3. A circuit is composed of an inductance and a capacity in parallel and this parallel circuit is in series with a resistance R_1 and an e.m.f., E. If the im-

pedances of the inductance and capacity branches are designated respectively by $z_2(p)$ and $z_3(p)$ and the currents by I_2 and I_3, derive the equations

$$I_2 = z_3/[R_1(z_2 + z_3) + z_2 z_3] \rightarrow E \ ,$$
$$I_3 = z_2/[R_1(z_2 + z_3) + z_2 z_3] \rightarrow E \ .$$

If I_1 is the current in the branch with the resistance R_1, how may it be computed?

4. By an *electric filter* is meant a system of circuits connected in tandem as in figure 1. The impedances $z_1(p)$ and $z_2(p)$ for the circuit in figure 1 we shall call the *series impedances* and the impedance $z_3(p)$ the *shunt impedance.*

Let us consider a filter of n circuits in which the shunt impedances are all equal to $z_2(p)$ and the series impedances are all equal to $z_1(p)$, except in the first and last circuits where they are each equal to $\frac{1}{2}z_1(p)$. If the first circuit contains an e.m.f., E, show that the circuits are governed by the following equations:

$$\tfrac{1}{2}z_1(p) \rightarrow I_1 + z_2(p) \rightarrow (I_1 - I_2) = E \ ,$$
$$\tfrac{1}{2}z_1(p) \rightarrow I_n - z_2(p) \rightarrow (I_{n-1} - I_n) = 0 \ ,$$
$$z_1(p) \rightarrow I_r + z_2(p) \rightarrow (I_r - I_{r+1}) - z_2(p) \rightarrow (I_{r-1} - I_r) = 0 \ ,$$
$$r \neq 1, n.$$

5. Discuss the electrical filter of which the following problem is the mechanical analogue:

Consider the motion of an elastic string on which are fastened n beads, each of mass m, whose distances apart are equal to the constant length a. If y_1, y_2, $\cdots$, y_n represent the displacements of the respective beads from the position of equilibrium of the string, if these displacements are supposed to be at right angles to the string and in the same plane, and if S is the tension of the string, show that the following system of differential equations govern the motion of the beads:

$$my_1'' + (S/a)(y_1 - 0 + y_1 - y_2) = 0 \ ,$$
$$my_2'' + (S/a)(y_2 - y_1 + y_2 - y_3) = 0 \ .$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$my_n'' + (S/a)(y_n - y_{n-1} + y_n - 0) = 0 \ .$$

[This problem was originally treated by J. Lagrange: *Mécanique analytique*, vol. 1, Paris (1788), p. 390. See also: A. G. Webster: *The Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies*, Leipzig (1904), pp. 164-173].

3. Fundamental Theorems. We proceed next to a discussion of the solution of systems of equations of the type derived in the last section. It will be convenient to limit the exposition to the case of a system of two equations, but no essential impairment of generality is thus introduced. We shall thus consider the system

$$A(p) \rightarrow Q_1(t) + B(p) \rightarrow Q_2(t) = E_1(t) \ , \quad (3.1)$$
$$C(p) \rightarrow Q_1(t) + D(p) \rightarrow Q_2(t) = E_2(t) \ ,$$

where the operators are linear and of finite order with constant coefficients. Since we are concerned here particularly with the Heaviside problem, we shall assume that the two functions, $E_1(t)$ and $E_2(t)$, are zero prior to $t=0$. In the last chapter we showed that this *assumption is equivalent to the determination of the functions $Q_1(t)$ and $Q_2(t)$ so that they shall vanish to as high an order as possible at* $t=0$. In the actual circuit problem the operators are polynomials of second degree, but there is no reason in the ensuing analysis to make this limiting assumption.

As a simplification which does not impair generality we will note that the solutions of the system can be obtained from the addition of the solutions of the two problems:

$$\begin{aligned} A(p) &\to V_1 + B(p) \to V_2 = E_1(t) \ , \\ C(p) &\to V_1 + D(p) \to V_2 = 0 \ ; \end{aligned} \tag{3.2a}$$

$$\begin{aligned} A(p) &\to W_1 + B(p) \to W_2 = 0 \ , \\ C(p) &\to W_1 + D(p) \to W_2 = E_2(t) \ . \end{aligned} \tag{3.2b}$$

That is to say, $Q_1(t) = V_1(t) + W_1(t)$ and $Q_2(t) = V_2(t) + W_2(t)$. Hence no loss of generality is suffered if we consider a problem of the form (3.2a).

A further simplification of the problem is possible by replacing $E_1(t)$ with a unit e. m. f. By what is known as the *superposition theorem,* which we shall discuss later, it is possible to derive the solution of the general system (3.1) by means of a single quadrature of the solution of this system in which $E_1(t)$ and $E_2(t)$ have been replaced by 1. Let us therefore consider the following theorem:

Theorem 1. If $h_1(t)$ and $h_2(t)$ are solutions of the system

$$\begin{aligned} A(p) &\to h_1(t) + B(p) \to h_2(t) = 1 \ , \\ C(p) &\to h_1(t) + D(p) \to h_2(t) = 0 \ , \end{aligned} \tag{3.3}$$

*where $A(z)$ is a polynomial of degree a, B of degree b, C of degree c, and D of degree d, then solutions $h_1(t)$ and $h_2(t)$ exist which vanish together with their derivatives of orders up to and including $a-1$ and $b-1$ respectively provided $a+d > b+c$.**

Proof: If we replace $h_1(t)$ by H_1 in (3.3) and drop the operational symbols we obtain the generatrix equations

*No essential restriction is imposed by assuming that $a + d > b + c$ rather than $b + c > a + d$, because this assumption merely fixes the subscripts of h_1 and h_2.

$$A(p)H_1 + B(p)H_2 = 1 \ ,$$

$$C(p)H_1 + D(p)H_2 = 0 \ ,$$

which have for solutions

$$H_1 = D(p)/[A(p)D(p) - C(p)B(p)]$$

$$H_2 = -C(p)/[A(p)D(p) - C(p)B(p)] \ .$$

Let us write $\Delta(p) = A(p)D(p) - C(p)B(p)$ as a polynomial of degree n, i.e., $\Delta(p) = a_0p^n + a_1p^{n-1} + \cdots + a_n$, where it is clear that $n = a + d$.

Since $D(p)$ is of degree d we may write H_1 in the form

$$\begin{aligned} H_1 &= (d_0p^d + d_1p^{d-1} + \cdots + d_d)/(a_0p^n + a_1p^{n-1} + \cdots + a_n) \ , \\ &= (1/p^{n-d})(d_0 + d_1/p + \cdots + d_d/p^d)(a_0 + a_1/p \\ &\qquad\qquad + \cdots + a_n/p^n)^{-1} \ , \\ &= (1/p^{n-d})(A_0 + A_1/p + A_2/p^2 + A_3/p^3 + \cdots) \ . \end{aligned}$$

Giving to $1/p^n \to f(t)$ the customary interpretation

$$1/p^n \to f(t) = \int_0^t (t-s)^{n-1} f(s)\, ds/(n-1)! \ ,$$

we obtain as the value of $h_1(t)$ the function

$$\begin{aligned} h_1(t) &= H_1(p) \to 1 \\ &= t^{n-d}[A_0/(n-d)! + A_1t/(n-d+1)! \\ &\qquad + A_2t^2/(n-d+2)! + \cdots] \ . \qquad (3.4) \end{aligned}$$

It is clear that this function vanishes together with its $n-d-1$ derivatives at the point $t = 0$. Since $n-d-1 = a-1$, the part of the theorem relating to $h_1(t)$ is seen to be true. A similar argument applies also to $h_2(t)$ except now the solution and its $n-c-1$ derivatives vanish at $t = 0$. From the inequality $n-c-1 > b-1$, we establish that part of the theorem which applies to $h_2(t)$.

It should be noticed that these results might have been stated directly as corollaries of theorem 2, chapter 6, since the operational expansions which were employed were those of the outer Laurent annulus.

We have previously discussed the method of Carson as it applies to single differential equations. (See section 4, chapter 6). This method immediately generalizes for systems of equations and we may state it as it applies to system (3.3).

Theorem 2. The problem of solving system (3.3) *is formally equivalent to finding the solutions of the integral equations*:

$$
\begin{aligned}
D(p)/[p\,\Delta(p)] &= \int_0^\infty h_1(t)\, e^{-pt}\, dt \ , \\
-C(p)/[p\,\Delta(p)] &= \int_0^\infty h_2(t)\, e^{-pt}\, dt \ .
\end{aligned}
\tag{3.5}
$$

Proof: The proof of the theorem is easily effected by showing the formal equivalence of (3.4) and (3.5). Noting the integral

$$\int_0^\infty e^{-pt}\, t^n\, dt = n!/p^{n+1} \ ,$$

we have directly

$$
\begin{aligned}
\int_0^\infty e^{-pt} h_1(t)\, dt &= [1/(p \cdot p^{n-d})]\,[A_0 + A_1/p + A_2/p^2 + \cdots] \\
&= D(P)/[p\,\Delta(p)] \ .
\end{aligned}
$$

We are now in a position to derive the solution of problem (3.2a) which may be stated in the following *superposition theorem*:

Theorem 3. If $h_1(t)$ and $h_2(t)$ are solutions of system (3.3) *vanishing together with their derivatives of orders a—1 and b—1 respectively at $t = 0$, then solutions of* (3.2a) *are obtained from the formula*

$$V_i(t) = p \rightarrow \int_0^t E_1(s)\, h_i(t-s)\, ds \ , \quad i = 1, 2, \tag{3.6}$$

provided $a + d > b + c$. Moreover $V_1(t)$ and $V_2(t)$ will vanish together with their derivatives up to and including orders a—1 and b—1 respectively.

Proof: In section 4, chapter 6 we have identified the Carson solution of the equation

$$F(z) \rightarrow u(x) = f(x)$$

with the solution described in theorem 2, chapter 6. If the resolvent generatrix is of the form

$$1/F(z) = P_m(z)/Q_n(z) \ ,$$

where $P_m(z)$ and $Q_n(z)$ are polynomials of degrees m and n respectively, $m < n$, it will be recalled that the solution $u(x)$ together with its derivatives up to and including order $n - m - 1$ vanishes at $x = 0$, provided that the expansion of the generatrix in the region exterior to its poles is employed. Since the Carson solution is identical with

this one, it must vanish together with its first $(n-m-1)$ derivatives at the origin.

Hence we see that $V_i(t)$ are given formally by (3.6). Also from the explicit forms of H_1 and H_2 it is clear that $V_1(t)$ together with its first $(n-d-1)$ derivatives and $V_2(t)$ together with its first $(n-c-1)$ derivatives will vanish at $t = 0$. Since we have $n-d-1 = a - 1$ and $n-c-1 > b-1$, it follows that $V_1(t)$ and $V_2(t)$ will vanish together with their derivatives up to and including orders $a-1$ and $b-1$ respectively.

The solution of system (3.2b) is obtained now by a simple application of theorems 2 and 3 and is found to be

$$W_i(t) = p \rightarrow \int_0^t E_2(s)\, k_i(t-s)\, ds \;, \qquad i = 1, 2 \;,$$

where the functions $k_i(t)$, $i = 1, 2$, are inversions of the equations,

$$-B(p)/[p\,\Delta(p)] = \int_0^\infty k_1(t)\, e^{-pt}\, dt \;,$$

$$A(p)/[p\,\Delta(p)] = \int_0^\infty k_2(t)\, e^{-pt}\, dt \;.$$

It is also clear that $W_1(t)$ will vanish up to and including order $c-1$ and $W_2(t)$ up to and including order $d-1$.

From this it follows that $Q_1(t)$ of the complete system (3.1) will vanish up to and including order $p-1$, where p is the smaller of the numbers a and c and that $Q_2(t)$ will vanish up to and including order $q-1$ where q is the smaller of the numbers b and d.

Example: The following example will illustrate the application of these theorems.

Let us find the solution of the system

$$Q_1''(t) - 5Q_1'(t) + 13Q_1(t) \qquad + Q_2'(t) + 20Q_2(t) = e^t \;,$$
$$Q_1'(t) + 2Q_1(t) + Q_2''(t) + 3Q_2'(t) + 3Q_2(t) = e^{-t} \;,$$

which vanishes to as high an order as possible at the origin.

We first consider the reduced system

$$h_1'' - 5h_1' + 13h_1 + h_2' + 20h_2 = 1 \;,$$
$$h_1' + 2h_1 + h_2'' + 3h_2' + 3h_2 = 0 \;.$$

Then $h_1(t)$ and $h_2(t)$ will be solutions of the equations

$$\frac{p^2 + 3p + 3}{p(p-1)^3(p+1)} = \int_0^\infty e^{-pt} h_1(t)\, dt \;,$$

$$\frac{-(p+2)}{p(p-1)^3(p+1)} = \int_0^\infty e^{-pt} h_2(t)\, dt \;.$$

Since we have

$$[p^2+3p+3]/[p(p-1)^3(p+1)]$$
$$=-3/p+(1/8)/(p+1)+(23/8)/(p-1)$$
$$-(11/4)/(p-1)^2+(7/2)/(p-1)^3\ ,$$
$$-(p+2)/[p(p-1)^3(p+1)]$$
$$=2/p-(1/8)/(p+1)-(15/8)/(p-1)$$
$$+(7/4)/(p-1)^2-(3/2)/(p-1)^3\ ,$$

we can derive the values of $h_1(t)$ and $h_2(t)$ from the formula

$$\int_0^\infty e^{-pt}(t^n/n!)\,e^{\lambda t}\,dt=1/(p+\lambda)^{n+1}\ .$$

We then get

$$h_1(t)=-3+(1/8)\,e^{-t}+(23/8)\,e^t-(11/4)\,t\,e^t+(7/4)\,t^2e^t$$
$$h_2(t)=2-(1/8)e^{-t}-(15/8)e^t+(7/4)t\,e^t-(3/4)t^2e^t\ ,$$

and we observe that

$$h_1(0)=h_1'(0)=h_2(0)=h_2'(0)=h_2''(0)=0\ .$$

Similarly for the system

$$k_1''-5k_1'+13k_1+k_2'+20k_2=0\ ,$$
$$k_1'+2k_1+k_2''+3k_2'+3k_2=1\ ,$$

we get

$$k_1(t)=20-(19/8)e^{-t}-(141/8)e^t+(64/4)t\,e^t-(21/4)\,t^2e^t\ ,$$
$$k_2(t)=-13+(19/8)e^{-t}+(85/8)e^t-(33/4)\,t\,e^t+(9/4)t^2e^t\ ,$$

where $k_1(0)=k_1'(0)=k_1''(0)=k_2(0)=k_2'(0)=0$.

Making use of theorem 3 we obtain the complete solution of the system in the form

$$Q_1(t)=\frac{d}{dt}\Big[\int_0^t e^s\,h_1(t-s)\,ds+\int_0^t e^{-s}\,k_1(t-s)\,ds\Big]\ ,$$
$$=e^t\Big(-\frac{15}{4}+\frac{41}{8}t-\frac{9}{4}t^2+\frac{7}{12}t^3\Big)+e^{-t}\Big(\frac{15}{4}+\frac{19}{8}t\Big)\ ,$$
$$Q_2(t)=e^t\Big(\frac{11}{4}-\frac{25}{8}t+\frac{5}{4}t^2-\frac{1}{4}t^3\Big)+e^{-t}\Big(-\frac{11}{4}-\frac{19}{8}t\Big)\ .$$

We also note that $Q_1(0)=Q'_1(0)=Q_2(0)=Q'_2(0)=0$.

4. The Heaviside Expansion Theorem. Central to many applications of the Heaviside calculus is the so called *expansion theorem.* Numerous proofs of this theorem have been given and Heaviside himself gave two as has been pointed out by M. S. Vallarta (See *Bibliography*).

The expansion theorem is easily written down from the results which we have previously obtained, being generalized, as a matter of fact, by formula (2.2) of chapter 6.

To put the theorem in the form in which it appears in the theory of electric circuits, we consider the operational symbol

$$G(p) = P_m(p)/Q_n(p) \ , \qquad m < n \ ,$$

where $P_m(p)$ and $Q_n(p)$ are polynomials of degrees m and n respectively.

Referring to formula (2.2) of chapter 6 we see that we may write the expansion of the outer Laurent annulus, that is to say, the Heaviside expansion, in the form

$$\begin{aligned} G(p) \rightarrow E(t) = \sum_{j=1}^{n} \{P_m(a_j)/[a_j\, Q'_n(a_j)]\} e^{a_j t} \{E(0) + E'(0)/a_j \\ + E''(0)/a_j^2 + \cdots\} + \{G(0) + G'(0)p + G''(0)p^2/2! \\ + \cdots\} \rightarrow E(t) \ . \qquad (4.1) \end{aligned}$$

where $a_1, a_2, \cdots, a_n$ are zeros of $Q_n(p)$.

This may be formulated in the following theorem:

Theorem 5. If $G(p)$ is the operational symbol,

$$G(p) = P_m(p)/Q_n(p) \ , \quad m < n \ ,$$

then $G(p) \rightarrow E(t)$ as defined by (4.1) is equivalent to the operation upon $E(t)$ of the expansion of $G(p)$ in its outer annulus of convergence.

Two special applications are to be noted, the first where $E(t)$ is a unit *e. m. f.*, $E(t) = 1$, and the second where $E(t)$ is an alternating *e.m.f.*, $E(t) = e^{i\lambda t}$.

In the first case the expansion theorem reduces to

$$G(p) \rightarrow 1 = \sum_{j=1}^{n} \{P_m(a_j) e^{a_j t}/[a_j Q'(a_j)]\} + G(0) \ ,$$

and in the second to

$$G(p) \rightarrow e^{i\lambda t} = \sum_{j=1}^{n} \{P_m(a_j) e^{a_j t}/[Q_n'(a_j)\,(a_j - \lambda i)]\} + G(\lambda i) e^{\lambda i t} \ . \qquad (4.2)$$

This formula must be modified when λi equals some one of the values of a_i, let us say a. This modification is made by means of

formula (2.4) of chapter 6. Referring to this formula and noting that

$$F'(a) = Q'_n(a)/P_m(a) \ ,$$

$$F''(a) = [P_m(a)\ Q''_n(a) - 2P'_m(a)\ Q'_n(a)]/P_m{}^2(a) \ ,$$

we modify (4.2) to read

$$G(p) \rightarrow e^{i\lambda t} = \sum_{j=1}^{n}{}' \{P_m(a_j)\, e^{a_j t}/[Q_n'(a_j)\,(a_j - a)]\}$$
$$+ \{e^{at} P_m(a)/Q_n'(a)\}\{t - [P_m(a)\, Q_n''(a)$$
$$- 2P_m'(a)\, Q_n'(a)]/2P_m(a)\, Q_n'(a)\} \ ,$$

where Σ' means that $\lambda i = a$ has been omitted from the sum.

Example: As an example let us solve the following equation,

$$(L\,p^2 + R\,p + 1/C) \rightarrow Q(t) = 1 \ ,$$

which is the equation of the charge in a simple circuit upon which has been imposed a unit *e.m.f.*

Making the assumption that $4L/C > R^2$, we obtain as the roots of the equation, $Lp^2 + Rp + 1/C = 0$, the values $a_1 = -a + \omega i$, $a_2 = -a - \omega i$, where we employ the abbreviations, $a = R/2L$, $\omega = (1/CL - R^2/4L^2)^{\frac{1}{2}}$. Noting that $Q(p) = Lp^2 + Rp + 1/C$, $Q'(p) = 2Lp + R$, we obtain $Q'(a_1) = 2L\omega$ and $Q'(a_2) = -2L\omega$.

Employing these values in the expansion theorem we at once obtain,

$$G(p) \rightarrow 1 = [e^{(-a+\omega i)t}/(2L\, i\omega a_1) + e^{-(a+\omega i)t}/(-2L\, i\omega a_2)] + C$$
$$= (e^{-at}/2L)\,[-e^{\omega i t}\,\frac{\omega^2 - a\omega i}{\omega^4 + a^2\omega^2} - e^{-\omega i t}\,\frac{\omega^2 + a\omega i}{\omega^4 + a^2\omega^2}] + C \ .$$

By means of $\omega^2 + a^2 = 1/CL$, this expansion reduces to,

$$G(p) \rightarrow 1 = Q(t) = C - C\, e^{-at}(\cos\omega t + a \sin\omega\, t/\omega) \ .$$

In order to obtain the current, $I(t)$, we compute $Q'(t)$ and thus find,

$$I(t) = Q'(t) = (1/L\omega)\, e^{-at} \sin\omega\, t \ .$$

If instead of the assumption $4L/C > R^2$, we have $4L/C < R^2$, the formulas for $Q(t)$ and $Q'(t)$ are found to be

$$Q(t) = C - C \cdot e^{-at}(\cosh\omega' t + a \sinh\omega' t/\omega') \ ,$$

$$Q'(t) = (1/L\omega')\, e^{-at} \sinh\omega' t \ ,$$

where we abbreviate, $\omega' = (R^2/4L^2 - 1/CL)^{1/2}$.

It is interesting to note that considerable algebraic simplification is attained if we compute the current directly. Thus we should have

$$
\begin{aligned}
I(t) &= pG(p) \to 1 \\
&= p/(Lp^2 + Rp + 1/C) \to 1 \\
&= [e^{(-a+\omega i)t} - e^{-(a+\omega i)t}]/2Li = (1/L\omega)\, e^{at} \sin \omega t \ .
\end{aligned}
$$

PROBLEMS

1. Solve the equation

$$(Lp^2 + Rp + 1/C) \to Q(t) = E \cos mt \ ,$$

and obtain the solution

$$Q(t) = \frac{E}{\sqrt{a^2+b^2}} \cos (mt - \alpha) + F(t)$$

where $a^2 + b^2 = R^2m^2 + [(1/C) - m^2L]^2$, $\tan \alpha = Rm/[(1/C) - m^2L]$, and $F(t)$ is a function which damps to zero.

This problem furnishes a simple example of *forced vibration*, since the system is forced to assume the same period as the impressed force. It will be further observed that if $(1/C) - m^2L = 0$ and if R is small, the amplitude of the vibration will be large. This is the phenomenon of *resonance*, which is found in the heavy rolling of ships, the vibration of bridges under marching troops, etc.

2. Solve the system

$$
\begin{aligned}
L_1(p) \to Q_1 + L_2(p) \to Q_2 &= E_1 \cos mt \ , \\
L_3(p) \to Q_1 + L_4(p) \to Q_2 &= E_2 \cos mt \ ,
\end{aligned}
$$

and discuss the condition for resonance.

3. Employing the methods of the Heaviside calculus, solve the following system:

$$
\begin{aligned}
3y'' - 2y' + y + 6z'' - 3z' + 4z &= e^{2t} \ , \\
y'' + 4y' - 3y + 2z'' - 2z' + z &= e^{-2t} \ .
\end{aligned}
$$

4. A particle is moving in the xy-plane under the influence of a central force directed toward the origin and proportional to the distance of the particle from the origin.

Show that the equations of motion are

$$x'' = -a^2x, \qquad y'' = -a^2y \ ,$$

and hence deduce that the path is an ellipse.

5. We are given a set of data forming a time series

$$y: \quad y_1, y_2, y_3, \cdots, y_n \ ,$$

and from this set we derive the first and second difference series

$$
\begin{aligned}
&\Delta y: \quad \Delta y_1, \Delta y_2\ \Delta y_3, \cdots, \ \Delta y_n \ , \\
&\Delta^2 y: \ \Delta^2 y_1, \Delta^2 y_2, \Delta^2 y_3, \cdots, \ \Delta^2 y_n \ .
\end{aligned}
$$

If r_{12} is the correlation coefficient of the second derived series with the first, r_{13} the coefficient of the second derived series with the original series, and r_{23} the

coefficient of the first derived series with the original, show that the equation connecting the three series is the following:

$$\frac{1-r^2_{23}}{\sigma_1}(\Delta^2 y-\Delta^2 y_0)+\frac{r_{23}r_{13}-r_{12}}{\sigma_2}(\Delta y-\Delta y_0)+\frac{r_{23}r_{12}-r_{13}}{\sigma_3}(y-y_0)=0\ ,$$

where σ_1, σ_2, and σ_3 are the standard deviations respectively of $\Delta^2 y$, Δy and y, and $\Delta^2 y_0$, Δy_0, and y_0 are the respective averages of the series.

Replacing Δy by $(e^p - 1) \rightarrow y$ and $\Delta^2 y$ by $(e^p - 1)^2 \rightarrow y$, solve the equation for y. Obtain the particular solution $y = e^{\lambda t}$, where

$$\lambda = \tfrac{1}{2} \log \left(\frac{A - B + C}{A}\right) \pm \arctan \frac{(4AC - B^2)^{\frac{1}{2}}}{2A - B}\ ,$$

in which we designate the coefficients of the difference equation by A, B, C respectively.

6. The Dow Jones industrial stock market averages by months from 1897 to 1914, when subjected to the computations described in problem 5, yield the following numerical values:

$$\sigma_1(\text{for } \Delta^2 y) = .512\ ,$$

$$\sigma_2(\text{for } \Delta y) = 2.358\ ,$$

$$\sigma_3(\text{for } y) = 15.150\ ,$$

$r_{12} = .129$, $r_{13} = -.552$, $r_{23} = -.234$.

Under the assumption that the averages of all the series are zero, show that the difference equation for the original series is the following:

$$1.847\Delta^2 y + .0001\Delta y + .034y = 0\ .$$

Hence show that the period of y is 46 months. (Data furnished by the Cowles Commission for Research in Economics).

7. Since the operator $\Delta = e^p - 1 = p + \frac{1}{2}p^2 + \cdots$, and since $\Delta^2 = (e^p - 1)^2 = p^2 + \cdots$, replace the difference equation of problem 6 by a differential equation of second order on the assumption that derivatives of order higher than 2 can be neglected. Determine the period of y from this equation. What criterion is suggested by this problem for the justification of the neglect of higher differences in a statistical problem of this type?

8. If the initial values of x, x' and y, y' are respectively designated by x_0, x_1, and y_0, y_1, show that the value of x determined by the following system:

$$(p^2 - 4p) \rightarrow x - (p-1) \rightarrow y = 0\ ,$$

$$(p+6) \rightarrow x + (p^2 - p) \rightarrow y = 0\ ,$$

is given by $x = (1/12)e^{-t}(6x_0 - x_1 - y_0 + y_1) - (1/3)e^{2t}(-3x_0 + 2x_1 - y_0 + y_1) + (1/4)e^{3t}(-2x_0 + 3x_1 - y_0 + y_1)$.

[E. J. Routh's *Rigid Dynamics*, vol. 2 (1892), art. 367; Bromwich: (1), p. 407].

9. Using the notation of problem 8, show that the value of x determined from the following system:

$$(p^2-2p) \to x - y = 0 \ ,$$

$$(2p-1) \to x + p^2 \to y = 0 \ ,$$

is given by $x = (M_0 + M_1 t + \tfrac{1}{2} M_2 t^2)\, e^t + N e^{-t}$, where the values of M_0 M_1, M_2 and N are determined by the equations

$$2M_2 = x_0 + x_1 + y_0 + y_1 \ , \qquad 2M_0 + M_1 = x_0 + x_1 \ ,$$

$$2M_1 + M_2 = -x_0 + 2x_1 + y_0 \ , \qquad M_0 + N = x_0 \ .$$

[Routh: *loc. cit.*, art. 373; Bromwich: (1), pp. 407-408].

10. Using the notation of problem 8, show that the solution of the following system:

$$(p^2+1) \to x + (p^2-2p) \to y = 0 \ ,$$

$$(p^2+p) \to x + p^2 \to y = 0 \ ,$$

is given by

$$x = (x_0 + x_1/3) - (1/3) x_1 e^{-3t} \ ,$$

$$y = (y_0 - 2x_1/9) + (y_1 + 2x_1/3)\, t + (2/9) x_1 e^{-3t},$$

provided we assume that $x_0 = x_1 + 2y_1$. [Bromwich: (1), pp. 409-410].

11. Prove that the system

$$(p^2+1) \to x + (p^2+p+1) \to y = t \ ,$$

$$p \to x + (p+1) \to y = e^t \ ,$$

has the unique solution (without arbitrary constants)

$$x = 1 + t - 3e^t \ , \qquad y = 2e^t - 1 \ .$$

[E. L. Ince: *Ordinary Differential Equations*, London (1927), p. 145].

12. Solve the following difference system:

$$u(t+2) - 3\, u(t+1) + 2\, u(t) + 3\, v(t+1) = t+1 \ ,$$

$$u(t+1) - 2\, v(t+2) + v(t+1) - 2\, v(t) = t^2 - t - 2 \ .$$

13. Find the Heaviside solution of the equation

$$(p^3 - 7p + 6) \to u(t) = 1 \ .$$

14. Find the Heaviside solution of the equation

$$(p^4 - 1) \to u(t) \cos 2t \ .$$

5. Applications to Certain Partial Differential Equations of Mathematical Physics. One of the most striking applications of the Heaviside calculus is to be found in the solution of certain partial differential equations of the general type,

$$A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = f(x, y) \ , \qquad (5.1)$$

where A, B, and C are constants and the solution $u(x, y)$ is subject to boundary conditions suggested by physical or geometrical considerations.

The actual method of procedure can best be illustrated by certain physical examples.

a. The Fourier Heat Problem

We shall begin by considering Fourier's problem of determining the steady state of temperature in a thin, rectangular plate of breadth L and of infinite length with faces impervious to heat. We shall suppose that the two long faces of the plate, AB and $A'B'$, are kept at zero temperature and that the distribution of temperature along the short face, AA', is a given function of x, $f(x)$. The temperature is also assumed to approach zero as we proceed indefinitely far from the base. A special case of this problem has already been solved by other means in section 8 of chapter 3.

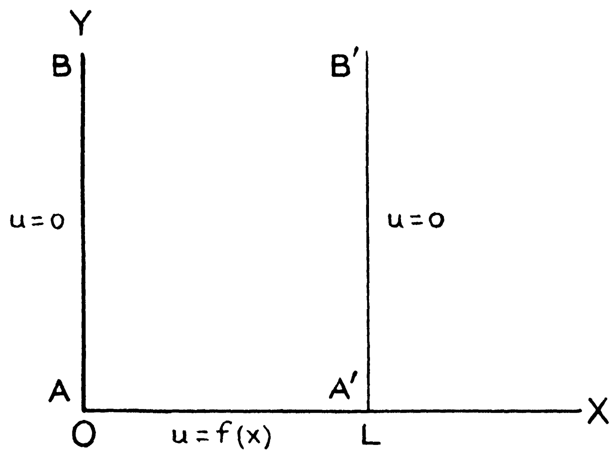

FIGURE 3

The problem, then, is to determine a solution of the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 , \tag{5.2}$$

where $u(x, y)$ is subject to the boundary conditions:

$$u(0,y)=0 , \ u(L,y)=0 , \ u(x,0)=f(x) , \ \lim_{y=\infty} u(x,y)=0 .$$

To begin with, let us now regard this as a problem in the variable y, replacing $\partial^2/\partial x^2$ by z^2. We then are led to consider the differential system,

$$\frac{\partial^2 u}{\partial y^2}+z^2 u=0 ,$$

$$u(0)=f(x), u'(0)=0 .$$

Employing the abbreviation, $\partial^2/\partial y^2 = q^2$, and referring to formula (2.13) of chapter 6, we see that this statement of the problem is formally equivalent to solving the algebraic equation,

$$(q^2 + z^2)\,U = q^2 f(x) \qquad (5.3)$$

from which the desired solution, $u(x,y)$, is to be attained by interpreting the operation $U(q,x) \rightarrow 1$ by means of equation (2.8) of chapter 6.

We now come to the magical aspect of formula (5.3) which is to be regarded as a differential equation in x by replacing z^2 by its equivalent symbol $\partial^2/\partial x^2$. Hence we turn to the solution of the differential equation

$$\left(\frac{\partial^2}{\partial x^2} + q^2\right) U = q^2 f(x) \ , \qquad (5.4)$$

where $U(x)$ is subject to the boundary conditions:

$$U(0) = U(L) = 0 \ . \qquad (5.5)$$

In order to solve this problem, we must have recourse to the theory of Green's functions of a single variable, a theory which is extensively developed in section 5 of chapter 11. We first observe that a set of fundamental solutions of the homogeneous equation,

$$\frac{d^2U}{dx^2} + q^2 U = 0 \ ,$$

is furnished by $U_1(x) = \cos qx$, $U_2(x) = \sin qx$. From a linear combination of these solutions we must construct a continuous function which satisfies the boundary conditions (5.5). This is the desired Green's function.

With reference to equation (5.4) of chapter 11, we then compute the Green's function to be,

$$G(x,s) = \sin qx \sin q(L-s)/(q \sin qL) \ , \qquad x \leqq s \ ,$$

$$G(x,s) = \sin qs \sin q(L-x)/(q \sin qL) \ , \qquad x \geqq s \ .$$

If we designate the first of these functions by $G_1(x,s)$ and the second by $G_2(x,s)$, the following limit is easily verified:

$$\lim_{x=s} \left[\frac{\partial}{\partial x} G_1(x,s) - \frac{\partial}{\partial x} G_2(x,s)\right] = 1 \ .$$

In terms of $G(x,s)$ we express the solution of the system (5.4), (5.5) as the integral

$$U(x) = -\int_0^L G(x,s)\ q^2 f(s)\, ds \ ,$$

$$U(x) = -\int_0^x G_2(x,s)\,q^2 f(s)\,ds - \int_x^L G_1(x,s)\,q^2 f(s)\,ds$$

$$= \int_0^x [q \sin qs \sin q(x-L)/\sin qL] f(s)\,ds$$

$$+ \int_x^L [q \sin qx \sin q(s-L)/\sin qL] f(s)\,ds \ .$$

Finally, in order to attain the desired solution, we invoke the principle enunciated above. $U(x) = U(q,x)$ is to be regarded as an operator in which $q = \partial/\partial y$, and the solution, $u(x,y)$, of the original problem is supplied from the operation,

$$u(x,y) = U(q,x) \to 1 \ .$$

This expression is interpreted as the expansion (2.8) of chapter 6.

To arrive at the explicit development we first abbreviate,

$$F_1(q) = \sin qL/[q \sin qs \sin q(x-L)] \ ,$$

$$F_2(q) = \sin qL/[q \sin qx \sin q(s-L)] \ ,$$

and note that the zeros, a_n, of each function are given by $a_n = n\pi/L$. Hence we compute,

$$F_1'(a_n) = F_2'(a_n) = L^2/[n\pi \sin(n\pi s/L) \sin(n\pi x/L)] \ .$$

We next take account of the fourth boundary condition of our problem, namely, $\lim_{y=\infty} u(x,y) = 0$, which immediately excludes all terms of the form e^{ay}, $a > 0$, from our summation.

In order to avoid this obvious difficulty and to obtain a convergent series we replace y by $-y$ for all positive values of n and hence reach the following solution:

$$u(x,y) = \sum_{n=-\infty}^{\infty} e^{-\delta n\pi y/L} \left[\int_0^x \frac{1}{a_n F_1'(a_n)} f(s)\,ds + \int_x^L \frac{1}{a_n F_2'(a_n)} f(s)\,ds \right],$$

where δ is positive or negative as n is positive or negative.

From obvious symmetry, this reduces to the familiar solution,

$$u(x,y) = (2/L) \sum_{n=1}^{\infty} e^{-n\pi y/L} \sin(n\pi x/L) \int_0^L \sin(n\pi s/L) f(s)\,ds \ .$$

b. The Problem of the Elastic String

It is illuminating to begin our discussion with a derivation of the equations which govern the vibrations of an elastic string. We shall assume that the position of equilibrium of the string is along the axis of X and that the ends of the string are attached at the points $x = 0$ and $x = L$. Also let T be the initial tension in the string,

that is to say, T is the force necessary to hold a point P in position if the string were severed at that point. We shall assume further that the weight of the string is W and that its *modulus of elasticity* is E (Young's modulus), that is to say, E measures the force necessary to increase the length of the string from L to $L + E\,\delta\,L$. This law asserting the linear increase in the tension with the extension of the string was enunciated by Robert Hooke (1635-1703) in 1676 and is assumed to hold within the elastic limits of the material.

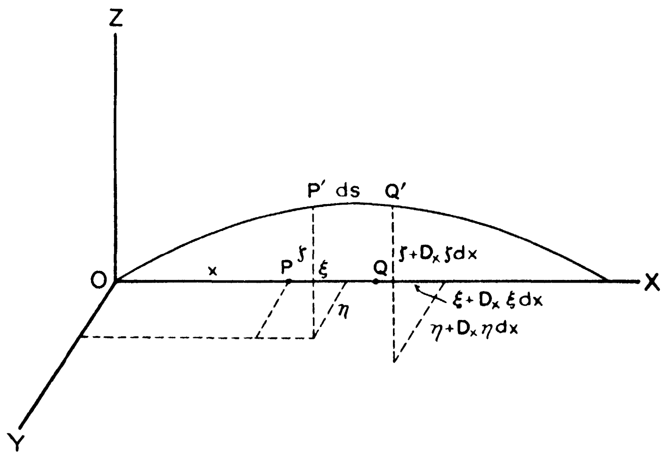

FIGURE 4

We now let the string be deformed in such a manner that the point P with coordinates $(x,0,0)$ goes into a point P' with coordinates $x + \xi$, η, ζ, in which ξ, η ζ, are functions of the variables x and t, t denoting time. We shall regard ξ, η, ζ together with their derivatives $D_x\xi$, $D_x\eta$, $D_x\zeta$ as small variations of the first order the higher powers of which can be neglected in comparison with lower powers. Let $T_{p'}$ be the force necessary to accomplish the described deformation, that is to say, $T_{p'}$ will be the tension acting in each direction from the point P'.

Let us now resolve the tension $T_{p'}$ into its components $X_{p'}$, $Y_{p'}$, $Z_{p'}$. If λ, μ, ν are the direction cosines of the tangent to the curve at the point P', we have,

$$X_{p'} = -\lambda\, T_{p'}\,, \quad Y_{p'} = -\mu\, T_{p'}\,, \quad Z_{p'} = -\nu\, T_{p'}\,.$$

To x we now give an increment dx and obtain the point Q. In the deformation just described the point Q is displaced to Q' with coordinates $x + dx + \xi + D_x\xi\, dx$, $\eta + D_x\eta\, dx$, $\zeta + D_x\zeta\, dx$.

Denoting the tension at Q' by $T_{Q'}$, we see that it can be resolved into the components,

$$X_{Q'} = (X_{P'} + D_x X_{P'}\, dx),$$
$$Y_{Q'} = (Y_{P'} + D_x Y_{P'}\, dx),$$
$$Z_{Q'} = (Z_{Q'} + D_x Z_{P'}\, dx)\ .$$

Hence the force acting upon the elementary segment ds of the elastic curve will be,

$$F_x = X_{P'} + X_{Q'} = D_x X_{P'}\, dx\ ,$$
$$F_y = Y_{P'} + Y_{Q'} = D_x Y_{P'}\, dx\ , \qquad (5.6)$$
$$F_z = Z_{P'} + Z_{Q'} = D_x Z_{P'}\, dx\ .$$

By Newton's second law of motion these forces must be equal to the components of acceleration,

$$\Delta M \frac{\partial^2 \xi}{\partial t^2}\ ,\ \Delta M \frac{\partial^2 \eta}{\partial t^2}\ ,\ \Delta M \frac{\partial^2 \zeta}{\partial t^2}\ , \qquad (5.7)$$

where ΔM is the mass of ds, $\Delta M = W\, ds/gL$, g denoting the acceleration of gravity.

We now observe that $ds = (1 + D_x\xi)\, dx$, provided differentials of a higher order than the first are neglected, and that the direction cosines are respectively: $\lambda = 1$, $\mu = D_x\eta$, $\nu = D_x\zeta$.

Hence by an application of Hooke's law and the neglect of terms involving differentials of higher order than the first, we get

$$X_{P'} = -(T + E\, D_x\xi),\quad Y_{P'} = -T\, D_x\eta,\quad Z_{P'} = -T\, D_x\zeta\ .$$

Substituting these values in equations (5.6), we readily obtain,

$$F_x = ED_x{}^2\xi\ ,\ F_y = TD_x{}^2\eta\ ,\ F_z = TD_x{}^2\zeta\ . \qquad (5.8)$$

Since we have also, $\Delta M = (W/gL)\,(1 + D_x\xi)\, dx$, we may substitute this value in (5.7) and upon neglecting second order terms, obtain by equating (5.7) and (5.8), the equations:

$$\frac{\partial^2 \xi}{\partial t^2} = w^2 \frac{\partial^2 \xi}{\partial x^2}\ ,\ \frac{\partial^2 \eta}{\partial t^2} = v^2 \frac{\partial^2 \eta}{\partial x^2}\ ,\ \frac{\partial^2 \zeta}{\partial t^2} = v^2 \frac{\partial^2 \zeta}{\partial x^2}\ , \qquad (5.9)$$

where we abbreviate: $w^2 = ELg/W$, $v^2 = TLg/W$.

The first of these equations defines the longitudinal vibration of the string and the other two the components of the transverse vibration. The longitudinal vibration can usually be neglected in comparison with the transverse motion and this assumption we shall make here.

In order to simplify the problem for the sake of the application which we contemplate, we shall limit our discussion to a deformation

in a single plane. Hence we shall consider the solution of the single equation,

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2} , \tag{5.10}$$

imposing upon the function $u(x,t)$ the following boundary conditions:

$$\begin{aligned}
&(1) \quad u(0,t)=u(L,t)=0 , \\
&(2) \quad u(x,0)=f(x) , \\
&(3) \quad D_t u(x,0)=g(x) .
\end{aligned} \tag{5.11}$$

It will be readily observed that setting the constant v^2 equal to unity imposes no essential restriction on the problem, since it may be absorbed in a transformation of the variable t.

In order to solve this fundamental problem by the operational methods of the Heaviside calculus, we now proceed exactly as in the case of the flow of heat discussed in section (a).

Abbreviating $\partial/\partial t$ by p, we replace the differential equation (3.10) and the boundary conditions (2) and (3), by the single operational equation

$$\frac{d^2U}{dx^2}-p^2U=-[p^2f(x)+pg(x)] , \tag{5.12}$$

where U must be determined so as to satisfy the conditions:

$$U(0)=U(L)=0 . \tag{5.13}$$

A set of fundamental solutions of the equation, $d^2U/dx^2 - p^2U = 0$, is given by $U_1(x) = \cosh px$, $U_2(x) = \sinh px$. From these, with reference to equation (5.4) of chapter 11, we then compute the Green's function for the system (5.12) (5.13) to be,

$$\begin{aligned}
G(x,s)&=\sinh px \sinh p(L-s)/(p \sinh pL) , \quad x \leqq s , \\
G(x,s)&=\sinh ps \sinh p(L-x)/(p \sinh pL) , \quad x \geqq s .
\end{aligned}$$

In terms of $G(x,s)$ we express the solution of the system as the integral,

$$\begin{aligned}
U(x)&=\int_0^L G(x,s)[p^2f(s)+pg(s)]ds , \\
&=\int_0^x [\sinh px \sinh p(L-x)/\sinh pL][pf(s)+g(s)]ds \\
&\quad+\int_x^L [\sinh px \sinh p(L-x)/\sinh pL][pf(s)+g(s)]ds .
\end{aligned}$$

As in the previous example we now regard the function $U(x) = U(p,x)$ as an operator and obtain the desired function $u(x,t)$ from an interpretation of the symbol,

$$u(x,t) = U(p,x) \rightarrow 1 \ .$$

This interpretation, as before, is made by means of the expansion (2.8) of chapter 6.

To attain the explicit development we first abbreviate,

$$F_1(p) = \sinh pL/[p \sinh ps \sinh p(L-x)] \ ,$$

$$F_2(p) = \sinh pL/[p \sinh px \sinh p(L-x)] \ ,$$

and note that the zeros, a_n, of each function are given by $a_n = n\pi i/L$. Hence we compute,

$$F'(a_n) = F'(a_n) = L^2/[n\pi i \sin(n\pi s/L) \sin(n\pi x/L)] \ ,$$

$$\left.\frac{d\,p\,F_1(p)}{dp}\right|_{p=a_n} = \left.\frac{d\,p\,F(p)}{dp}\right|_{p=a_n} = L/[\sin(n\pi s/L)\sin(n\pi x/L)] \ ,$$

$$= a_n F'(a_n) = a_n F'(a_n) \ .$$

The desired solution is thus immediately expressed in the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} e^{n\pi i t} \int_0^L \left[\frac{1}{aF'(a_n)} f(s) + \frac{1}{a^2F'(a_n)} g(s) \right] ds$$

From obvious symmetry this reduces to the explicit solution:

$$u(x,t) = (2/L) \sum_{n=1}^{\infty} \sin(n\pi x/L) \cos(n\pi t/L) \int_0^L \sin(n\pi s/L) f(s)\, ds$$

$$+ (2/L) \sum_{n=1}^{\infty} (1/n) \sin(n\pi x/L) \sin(n\pi t/L) \int_0^L \sin(n\pi s/L) g(s)\, ds \ .$$

PROBLEMS

1. A potential function, $V(x,y)$, belonging to an electrostatic field satisfies Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \ .$$

Set up the operational equation for the potential of a plane, rectangular plate, which is kept at zero potential on three sides and at a potential $f(x)$ on the fourth side. That is, assume

$$V(0,y) = V(a,y) = V(x,b) = 0 \ , \quad V(x,0) = f(x) \ .$$

2. Show that the solution of problem 1 is given by

$$V(x,y) = \frac{2}{a} \sum_{m=1}^{\infty} \frac{\sinh\{m\pi(b-y)/a\} \sin(m\pi x/a)}{\sinh(m\pi b/a)} \int_0^a f(t) \sin(m\pi t/a)\, dt \ .$$

3. Solve the problem just proposed for $f(x) = 1$.

4. Solve the above problem for $f(x) = x$.

5. Solve the above problem if $f(x) = e^{rx}$.

6. Determine the value of $V(x,y)$ when the plate is of infinite dimensions in the direction of the y-axis, namely, if $b = \infty$.

7. If a string is vibrating in a resisting medium, equation (5.10) and the boundary conditions (5.11) are replaced by

$$\frac{\partial^2 u}{\partial t^2} + 2k\frac{\partial u}{\partial t} = v^2 \frac{\partial^2 u}{\partial x^2} ,$$

$$u(0,t) = u(L,t) = 0 ,$$

$$u(x,0) = f(x) ,$$

$$D_t(x,0) = 0 .$$

Set up the operational equation for this problem.

8. If an elastic string of length L, fastened at each end, is displaced at its center through a distance k, show that its subsequent vibration is given by the series

$$u(x,t) = (8k/\pi^2) \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \tfrac{1}{2}n\pi \sin(n\pi x/L) \cos(n\pi t/L)$$

9. Show that the actual motion of the elastic string of problem 8 consists of three straight pieces, the center piece always moving parallel to the axis of x. [For a diagram of the motion see Lord Rayleigh: *Theory of Sound*, London (1894-1896), vol. 1, art. 146].

10. If the elastic string of problem 8 is displaced through a distance k at a point $(1/m)$th the length of the string, show that the motion is given by the series

$$u(x,t) = \frac{2\,km^2}{(m-1)\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi/m) \sin(n\pi x/L) \cos(n\pi t/L) .$$

6. Applications in the Theory of Electrical Conduction. In the last section we discussed the classical problem of determining the vibrations of an elastic string which is subject to a given initial distortion. This problem is closely related to certain problems of electrical communication through a conducting cable which we shall now discuss. The situation with respect to conducting cables enjoys considerable simplification, however, due to the fact that the flow of electricity is set up by a known impressed voltage $V(t)$ which is zero for values of the time prior to the time, conveniently chosen at $t = 0$, when it is impressed upon the cable. This simplification of boundary conditions makes the Heaviside calculus an unusually effective tool for the development of this problem.

We shall begin by considering the case of the flow of electricity in a non-inductive cable with distributed resistance R and capacity

C per unit length subject to an impressed voltage $V_0(t)$ at the point $x = 0$.

The differential equations of the cable are*

$$RI = -\frac{\partial}{\partial x} V \ ,$$

$$C \frac{\partial}{\partial t} V = -\frac{\partial}{\partial x} I \ , \tag{6.1}$$

where x is the distance measured along the cable from a fixed point $(x = 0)$, I is the current at the point x, and V the corresponding potential.

Let us now replace $\partial/\partial t$ by p and through the elimination of I obtain the operational equation,

$$p\,R\,C\,V = \frac{\partial^2 V}{\partial x^2} \ .$$

Then symbolically we have

$$V(x,t) = e^{-ax} \to V_1(t) + e^{ax} \to V_2(t) \ , \tag{6.2}$$

where we abbreviate $a = (pRC)^{\frac{1}{2}}$ and $V_1(t)$ and $V_2(t)$ are arbitrary functions.

Since $I(x,t) = -\frac{\partial}{\partial x} V(x,t)$, we obtain from the differentiation of (6.2) the symbolic equation

$$I(x,t) = \sqrt{\frac{p\,c}{R}} \to [e^{-ax} \to V_1 - e^{ax} \to V_2] \ .$$

Assuming that the cable is infinitely long so that the reflected wave is absent we may set $V_2 \equiv 0$ and have as the symbolical solution of our problem

$$I = \sqrt{\frac{pC}{R}}\, e^{\,ax} \to V_1(t) \ ,$$

where $V_1(t)$ is to be determined from the boundary conditions. Since this asserts that $V(x,t)|_{x=0} = V_0(t)$, it is clear that we may set $V_1(t) \equiv V_0(t)$. Hence the complete solution of the problem is found in the expansion:

$$I(x,t) = \sqrt{\frac{pC}{R}}\, e^{-ax} \to V_0(t) \ ,$$

*For a discussion of the derivation and significance of these equations see J. H. Jeans: *The Mathematical Theory of Electricity and Magnetism*, Cambridge (1915) pp. 332-335. See also H. Bateman: *Partial Differential Equations of Mathematical Physics*, Cambridge (1932), pp. 73-76.

$$I(x,t) = \sqrt{\frac{C}{R}}\{p^{\frac{1}{2}} + \frac{(Ax)^2}{2!}p^{3/2} + \frac{(Ax)^4}{4!}p^{5/2} + \cdots\}$$

$$- \sqrt{\frac{C}{R}}\{p\,Ax + \frac{(Ax)^3}{3!}p^2 + \frac{(Ax)^5}{5!}p^3 + \cdots\}] \to V_0(t) \ ,$$

where for brevity we have written $A = \sqrt{RC}$. Therefore we get

$$I(x,t) = \sqrt{\frac{C}{R\pi}}\{\frac{d}{dt} + \frac{(Ax)^2}{2!}\frac{d^2}{dt^2}$$

$$+ \frac{(Ax)^4}{4!}\frac{d^3}{dt^3} + \cdots\} \to \int_0^t \frac{V_0(s)}{(t-s)^{\frac{1}{2}}}ds - \sqrt{\frac{C}{R}}\{(Ax)\frac{d}{dt}$$

$$+ \frac{(Ax)^3}{3!}\frac{d^2}{dt^2} + \frac{(Ax)^5}{5!}\frac{d^3}{dt^3} + \cdots\} \to V_0(t) \ .$$

If $V_0(t) = 1$, that is to say if a unit *e.m.f.* is impressed at $x = 0$, we get the well known solution*

$$I(x,t) = (C/R\pi t)^{\frac{1}{2}}\, e^{-RCx^2/4t} \ .$$

If $V_0(t) = \sin \omega t$, that is to say, if an alternating *e.m.f* is impressed on the circuit, we have at $x = 0$,

$$I(0,t) = (\frac{C}{R\pi})^{\frac{1}{2}}\frac{d}{dt}\int_0^t \frac{\sin \omega s}{(t-s)^{\frac{1}{2}}}ds \ ,$$

$$= (\frac{C}{R\pi})^{\frac{1}{2}}\,\omega\int_0^t \frac{\cos \omega s}{(t-s)^{\frac{1}{2}}}ds \ .$$

In establishing the convergence of the series representing $I(x,t)$ in the general case for real values of t greater than 0, it will be sufficient to assume that $V_0(t)$ is analytic in the neighborhood of $t = 0$. Let us first consider $I(x,t)$ as a function of x. If we replace $V_0(s)$ by a series $\sum_{n=0}^{\infty} a_n s^n$ and make the transformation $y = s/(t-s)$, we shall have

$$A(t) = \int_0^t [V_0(s)/(t-s)^{\frac{1}{2}}]\,ds = \int_0^\infty [\sum_{n=0}^{\infty} t^{n+\frac{1}{2}}\,a_n\,y^n/(1+y)^{n+\frac{3}{2}}]\,dy \ ,$$

$$= t^{\frac{1}{2}}\sum_{n=0}^{\infty}[\Gamma(n+1)\,\pi^{\frac{1}{2}}/\Gamma(n+3/2)]\,a_n t^n$$

$$= t^{\frac{1}{2}}\,\varphi(t) \ ,$$

where $\varphi(t)$ is an analytic function without singularities at the origin.

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*Jeans: *Loc. cit.*, p. 334.

Hence for values of $t = 0$, and for n sufficiently large, we have the inequality,

$$\left| \frac{d^n}{dt^n} \{ t^{\frac{1}{2}} \varphi(t) \} \right| < k^n n! \,, \tag{6.3}$$

where k is a constant independent of n. Making use of Stirling's approximation for $n!$, we obtain

$$\lim_{n=\infty} \left[\frac{|A x|^{2n-2}}{(2n-2)!} \left| \frac{d^n}{dt^n} \{ t^{\frac{1}{2}} \varphi(t) \} \right| \right]^{1/n} = 0 \,.$$

Similarly the nth root of the general term of the second series leads to the same limit and the convergence of $I(x,t)$ regarded as a function of x is established. Regarding it as a function of t, we may use

$$\sum_{n=1}^{\infty} |A x|^{2n-2} \, n! \, k^n / (2n+2)!$$

as a dominating series for the first series and

$$\sum_{n=1}^{\infty} |A x|^{2n-1} \, n! \, k^n / (2n-1)!$$

as a dominating series for the second.

The case of a cable with distributed resistance R and capacity C with distributed leakage G per unit length leads similarly to the differential equations:

$$R I = -\frac{\partial}{\partial x} V \,,$$

$$(C p + G) \rightarrow V = -\frac{\partial}{\partial x} I \,.$$

If the cable is infinitely long we shall then have

$$V = e^{-x[R(Cp+G)]^{1/2}} \rightarrow V_0(t) =$$

$$\{ 1 + \frac{x^2 (RB)}{2!} + \frac{x^4 (RB)^2}{4!} + \frac{x^6 (RB)^3}{6!} + \cdots \} \rightarrow V_0(t)$$

$$- \{ x (RB)^{\frac{1}{2}} + \frac{x^3 (RB)^{3/2}}{3!} + \frac{x^5 (RB)^{5/2}}{5!} + \cdots \} \rightarrow V_0(t) \,,$$

where we have used the abbreviation $B = Cp + G$.

The problem is thus reduced to that of determining the meaning to be attached to the operations,

$$(Cp + G)^{n+\frac{1}{2}} \rightarrow V_0(t) \text{ and } (Cp + G)^n \rightarrow V_0(t) \,.$$

But from formula (12.9) of chapter 2, these expressions may be interpreted as the following:

$$(Cp+G)^\mu \to V_0(t) = C^\mu (p+G/C)^\mu \to V_0(t)$$
$$= C^\mu e^{-Gt/C} p^\mu \to [e^{Gt/C} V_0(t)] \ ,$$

where μ is either $n+\frac{1}{2}$ or n. In the former case the interpretation of the operation is finally effected by means of the formulas of fractional differentiation.

Hence employing the abbreviation $\lambda = G/C$, we obtain the expansion,

$$V = e^{-\lambda t}\left[1+\frac{x^2(RC)}{2!}p+\frac{x^4(RC)^2}{4!}p^2+\cdots\right] \to \{e^{\lambda t} V_0(t)\}$$
$$-\frac{e^{-\lambda t}}{\sqrt{\pi}}\left[x(RC)^{\frac{1}{2}}p+\frac{x^3(RC)^{3/2}}{3!}p^2+\cdots\right] \to \int_0^t \frac{e^{-\lambda s} V_0(s)}{\sqrt{t-s}}ds \ . \tag{6.4}$$

The solution for the case where unit *e. m. f.* is impressed at $x = 0$ can be obtained with considerable ease by this formula. We first recall that:

$$\int_0^t \frac{e^{-\lambda s}}{\sqrt{t-s}}ds = \int_{-\infty}^t \frac{e^{\lambda s}}{\sqrt{t-s}}ds - \int_{-\infty}^0 \frac{e^{\lambda s}}{\sqrt{t-s}}ds \ ,$$
$$= \sqrt{\pi/\lambda}\, e^{\lambda t} - \int_{-\infty}^0 \frac{e^{\lambda s}}{\sqrt{t-s}}ds \ .$$

Substituting this value for the integral in (6.4) and taking the indicated derivatives, we get

$$V = e^{-x(RG)^{1/2}} + \int_{-\infty}^0 \left[\frac{1}{2}\frac{x(RC)^{\frac{1}{2}}}{(t-s)^{3/2}} - \frac{1\cdot 3}{2\cdot 2}\frac{x^3(RC)^{3/2}}{3!(t-s)^{5/2}}\right.$$
$$\left.+\frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}\frac{x^5(RC)^{5/2}}{5!(t-s)^{7/2}} - \cdots\right]\frac{e^{\lambda(s-t)}}{\sqrt{\pi}}dt \ . \tag{6.5}$$

Making the transformation $t - s = \xi$, we reduce (6.5) to

$$V = e^{-x(RG)^{1/2}} + \frac{\sqrt{RC}}{2\sqrt{\pi}}\int_t^\infty \frac{x}{\xi^{3/2}}\left[1-\frac{(RC)x^2}{4\xi}\right.$$
$$\left.+\frac{(RC)^2x^4}{2!(4\xi)^2}-\frac{(RC)^3x^6}{3!(4\xi)^3}+\cdots\right] e^{\lambda\xi}\, d\xi$$
$$= e^{-x(RG)^{1/2}} + \frac{\sqrt{RC}}{2\sqrt{\pi}}\int_t^\infty \frac{x}{\xi^{3/2}} e^{-(RCx^2/4\xi)}\, e^{-\lambda\xi}\, d\xi \ .$$

That this function, together with the value of $I(x,t)$ derived from it by differentiation, forms a solution of the original differen-

tial equations can be verified by direct substitution. The boundary condition is obviously satisfied.

A third problem of similar kind is that of determining the voltage at a cable terminal when an *e.m.f.*, $f(t)$, is impressed on a long cable of distributed resistance R and capacity C per unit length through a condenser of capacity C_0. We have just seen that the current entering a cable, the terminal voltage of which is V, is given by $(Cp/R)^{\frac{1}{2}} \rightarrow V$. The current flowing into the condenser will be $C_0 p \rightarrow [f(t) - V]$ since $f(t) - V$ is the voltage across the condenser.

Equating these two values and dividing by p we obtain the following equation for the determination of V:

$$[(C/R)^{\frac{1}{2}} p^{-\frac{1}{2}} + C_0] \rightarrow V(t) = C_0 f(t) \ .$$

The solution of this equation is easily found since it is identical with equation (6.17) of chapter 6. Properly specializing the solution of this equation we then obtain

$$V(t) = f(0)\, e^{Ct/RC_0^2}$$
$$+ \int_0^t [f'(s) - (C/RC_0^2)^{\frac{1}{2}}\, {}_0D_s^{\frac{1}{2}} f(s)]\, e^{C(t-s)/RC_0^2}\, ds \ .$$

If $f(t) = 1$, this reduces to

$$V(t) = e^{Ct/RC_0^2} - e^{Ct/RC_0^2} (C/RC_0^2\pi)^{\frac{1}{2}} \int_0^t [e^{-Cs/RC_0^2}/s^{\frac{1}{2}}]\, ds \ .$$

The general telegrapher's problem may be similarly treated, but the details will be left to the reader. If distributed inductance L is included, then the general equations become

$$(Lp + R) \rightarrow I = -\frac{\partial V}{\partial x} \ ,$$
$$(Cp + G) \rightarrow V = -\frac{\partial I}{\partial x} \ . \qquad (6.6)$$

The elimination of I from the system leads to the *equation of telegraphy*, which E. T. Whittaker suggests might properly be called *Heaviside's equation*:

$$\frac{\partial^2 V}{\partial x^2} = CL \frac{\partial^2 V}{\partial t^2} + (RC + GL) \frac{\partial V}{\partial t} + RG\, V \ . \qquad (6.7)$$

If we employ the transformation:

$$I = i\, e^{-Rt/L} \ , \ V = v\, e^{-Rt/L} \ ,$$

both the system (6.6) and the equation (6.7) take simpler forms.

$$Lp \rightarrow i = -\frac{\partial v}{\partial x} \ , \ [Cp + (G - RC/L)] \rightarrow v = -\frac{\partial i}{\partial x} \ ,$$

and

$$\frac{\partial^2 v}{\partial x^2} = CL \frac{\partial^2 v}{\partial t^2} + (GL - RC) \frac{\partial v}{\partial t} \ .$$

PROBLEMS

1. If a conducting cable is of finite length L, then the second term of the right hand member of equation (6.2) cannot be neglected. Hence assuming that at $x = 0$, $V = E$ and at $x = L$, $V = 0$, show that

$$V(x,t) = \frac{\sinh a(L - x)}{\sinh aL} \rightarrow E \ .$$

If E is constant, derive the expansion

$$V(x,t) = E(1 - x/L) - (2E/\pi) \sum_{m=1}^{\infty} (1/m) \sin(m\pi x/L)\, e^{-(m^2\pi^2 t/RCL^2)} \ .$$

2. If for a conducting cable of finite length L, we have the following terminal conditions:

$$(1) \text{ at } x = 0 \ , \ V = E \ ; \ (2) \text{ at } x = L \ , \ dV/dx = 0 \ ,$$

show that $V(x,t)$ is determined from the operational equation

$$V(x,t) = \frac{\cosh a(L - x)}{\cosh aL} \rightarrow E \ .$$

If E is a constant, derive the expansion

$$V(x,t) = E - (4E/\pi) \sum_{m=1}^{\infty} (1/m) \sin(m\pi x/2L)\, e^{-(m^2\pi^2 t/4RCL^2)} \ .$$

3. If in problem 2 we have the conditions

$$(1) \text{ at } x = 0 \ , \ V = E \cos \omega t \ ; \ (2) \text{ at } x = L \ , \ dV/dx = 0 \ ,$$

show that $V(x,t)$ is given by the expansion

$$V(x,t) = \text{Real part of } E\, e^{\omega i t} \frac{\cosh [\frac{1}{2}(RC\omega)^{\frac{1}{2}}(1+i)(L - x)]}{\cosh [\frac{1}{2}(RC\omega)^{\frac{1}{2}}(1+i)L]}$$

$$- 4E\pi^3 \sum_{m=1}^{\infty} \frac{m^3 \sin(m\pi x/2L)\, e^{-(m^2\pi^2 t/4RCL^2)}}{m^4\pi^4 + 16\, RCL^2\omega^2} \ .$$

4. Show that the voltage, V, at a cable terminal when an e. m. f., $f(t)$, is impressed on a long cable of distributive resistance R and capacity C per unit length through a terminal resistance R_0, is given by the equation

$$[1 + R_0(C/R)^{\frac{1}{2}} p^{\frac{1}{2}}] \rightarrow V(t) = f(t) \ .$$

Show that the solution for $f(t) = E$ (a constant) may be put into either of the following forms:

$$V(t) = E - E R_0 (C/R\pi t)^{\frac{1}{2}} \{1 - (R_0^2 C/2Rt) + 1 \cdot 3 (R_0^2 C/2Rt)^2 - \cdots\}$$

$$= E [1 - \exp(Rt/R_0^2 C)]$$

$$+ 2E (Rt/R_0^2 \pi C) \{1 + (2Rt/R_0^2 C)/3 + (2Rt/R_0^2 C)/3 \; 5 + \cdots\} .$$

5. Let an impedance z_1 be connected at the transmitting end, namely, with an e. m. f., $f(t)$, of a cable of finite length L with distributive resistance R and capacity C, and let an impedance z_2 be connected at the receiving end. Employing the abbreviation $a = (RCp)^{\frac{1}{2}}$, show that the voltage in the cable is determined from the following equation:

$$\frac{(R^2 + z_1 z_2 a^2) \sinh aL + (aR)(z_1 + z_2) \cosh aL}{aR \sinh a(L - x) + a^2 z_2 \cosh a(L - x)} \rightarrow V(t) = f(t) .$$

6. Show that when $z_2 = 0$, $z_2 = R_0$, the current at $x = L$, under the conditions of problem 5 and for $f(t) = E$ (a constant), is given by

$$I_L = E/(RL + R_0) + 2E \sum \frac{\lambda_n \cos \lambda_n \exp(-\lambda_n^2 t/RCL^2)}{RL(\lambda_n - \frac{1}{2} \sin 2\lambda_n)}$$

where the summation is over the roots of the equation

$$\tan \lambda = -(R_0/RL)\lambda .$$

7. If the initial temperature of an infinite solid is given as a function, $f(x)$, of the distance from some fixed origin at an initial time $t = 0$, and if the heat flows in one direction only, then the temperature at any subsequent time is given by

$$\frac{\partial T}{\partial t} = a^2 \frac{\partial^2 T}{\partial x^2} ,$$

where a is an experimentally determined constant and $T(x, t)$ is subject to the boundary condition: $T(x, 0) = f(x)$.

Show that the solution is given by

$$T(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2a\sqrt{t}\, s)\, e^{-s^2} ds .$$

Hint: Note that a formal solution is given by

$$T(x, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} z^{2n} \rightarrow f(x) .$$

Now employ the integral

$$\int_0^{\infty} e^{-r^2} r^{2n} dr = \frac{1}{2} \int_0^{\infty} e^{-s} s^{n-\frac{1}{2}} ds = \frac{1}{2} \Gamma(n + \frac{1}{2}) .$$

8. If a sphere of radius R cools in air, the temperature, $T(r, t)$, where r is the distance from the center and t is the time, is subject to the equation

$$\frac{\partial (Tr)}{\partial t} = a^2 \frac{\partial^2 (Tt)}{\partial r^2} .$$

If the initial distribution of temperature is given as a function $f(r)$ of the distance from the center of the sphere, and if the surface of the sphere is initially kept at a constant temperature T_0, then the boundary conditions are: $T(r,0) = f(r)$, $T(R,t) = T_0$, where R is the radius of the sphere.

Show that

$$T(r,t) = \frac{2}{rR}\sum_{n=1}^{\infty} \exp(-n^2 a^2 \pi^2 t/R^2) \sin(n\pi r/R) \int_0^R s f(s) \sin(n\pi s/R)\, ds$$

$$+ T_0 \{1 + \frac{2R}{r\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \exp(-n^2 a^2 \pi^2 t/R^2) \sin(n\pi r/R)\} .$$

9. If the sphere of the preceding problem is placed in air at a constant temperature of zero into which it may radiate, the boundary conditions become

$$T(r,0) = f(r) ,$$

$$\frac{\partial}{\partial r} T(R,t) + h\,T(R,t) = 0 ,$$

where h is a constant which depends upon the conductivity of the surface of the sphere.

Derive the operational equivalent of this problem.

10. From the operator of the preceding problem, show that the solution may be expressed in the form

$$T(r,t) = (1/r) \sum_{n=1} c_n \exp(-a^2 \lambda_n^2 t) \sin \lambda_n r ,$$

where we abbreviate

$$c_n = \frac{2}{R} \frac{\lambda_n^2 R^2 + (hR-1)^2}{\lambda_n^2 R^2 + hR(hR-1)} \int_0^R s f(x) \sin \lambda_n s\, ds ,$$

in which λ_n is a root of the equation

$$\lambda R \cos \lambda R + (hR-1) \sin \lambda R = 0 .$$

7. *Infinite Systems of Equations with Constant Coefficients.* In the discussions which have preceded this section, attention was devoted mainly to the formal machinery of inverting systems of equations with constant coefficients. We shall now consider existence theorems for such systems in the general case where the differential equations are of infinite order and the variables infinite in number. The principal paper on this subject is due to I. M. Sheffer [See *Bibliography*: Sheffer (4)]. His results, however, are an almost immediate corollary of the Cauchy-Bromwich method described in section 3, chapter 6 upon which one imposes properly specialized conditions derived from the existence theorem for infinite determinants, that is, from theorem 1, section 3, chapter 3.

Let us assume a system of the form

$$\sum_{i=-\infty}^{\infty} A_{ij}(p) \to v_i(t) = g_j(t) , \quad p = d/dt, \ (j = -\infty, \cdots, \infty) \tag{7.1}$$

where we abbreviate

$$A_{ij}(p) = \sum_{k=-\infty}^{\infty} a_{ij}^{(k)} p^k \ . \tag{7.2}$$

We shall make the following assumptions:

(a) All the functions $A_{ij}(p)$ share a common annulus of convergence: $R < |p| < R'$.

(b) The elements of the determinant $\Delta(p) = |A_{ij}(p)|$ satisfy the conditions of theorem 1, section 3, chapter 3 for all values of p in the annulus of convergence.

(c) $\Delta(t)$ has only a finite number of zeros in $R < |p| < R'$.

(d) The functions $g_i(t)$ have the expansion

$$g_i(t) = \sum_{m=0}^{\infty} g_{im}\, t^m/m! \ ,$$

where $\sup\lim_{m=\infty} |g_{im}|^{1/m} \leqq r < R'$.

We consider first the homogeneous case of system (7.1), that is, the system

$$\sum_{i=-\infty}^{\infty} A_{ij}(p) \to v_i(t) = 0\ , \quad (j = -\infty, \cdots, \infty)\ , \tag{7.3}$$

for which we prove

Theorem 6. If functions $v_i(t)$ are defined by the Cauchy integral

$$v_i(t) = \frac{1}{2\pi i} \int_C \frac{e^{xt}}{t\,\Delta(t)} \Delta_i(t)\, dt \ , \tag{7.4}$$

where we define $\Delta(t) = |A_{ij}(t)|$ and $\Delta_i(t)$ is $\Delta(t)$ with the ith *column replaced by $P_1(t)$, $P_2(t)$, $\cdots$, where $P_1(t)$, $P_2(t)$, $\cdots$ are arbitrary polynomials which vanish at $t = 0$ and are so chosen that $\Delta_i(t)$ converges within the annulus of convergence, and if C is a path which lies within the annulus of convergence and includes the point $t = \lambda$, where $\Delta(\lambda) = 0$, in its interior, then the functions $v_i(t)$ furnish a solution for* (7.3).

Proof: From the operation

$$\sum_{i=-\infty}^{\infty} A_{ij}(p) \to v_i(t) = \frac{1}{2\pi i} \int_C [P_j(t)\ e^{xt}/t]\, dt$$

we obtain formally

$$\sum_{i=-\infty}^{\infty} A_{ij}(p) \to v_i(t) = 0 \ ,$$

since the right hand member is zero from the fact that $P_i(t)/t$ has no singularity on or within C.

The process of operation under the integral sign is justified by the corollary to theorem 1, section 3, chapter 3. Since the polynomials are bounded within the annulus of convergence, the expansion of $\Delta_i(t)$ with respect to the ith column would converge uniformly within the annulus and hence, in particular, upon the path C.

Considering now the non-homogeneous equation (7.1), we shall prove

Theorem 7. If the functions $v_i(t)$ are defined by

$$v_i(t) = \frac{1}{2\pi i}\int_c \frac{e^{xt}}{t\,\Delta(t)}\,\Delta_i(t)\,dt \ , \tag{7.5}$$

where $\Delta_i(t)$ is the determinant $\Delta(t)$ with the ith column replaced by

$$G_i(t) = \sum_{m=0}^{\infty} g_{im}/t^m \ ,$$

and if C is a path in the annulus of convergence which contains no zero of $\Delta(t)$ upon it and whose modulus exceeds r [see condition (d) above], then the functions $v_i(t)$ furnish a solution for equation (7.1).

Proof: Operating upon (7.5) with $A_{ij}(p)$ and summing with respect to i, we get formally

$$\begin{aligned}\sum_{i=-\infty}^{\infty} A_{ij}(p) \to v_i(t) &= \frac{1}{2\pi i}\int_c e^{xt}\sum_{i=-\infty}^{\infty}\frac{A_{ij}(t)\Delta_i(t)}{t\,\Delta(t)}\,dt \\ &= \frac{1}{2\pi i}\int_c [e^{xt}\,G_j(t)/t]\,dt \\ &= g_i(t) \ .\end{aligned}$$

The justification for this formal summation under the sign of integration is derived immediately from the corollary to theorem 1, section 3, chapter 3. We first observe, from condition (d) above, that $|G(t)|$ is dominated by the function

$$M\sum_{m=0}^{\infty} r^m/|t|^m = M[1/(1-r/|t|)], \quad t > r \ .$$

Hence $G(t)$ converges uniformly along C and thus by the corollary to theorem 1, section 3, chapter 3, the expansion of $\Delta(t)$ according to the ith column will converge uniformly along C. This is sufficient to justify the formal processes which we have employed.

Corollary. The solutions obtained in theorem 7 will be of grade less than R'.

This may be derived from theorem 6, chapter 5.

CHAPTER VIII

The Laplace Differential Equation of Infinite Order

1. *Introduction.* The object of the present study is the Laplace differential equation of infinite order

$$\sum_{n=0}^{\infty}(a_{n0}+a_{n1}x+a_{n2}x^2+\cdots+a_{np}x^p)u^{(n)}(x)=f(x) \qquad (1.1)$$

where p is a positive integer and not all the quantities a_{np} are zero. If we employ the abbreviations $z=d/dx$ and

$$a_n(x)=a_{n0}+a_{n1}x+a_{n2}x^2+\cdots+a_{np}x^p\ , \qquad (1.2)$$

we can write (1.1) in the abbreviated form,

$$[a_0(x)+a_1(x)z+a_2(x)z^2+\cdots$$
$$+a_n(x)z^n+\cdots]\to u(x)=f(x)\ . \qquad (1.3)$$

The history and the present status of the theory of the solution of the Laplace differential equation of infinite order have already been summarized in section 6 of chapter 1. We shall attempt in this chapter to give the details of the formal solution of both the homogeneous and the non-homogeneous cases of this equation and present in so far as possible existence theorems which apply to a broad class of functions.

It will be observed immediately that the general theory of equation (1.1) formally unifies the theories of the following essentially different types of linear functional equations in which the functions $p_i(x)$ are polynomials of degree not greater than p:

(a) $\qquad u(x)+\int_x^{\infty}\sum_{i=1}^{\infty}p_i(x)\varphi_i(t-x)u(t)\,dt=f(x)\ ,$

in which we assume that the $\varphi_i(x)$ approach infinity in such a manner that $\int_0^{\infty}\varphi_i(s)s^n ds$ exists for all values of n;

(b) $\qquad u(x)+\int_a^b\sum_{i=1}^{m}p_i(x)q_i(t)u(x+ct)\,dt=f(x)\ ;$

(c) $\qquad p_m(x)u(x+m)+p_{m-1}(x)u(x+m-1)$
$$+\cdots+p_0(x)u(x)=f(x)\ ;$$

(d) *The Laplace differential equation of finite order.*

The formal equivalence of these types with equation (1.1) is exhibited by means of the Taylor's expansion,

$$u(T)=u(x)+(T-x)u'(x)+(T-x)^2u''(x)/2!+\cdots\ .$$

If we replace T by t and substitute in (a) we obtain an equation of the form:

$$P_0(x)u(x) + P_1(x)u'(x) + \cdots + P_n(x)u^{(n)}(x) + \cdots = f(x) \ , \tag{1.4}$$

where the coefficients are,

$$P_0(x) = 1 + \sum_{i=1}^{m} p_i(x) \int_0^{\infty} q_i(s)\,ds,$$

$$P_n(x) = \sum_{i=1}^{m} p_i(x) \int_0^{\infty} q_i(s) s^n ds/n! \qquad n > 0 \ .$$

Similarly for (b) we let $T = x + ct$ and obtain equation (1.4) with the coefficients,

$$P_0(x) = 1 + \int_a^b \sum_{i=1}^{m} p_i(x) q_i(t)\,dt \ ,$$

$$P_n(x) = \int_a^b \sum_{i=1}^{m} p_i(x) q_i(t) (ct)^n dt/n! \ , \qquad n > 0 \ .$$

Equation (c) is transformed into the desired type by replacing T by $x + r$, $r = 0, 1, 2, \ , \cdots, m$. We thus obtain the coefficients,

$$P_0(x) = \sum_{i=1}^{m} p_i(x) \ , \quad P_n(x) = \sum_{i=1}^{m} i^n p_i(x)/n! \ , \quad n > 0 \ .$$

The Laplace equation of finite order is derived from (1.1) merely by assuming that $a_{ni} = 0$ for all values of n greater than a fixed n'.

It should be noted, incidentally, that the case for which $p = 0$ has already been extensively treated in chapter 6.

2. *Calculation of the Resolvent Generatrix for the Laplace Equation.* We shall first consider the formal aspects of the inversion of equation (1.1). It will be convenient to employ the abbreviation $F(x,z)$, which we define as the operational series

$$F(x,z) = \sum_{n=0}^{\infty} (a_{n0} + a_{n1}x + a_{n2}x^2 + \cdots + a_{np}x^p) z^n \ .$$

It is clear from section 10 of chapter 4 that the formal solution of equation (1.1) depends upon the determination of a *resolvent generatrix*, $X(x,z)$, which will satisfy the equation,

$$X(x,z) \rightarrow F(x,z) = 1 \ ,$$

or more explicitly,

$$F(x,z)X(x,z) + (\partial F/\partial x)(\partial X/\partial z) + (\partial^2 F/\partial x^2)(\partial^2 X/\partial z^2)/2! + \cdots + (\partial^n F/\partial x^n)(\partial^n X/\partial z^n)/n! + \cdots = 1 \ . \tag{2.1}$$

If we make the abbreviation,

$$A_r(x,z) = \sum_{n=0}^{\infty} \sum_{m=r}^{\infty} a_{nm} m! x^{m-r} z^n / (m-r)! r! ,$$

we can write (2.1) in the form

$$A_0(x,z) X(x,z) + A_1(x,z) \partial X/\partial z + A_2(x,z) \partial^2 X/\partial z^2$$
$$+ \cdots + A_p(x,z) \partial^p X/\partial z^p = 1 . \qquad (2.2)$$

In order to integrate (2.2) let us assume that $X(x,z)$ is a function of the form $X(x,z) = e^{-xz} X(z)$. Taking successive derivatives with respect to z we obtain,

$$\partial^n X(x,z)/\partial x^n = e^{-xz}[X^{(n)}(z) - nx\, X^{(n-1)}(z)$$
$$+ n(n-1) x^2 X^{(n-2)}(x)/2! - \cdots + (-1)^n x^n X(z)] ,$$
$$n = 1, 2, 3, \cdots .$$

When these values are substituted in (2.2) an interesting simplification takes place and the generatrix equation reduces to

$$\{A_0(z) X(z) + A_1(z) X'(z) + A_2(z) X''(z)$$
$$+ \cdots + A_p(z) X^{(p)}(z)\} e^{-xz} = 1 , \qquad (2.3)$$

where the $A_i(z)$ are the functions,

$$A_i(z) = \sum_{n=0}^{\infty} a_{ni} z^n .$$

We shall make the assumption at this point that a_{0p} is different from zero. It will be noted that this is a restrictive condition, which will be discussed in greater detail in section 15. The significance of this restriction here is found in the fact that the homogeneous equation,

$$A_0(z) X(z) + A_1(z) X'(z) + A_2(z) X''(z)$$
$$+ \cdots + A_p(z) X^{(p)}(z) = 0 , \qquad (2.4)$$

will have $z = 0$ as a regular point provided $a_{0p} \neq 0$. There will then exist p linearly independent solutions, *regular at the origin,* which we shall designate by $X_1(z), X_2(z), X_3(z), \cdots, X_p(z)$.

Similarly the adjoint of equation (2.4), which we shall call the *adjoint resolvent,*

$$A_0(z) Y(z) - \frac{d}{dz}[A_1(z) Y(z)] + \cdots$$
$$+ (-1)^p \frac{d^p}{dz^p}[A_p(z) Y(z)] = 0 , \qquad (2.5)$$

will have a set of p linearly independent solutions, $Y_1(z)$, $Y_2(z)$, $Y_3(z)$, $\cdots$, $Y_p(z)$, regular at the origin. As is well known,* these solutions can be calculated in terms of $X_i(z)$ by means of the formula,

$$Y_i(z) = (\partial \log W/\partial X_i^{(p-1)})/A_p(z) \ , \qquad (2.6)$$

where W is the Wronskian,

$$W = \begin{vmatrix} X_1 & X_1' & \cdots & X_1^{(p-1)} \\ X_2 & X_2' & \cdots & X_2^{(p-1)} \\ \cdot & \cdot & \cdot & \cdot \\ X_p & X_p' & \cdots & X_p^{(p-1)} \end{vmatrix} .$$

The solution of equation (2.3) may be written in terms of the sets, $\{X_i(z)\}$, $\{Y_i(z)\}$, and we thus obtain as the formal value of the operator $X(x,z)$ the function,

$$X(x,z) = e^{-xz}[C_1(x)X_1(z) + C_2(x)X_2(z) + \cdots + C_p(x)X_p(z)] + \int_0^z e^{xt}W(z,t)\,dt \ , \qquad (2.7)$$

where we make use of the abbreviation,

$$W(z,t) = X_1(z)Y_1(t) + X_2(z)Y_2(t) + \cdots + X_p(z)Y_p(t) \ .$$

The functions $C_i(x)$ thus far enter the solution arbitrarily, but we shall see that they are uniquely defined.

It is well known that $W(z,t)$ has the following properties:†

$$\begin{aligned} &\partial^{i+j}W(z,t)/\partial z^i \partial t^j|_{t=z} = 0 \ , && i+j < p-1 \ , \\ &\partial^{i+j}W(z,t)/\partial z^i \partial t^j|_{t=z} = (-1)^j/A_p(z) && i+j = p-1 \ . \end{aligned} \qquad (2.8)$$

From these relations it is clear that we shall have,

$$\begin{aligned} \partial^r X(x,z)/\partial z^r|_{z=0} &= (\partial^r/\partial z^r)\{e^{-xz}[C_1(x)X_1(z) \\ &\qquad + \cdots + C_p(x)X_p(z)]\}_{z=0} \\ &= \left(\frac{d}{dz} - x\right)^r \to \{C_1(x)X_1(z) \\ &\qquad + \cdots + C_p(x)X_p(z)]\}_{z=0} \end{aligned}$$

for r less than or equal to $p - 1$.

If this last equation be expanded and the coefficients of x compared with the coefficients of x in equation (12.5) of chapter 4 we obtain the following system of equations:

*For this and other relations between the adjoint functions see G. Darboux: *La théorie des surfaces*. Paris (1889), part 2, book 4, pp. 99-106. See also section 2, chapter 11 of this book.

†See Darboux: *loc. cit.*, p. 103. See also section 2, chapter 11 of this book.

$$C_1(x)X_1(0) + \cdots + C_p(x)X_p(0) = u_0(x) \ ,$$
$$C_1(x)X_1'(0) + \cdots + C_p(x)X_p'(0) = u_1(x) \ ,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \ ,$$
$$C_1(x)X_1^{(p-1)}(0) + \cdots + C_p(x)X_p^{(p-1)}(0) = u_{p-1}(x) \ ,$$

where $u_i(x)$ is the solution of (1.1) when $f(x) = x^i$.

If we further specialize the functions $X_i(z)$ so as to satisfy the conditions,

$$X_i^{(j-1)}(0) = \delta_{ij} \ , \tag{2.9}$$

where $\delta_{ij} = 1$, $\delta_{ij} = 0$, $i \neq j$, the arbitrary functions are identified with the special solutions $u_i(x)$, namely $C_i(x) = u_{i-1}(x)$.

We may formulate these conclusions in the following theorem:

Theorem 1. The resolvent generatrix $X(x,z)$ of equation (1.1) *is expansible in the form,*

$$X(x,z) = e^{-xz}[\sum_{i=1}^{p} u_{i-1}(x)X_i(z) + \int_0^z e^{xt}W(z,t)\,dt] \ , \tag{2.10}$$

where the $u_i(x)$, $i = 0, 1, 2, \cdots, p-1$, are solutions of equation (1.1) *when $f(x) = x^i$, $X_i(z)$ are solutions of equation* (2.4) *subject to the defining conditions, $X_i^{(j-1)}(0) = \delta_{ij}$, where $\delta_{ij} = 1$, $\delta_{ij} = 0$, $i \neq j$, and $W(z,t)$ is defined by* (2.8).

We proceed next to an explicit determination of the solutions $u_i(x)$. The result that we shall prove is contained in the following theorem:

Theorem 2. Let us designate by $H_j^{(i)}(t)$ and $I_k^{(i)}(t)$ the following functions:

$$H_j^{(i)}(t) = A_j(t)Y_i(t) - [A_{j+1}(t)Y_i(t)]'$$
$$+ \cdots + (-1)^{p-j}[A_p(t)Y_i(t)]^{(p-j)} \ , \tag{2.11}$$

$$I_k^{(i)}(t) = \int_0^t [(t-s)^{k-1}A_0(s)Y_i(s)/(k-1)!$$
$$- (t-s)^{k-2}A_1(s)Y_i(s)/(k-2)! + \cdots + (-1)^k A_{k-1}(s)Y_i(s)]\,ds \ , \tag{2.12}$$

where the $Y_i(t)$ are the adjoints of $X_i(t)$ as defined by (2.6) *and* (2.9).

Then the solutions $u_i(x)$ of theorem 1 *are explicitly determined from the formula,*

$$u_{i-1}(x) = U(x) - \int_{l_i} e^{xt}Y_i(t)\,dt \ , \tag{2.13}$$

where $U(x)$ is a solution of the homogeneous equation and l_i is a path of integration between zero and any other point in the complex plane for which $H_j^{(i)}(t)e^{xt}=0$, $j=i, i+1, \cdots, p$, and $I_k^{(i)}(t)e^{xt}=0$, $k=1, 2, \cdots, i-1$.

Proof: We first observe that our definition of the solutions $X_i(t)$ at $t=0$, condition (2.9), has uniquely determined the values of the adjoints $Y_i(t)$ at the same point. Thus setting $t=z$ in equations (2.8) we have the bilinear relations,

$$X_1(z)Y_1(z)+X_2(z)Y_2(z)+\cdots+X_p(z)Y_p(z)=0\ ,$$

$$X_1'(z)Y_1(z)+X_2'(z)Y_2(z)+\cdots+X_p'(z)Y_p(z)=0\ ,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$X_1^{(p-1)}(z)Y_1(z)+X_2^{(p-1)}(z)Y_2(z)+\cdots+X_p^{(p-1)}(z)Y_p(z)=1/A_p(z).$$

Setting $z=0$ and noting the initial values of $X_i(z)$ we obtain, $Y_i(0)=0$, $i=1, 2, \cdots, p-1$, $Y_p(0)=1/A_p(0)$. Similarly letting $j=1$ in equations (2.8) we get $Y_i'(0)=0$, $i=1, 2, \cdots, p-2$, $Y'_{p-1}(0)=-1/A_p(0)$; and in general,

$$Y_i^{(q)}(0)=0\ ,\quad i=1,2,\cdots,p-q-1\ ,$$
$$Y^{(q)}{}_{p-q}(0)=(-1)^q/A_p(0)\ . \qquad (2.14)$$

From these values and those obtained by successive differentiation of equations (2.8) we obtain the following numerical equations:

$$H_j^{(i)}(0)=A_j(0)Y_i(0)-[A_{j+1}(0)Y_i(0)]'$$
$$+\cdots+(-1)^{p-j}[A_p(0)Y_i(0)]^{(p-j)}=0\ ,$$
$$j=0,1,2,\cdots,i-1,i+1,\cdots,p\ ;H_i^{(i)}(0)=1\ . \qquad (2.15)$$

These equations can be used to compute the elements of the Y-matrix, $\|Y_i^{(j-1)}(0)\|$. For example, if $p=2$, we have

$$\|Y_i^{(j-1)}(0)\|=\begin{Vmatrix} Y_1(0) & Y_2(0) \\ Y_1'(0) & Y_2'(0) \end{Vmatrix} \qquad (2.16)$$
$$=\begin{Vmatrix} 0 & 1/A_2(0) \\ -1/A_2(0), & [A_1(0)-A_2'(0)]/A_2^2(0) \end{Vmatrix}.$$

Assuming proper convergence of the integral in (2.13), let us differentiate $v_{i-1}(x)=u_{i-1}(x)-U(x)$, n times. We thus obtain,

$$v_{i-1}^{(n)}(x)=-\int_{l_i} e^{xt}t^nY_i(t)\,dt\ .$$

Substituting this function in the expression $F(x,z) \to u(x)$ we get,

$$F(x,z) \to v_{i-1}(x) \tag{2.17}$$

$$= -\int_{l_i} e^{xt} Y_i(t)\{A_0(t) + A_1(t)x + \cdots + A_p(t)x^p\}\, dt \ .$$

Let us now define a new function,

$$U_k(A) = \int_0^t Y_i(s)A(s)(t-s)^{k-1}ds/(k-1)! \ .$$

Recalling the identity [see equation (6.4), chapter 2],

$$\int_0^t \cdots \int_0^s y(s)\,ds^n = \int_0^t y(s)(t-s)^{n-1}ds/(n-1)! \ ,$$

we note that

$$d^r U_k(A)/dt^r = U_{k-r}(A) \ , \quad k \geqq r \ .$$

Making use of this relationship and integrating by parts, we establish the identity

$$\int_{l_i} Y_i(t)A(t)e^{xt}dt = \{U_1(A) - xU_2(A) + \cdots + (-1)^{k-1}x^k U_k(A)\}\, e^{xt}|_{l_i} + (-1)^k x^k \int_{l_i} e^{xt} U_k(A)\,dt \ .$$

Employing this identity and recalling the definition of $I_k(t)$, we obtain the following expansion:

$$\begin{aligned}
&\int_{l_i} e^{xt} Y_i(t)[A_0(t) + A_1(t)x + \cdots + A_{i-2}(t)x^{i-2}]\, dt \\
&\quad = \{U_1(A_0) - x[U_2(A_0) - U_1(A_1)] + x^2[U_3(A_0) - U_2(A_1) \\
&\quad + U_1(A_2)] - \cdots + (-1)^{i-2}x^{i-2}[U_{i-1}(A_0) - U_{i-2}(A_1) \\
&\quad + \cdots \pm U_1(A_{i-2})]\}e^{xt}|_{l_i} + (-1)^{i-1}x^{i-1}\int_{l_i} e^{xt}\{U_{i-1}(A_0) \\
&\quad - U_{i-2}(A_1) + \cdots \pm U_1(A_{i-2})\}dt \\
&= I_1^{(i)}(t) - xI_2^{(i)}(t) + x^2 I_3^{(i)}(t) - \cdots + (-1)^{i-2}x^{i-2}I_{i-1}^{(i)}(t)\}e^{xt}|_{l_i} \\
&\quad + (-1)^{i-1}x^{i-1}\int_{l_i} e^{xt} I_{i-1}^{(i)}(t)\,dt \ .
\end{aligned} \tag{2.18}$$

A different integration by parts results in the following identity:

$$\begin{aligned}
\int_{l_i} Y_i(t)A(t)e^{xt}dt = \{&Y_iA/x - (Y_iA)'/x^2 + (Y_iA)''/x^3 - \cdots \\
&+ (-1)^{k-1}(Y_1A)^{(k-1)}/x^k\}e^{xt}|_{l_i} \\
&+ [(-1)^k/x^k] \int_{l_i} e^{xt}(Y_iA)^{(k)}dt \ .
\end{aligned} \tag{2.19}$$

Making use of this expansion and recalling the definition (2.11) of $H_j^{(i)}(t)$ we obtain the following:

$$\int_{l_i} e^{xt} Y_i(t)\,[A_{i-1}(t)x^{i-1} + A_i(t)x^i + \cdots + A_p(t)x^p]\,dt$$
$$= \{x^{i-1}H_i^{(i)}(t) + x^i H_{i+1}^{(i)}(t) + \cdots + x^{p-1}H_p^{(i)}(t)\}\, e^{xt}\big|_{l_i} + x^{i-1}\int_{l_i} e^{xt} H_{i-1}^{(i)}(t)\,dt\ . \tag{2.20}$$

Combining expansions (2.18) and (2.20) we get the equation:

$$\int_{l_i} e^{xt} Y_i(t)\,\{A_0(t) + A_1(t)x + \cdots + A_p(t)x^p\}\,dt$$
$$= \{\sum_{k=1}^{i-1} (-1)^{k-1} I_k^{(i)} x^{k-1} + \sum_{j=1}^{p} H_j^{(i)}(t)x^{j-1}\}\, e^{xt}\big|_{l_i} \tag{2.21}$$
$$+ x^{i-1}\int_{l_i} e^{xt}\{H_{i-1}^{(i)}(t) + (-1)^{i-1} I_{i-1}^{(i)}(t)\}\,dt\,.$$

We must next establish the identity,

$$I_{i-1}^{(i)}(t) + (-1)^{i-1} H_{i-1}^{(i)}(t) = \int_0^t (t-s)^{i-2} H_0^{(i)}(s)\,ds/(i-2)!\ . \tag{2.22}$$

To prove this consider the integral,

$$\int_0^t H_0^{(i)}(s)\,ds = I_1^{(i)}(t) - H_1^{(i)}(t)\ .$$

From equation (2.15) we see that $H_1^{(2)}(0) = 0$ and the identity is established for $i = 2$. Similarly we have

$$\int_0^t (t-s) H_0^{(i)}(s)\,ds = I_2^{(i)}(t) + H_2^{(i)}(t)\big|_0^t\ .$$

From this integration and the observation that $H_2^{(3)} = 0$ the identity is established for $i = 3$. The extension to the general case follows in an identical manner.

Returning now to equation (2.21) and recalling (2.22) we see that the integral of the right hand member vanishes identically since $H_0^{(i)}(s)$ is identically zero. Furthermore we have $I_k(0) = 0$ for every k and from (2.15) $H_i^{(i)}(0) = 1$, $H_j^{(i)}(0) = 0$, $j \neq i$. Hence making the assumptions of theorem 2 that there exists a path l_i from 0 to some point a such that $\lim_{t=a} I_k^{(i)}(t)e^{xt} = 0$, $k = 1, 2, \cdots, i-1$, $\lim_{t=a} H_j^{(i)}(t)e^{xt} = 0$, $j = i, i+1, \cdots, p$, then it is clear that $u_{i-1}(x)$ defined by (2.13) is a solution of equation (1.1) in which $f(x) = x^{i-1}$. The negative sign is chosen for the integral of (2.13) since we have arbitrarily assumed that zero is the lower limit of the integration.

Corollary: If the conditions of theorem 1 *are satisfied and if there exists a unique limit a for all the paths l_i satisfying the condi-*

tions of theorem 2, *then the resolvent generatrix of equation* (1.1) *takes the form,*

$$X(x,z) = e^{-xz}U(x)\sum_{i=1}^{p} X_i(z) + e^{-xz}\int_a^z \sum_{i=1}^{p} X_i(z)\,Y_i(t)\,e^{xt}dt \ . \qquad (2.23)$$

If, moreover, $a = -\infty$, *the second term in* (2.23) *can be written*

$$X_0(x,z) = e^{-xz}\int_{-\infty}^{z} W(z,t)\,e^{xt}dt$$

$$= \int_0^{\infty} W(z,z-t)\,e^{-xt}dt \ , \qquad (2.24)$$

where we abbreviate as before

$$W(z,t) = X_1(z)\,Y_1(t) + X_2(z)\,Y_2(t) + \cdots + X_p(z)\,Y_p(t) \ .$$

3. The Homogeneous Equation. In theorem 2 of the preceding section the functions $u_i(x)$ which satisfy equation (1.1) for $f(x) = x^i$ were determined to within a solution of the homogeneous equation,

$$F(x,z) \equiv \{A_0(z) + A_1(z)x + A_2(z)x^2 + \cdots + A_p(z)x^p\} \rightarrow u(x) = 0 \ . \qquad (3.1)$$

It is the purpose of this section to show the efficacy of the Laplace transformation in the solution of this equation. The present status of the problem may be briefly summarized as follows. The classical case where all the $A_i(z)$ are polynomials was first systematically treated by H. Poincaré in 1885 and 1886.* Numerous memoirs have since contributed to the extension of these original ideas.† The case where the $A_i(z)$ are polynomials in e^z is that of difference equations and this has been the subject of extensive study by H. Galbrun, J. Horn, R. D. Carmichael, G. Birkhoff, N. E. Nörlund, and others.‡

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*Sur les équations linéaires aux différentielles ordinaires et aux différences finies. *American Journal of Mathematics*, vol. 7 (1885), pp. 203-258; Sur les intégrales irrégulières des équations linéaires. *Acta Mathematica*, vol. 8 (1886), pp. 295-344.

†See L. Schlesinger: *Handbuch der Theorie der linearen Differentialgleichungen*, vol. 1 (1895), pp. 409-414; E. L. Ince: *Ordinary Differential Equations*, London, (1927), chapter 18.

‡Galbrun: Sur la représentation des solutions d'une équation linéaire aux différences finies pour les grandes valeurs de la variable. *Acta Mathematica*, vol. 36 (1913), pp. 1-16; *Comptes Rendus*, vol. 148 (1909), p. 905; vol. 149 (1909), p. 1046; vol. 150 (1910), p. 206; vol. 151 (1910), p. 1114.

Horn: Integration linearer Differentialgleichungen durch Laplacesche Integrale und Fakultätenreihen. *Jahresbericht der Deutschen Math.-Vereinigung*, vol. 24 (1915), pp. 309-329; Laplacesche Integrale als Lösungen von Funktionalgleichungen. *Journal fur Mathematik*, vol. 146 (1916), pp. 95-115.

Carmichael: Linear Difference Equations and their Analytic Solutions. *Trans. Amer. Math. Soc.*, vol. 12 (1911), pp. 99-134.

Birkhoff: General Theory of Linear Difference Equations. *Trans. Amer. Math Soc.*, vol. 12 (1911), pp. 243-284.

Nörlund: See *Bibliography*.

A comprehensive discussion of this problem together with an inclusive bibliography is given in the well known treatise of N. E. Nörlund: *Vorlesungen über Differenzenrechnung,* in particular chapter 11.

The first example known to the writer from the point of view of the present chapter was given in 1908 by T. Lalesco [see *Bibliography*: Lalesco (1)], who solved the special case $(e^z - x) \rightarrow u(x) = 0$, and the domain of solutions was further extended by E. Hilb in 1920 [see *Bibliography*: Hilb (1)]. O. Perron (see *Bibliography*), employing results in the theory of equations of the form

$$\sum_{n=0}^{\infty} (a_n + b_{mn}) x_{m+n} = 0 \ , \quad m = 0, 1, 2, \cdots$$

gave an existence theorem for solutions of equation (3.1) which satisfy the restrictive condition that they shall be of finite grade.

Returning to the problem of this section let us first observe that

$$B(z)F(x,z) \rightarrow u(x) = \{F(x,z) \rightarrow B(z)\} \rightarrow u(x) \ .$$

This follows immediately from an application of (3.1) of chapter 4 since we have

$$F(x,z) \rightarrow B(z) = [F \cdot B](x,z) = F(x,z)B(z) \ .$$

We thus conclude that if $B(z)$ is a factor common to all the $A_i(z)$, then the set of solutions of equation (3.1) may contain as a sub-set the solutions of

$$B(z) \rightarrow u(x) = 0 \ . \tag{3.2}$$

If we designate by $v(x)$ a solution of the *reduced equation,*

$$F(x,z) \rightarrow u(x) = 0 \ ,$$

it is clear that corresponding solutions of the unreduced equation are immediately obtained from

$$B(z) \rightarrow u(x) = v(x) \ . \tag{3.3}$$

This non-homogeneous equation with constant coefficients has been discussed in chapter 6.*

Proceeding to more general considerations we now write,

$$u(x) = \int_L e^{xt} Y(t)\, dt \ , \tag{3.4}$$

*Let us note that the general solution of (3.3) which contains also the solutions of (3.2) in its complementary function may not always furnish a solution of the unreduced equation. Thus the equation

$$\{30z^2 - 6 + 12z(z^2-1)x + (z^2-1)^2x^2\} \rightarrow u(x) = 0$$

may be written

$$(z^2-1)\{30 + 24/(z^2-1) + 12zx + (z^2-1)x^2\} \rightarrow u(x) = 0 \ .$$

But the unrestricted solution of $(z^2-1) \rightarrow u(x) = v(x)$, where $v(x)$ is a function which satisfies the reduced equation, will not satisfy the original equation.

where L is a path in the complex plane, and substitute this function in $F(x,z) \to u(x)$. We then get

$$F(x,z) \to u(x) = \int_L e^{xt} Y(t) \{A_0(t) + A_1(t)x + \cdots + A_p(t)x^p\}\, dt \ .$$

Employing integration by parts, formula (2.20), and recalling the definition of $H_0(t)$, (2.11), we easily reduce this expression to

$$F(x,z) \to u(x) = \{H_1(t) + xH_2(t) + \cdots + x^{p-1}H_p(t)\}\, e^{xt}|_L + \int_L e^{xt} H_0(t)\, dt \ . \tag{3.5}$$

If we now define $Y(t)$ to be a solution of the equation

$$H_0(t) = 0 \ , \tag{3.6}$$

and choose L to be a path in the complex plane at the extremities of which we have simultaneously,

$$H_1(t)e^{xt} = H_2(t)e^{xt} = H_3(t)e^{xt} = \cdots = H_p(t)e^{xt}|_L = 0 \ , \tag{3.7}$$

then $u(x)$ as defined by (3.4) is a formal solution of (3.1)

If we note the following identity:

$$e^{-xz}X(z) \to f(x) = e^{xz} \to [X(z) \to f(x)] = X(z) \to f(x)|_{x=0} = \text{const.},$$

it is possible to give the solution $u(x)$ a more symmetric form than that of (3.4). Let $X_1(z), X_2(z), \cdots, X_p(z)$ be a fundamental set of solutions of the resolvent equation (2.4) and $Y_1(t), Y_2(t), \cdots, Y_p(t)$ the corresponding adjoints. Let L be any path in the complex plane for which equations (3.7) hold.

We may then write the solution of (3.1) in the form,

$$u(x) = e^{-xz} \int_L e^{xt} W(z,t)\, dt \to f(x) \ , \tag{3.8}$$

where $W(z,t) = X_1(z)Y_1(t) + X_2(z)Y_2(t) + \cdots + X_p(z)Y_p(t)$, and $f(x)$ is *arbitrary* to within the existence of the hight hand member.

The case of constant coefficients is immediately included in this formulization by writing,

$$W(z,t) = F(z)/F(t) \ ,$$

where we use the notation of section 2, chapter 6.

4. Some Particular Examples of the Solution of Homogeneous Equations. Before proceeding to more general considerations, it will be useful to examine some particular examples of the application of the theory set forth in the preceding section. It is a problem of much difficulty to formulate general theorems, since these theorems depend in an essential manner upon the singularities of $Y(t)$ and these singularities in turn depend upon the singularities of the adjoint resolvent generatrix equation which defines the function $Y(t)$.

The examples which are treated below illustrate many of the difficulties in the situation and will serve to introduce the more general discussion.

Example 1. Let us consider the equation,

$$\{(1+Ax)+z/a+z^2/a^2+z^3/a^3+\cdots\}\rightarrow u(x)=0 \ . \tag{4.1}$$

The adjoint resolvent is at once seen to be,

$$aY(t)/(a-t)-AY'(t)=0 \ ,$$

which has the particular solution, $Y(t)=(t-a)^\lambda$, $\lambda=-a/A$. We also find $H_1(t)=AY(t)$.

Let us first assume that λ is a negative integer, $\lambda=-n$. Then $H_1(t)e^{xt}|_L$ is zero provided L is a closed path and, in particular, a closed path around the point a. In this case the solution is,

$$\begin{aligned} u(x) &= \int_L e^{xt}(t-a)^{-n}dt \ , \\ &= 2\pi i\, e^{ax}\, x^{n-1}/(n-1)! \ . \end{aligned}$$

It is immediately obvious that when $n=1$, this function renders the original equation divergent. A similar conclusion is reached for all other values of n.

We shall now assume that λ is positive, but we shall not restrict it to integral values. Also assuming that the real part of x is greater than zero we see that $H_1(t)e^{xt}|_L$ is zero for the Laurent circuit of figure 1. Hence we obtain,

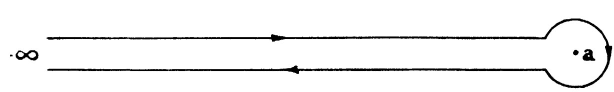

FIGURE 1

$$\begin{aligned} u(x) &= (1-e^{2\pi i\lambda})e^{ax}\int_0^\infty s^\lambda e^{-xs}ds \ , \\ &= (1-e^{2\pi i\lambda})\Gamma(1+\lambda)e^{ax}/x^{\lambda+1} \ . \end{aligned}$$

It is proper to inquire in what sense this function is a solution of the equation since it is obviously of infinite grade.

We observe first the operational identity, [see (12.9), chapter 2],

$$z^n \to e^{ax}x^{-m} = e^{ax}(z+a)^n \to x^{-m} = e^{ax}[\Gamma(m)a^n x^{-m}$$
$$-n\Gamma(m+1)a^{n-1}x^{-(m+1)} + n(n-1)\Gamma(m+2)a^{n-2}x^{-(m+2)}/2!$$
$$+\cdots]/\Gamma(m) \ .$$

Employing Borel summability we replace the factorials in the numerator by the integral

$$\int_0^\infty t^n e^{-t}dt = \Gamma(n+1) \ .$$

The identity then becomes,

$$z^n \to e^{ax}x^{-m} = e^{ax}x^{-m}\int_0^\infty e^{-t}t^{m-1}(a-t/x)^n dt/\Gamma(m) \ .$$

Hence if $\varphi(z)$ is a power series in either z or $1/z$, we may write,

$$\varphi(z) \to e^{ax}x^{-m} = e^{ax}x^{-m}\int_0^\infty e^{-t}t^{m-1}\varphi(a-t/x)dt/\Gamma(m) \ , \qquad (4.2)$$

provided the integral exists.

Applying the operation defined by (4.2) to our solution $u(x)$ we get,

$$\{1 + Ax + z/a + (z/a)^2 + (z/a)^3 + \cdots\} \to u(x)$$
$$= (1 - e^{2\pi i\lambda})(e^{ax}/x^{\lambda+1})\int_0^\infty e^{-t}t^\lambda(ax/t)dt + A\Gamma(\lambda+1)e^{ax}/x^\lambda \ .$$

If $\lambda > 0$, the integral converges and the right hand member of the equation becomes,

$$(1 - e^{2\pi i\lambda})(e^{ax}/x^\lambda)\{a\Gamma(\lambda) + A\Gamma(\lambda+1)\} = 0 \ .$$

Since the application of the operator defined in (4.1) may be open to some objection in the present instance, it is worth while justifying the solution on other grounds.

Since the operator $B(z) = 1 + z/a + (z/a)^2 + \cdots$ is equivalent to $1/(1-z/a)$ and since this function has the Laurent expansion $B_1(z) = -(a/z + a^2/z^2 + \cdots)$ we may consider the relationship of the latter to the differential operator. But we know from theorem 1, chapter 6, that $\{B_1(z) - B(z)\} \to f(x)$, where $f(x)$ is arbitrary to within the limits of the existence of the operator, is a solution of the equation $B(z) = 0$. It may be proved without difficulty that this solution is identically zero provided the integrals of $B_1(z)$ are taken over a proper path to infinity. In the present instance this path may be chosen from $-\infty$ to x. The essential equivalence of $B_1(z)$ and $B(z)$ is then assured.

Operating with $B_1(z)$ upon $u(x)$ and noting that

$$1/z^n \to u(x) = \int_{-\infty}^{x} (x-t)^{n-1} u(t)/(n-1)!\, dt \ ,$$

we get

$$B_1(z) \to u(x) = -(1-e^{2\pi i\lambda})\Gamma(1+\lambda)a \int_{-\infty}^{x} \{1 + a(x-t)$$
$$+ a^2(x-t)^2/2! + \cdots\}(e^{at}/t^{\lambda+1})dt$$
$$= -(1-e^{2\pi i\lambda})\Gamma(1+\lambda)ae^{ax} \int_{-\infty}^{x} t^{-\lambda-1}dt$$
$$= (1-e^{2\pi i\lambda})a\Gamma(\lambda)e^{ax}/x^{\lambda} \ , \quad \lambda > 0 \ .$$

Example 2. Let us examine the equation,

$$\{2(1+z^2+z^4+z^6+\cdots)+4x(z+z^3+z^5+\cdots)$$
$$-x^2\} \to u(x) = 0 \ . \qquad (4.3)$$

Since we have,

$$A_0(z) = 2/(1-z^2), \ A_1(z) = 4z/(1-z^2), \ A_2 = -1 \ ,$$

the equation, $H_0(t) = 0$, becomes,

$$[2/(1-t^2)]Y - (d/dt)[4t\,Y/(1-t^2)] - d^2Y/dt^2 = 0 \ .$$

A fundamental set of solutions is found to be,

$$Y_1(t) = t(t^2-1), \quad Y_2(t) = 1-t^2 \ ,$$

in terms of which we compute the four functions,

$$H_1^{(1)}(t) = -(t^2+1), \quad H_1^{(2)}(t) = 2t;$$
$$H_2^{(1)}(t) = -t(t^2-1), \quad H_2^{(2)}(t) = t^2-1 \ ,$$

where we have adopted the notation of (2.11).

In order to determine a point, $t = a$, for which we have $H_j(a) = 0$, $j = 1, 2$, we consider $Y(t) = a_1Y_1(t) + a_2Y_2(t)$. When this function is substituted in the equations $H_j(t) = 0$, we obtain the system,

$$a_1H_1^{(1)}(t) + a_2H_1^{(2)}(t) = 0 \ ,$$
$$a_1H_2^{(1)}(t) + a_2H_2^{(2)}(t) = 0 \ .$$

A necessary and sufficient condition for the existence of a_1, a_2 is that the determinant, $|H_j^{(i)}(t)|$ shall vanish,

$$D(t) = \begin{vmatrix} H_1^{(1)}(t) & H_1^{(2)}(t) \\ H_2^{(1)}(t) & H_2^{(2)}(t) \end{vmatrix} = 0 \ .$$

In the present case this gives,

$$D(t) = -(1-t^2)^2 ,$$

which yields the values $t = 1$ and $t = -1$. Corresponding to these values we compute $a_1 = a_2$ and $a_1 = -a_2$ respectively, and hence replace the original fundamental set of solutions by the following:

$$y_1(t) = (t-1)(t^2-1), \; y_2(t) = (t+1)(t^2-1) ,$$

for which $H_1(t)$ and $H_2(t)$ equal zero when $t = 1$ and $t = -1$, respectively. Since $\lim_{t=-\infty} e^{xt}y_1(t) = \lim_{t=-\infty} e^{xt}y_2(t) = 0$ when $R(x) > 0$, we obtain as solutions of (4.3) the two functions,

$$\begin{aligned} u_1(x) &= \int_{-\infty}^{1} (t^2-1)(t-1)e^{xt}dt = e^x(4/x^3 - 6/x^4) , \\ u_2(x) &= \int_{-\infty}^{-1} (t^2-1)(t+1)e^{xt}dt = -e^{-x}(4/x^3 + 6/x^4) . \end{aligned} \tag{4.4}$$

It is to be observed that neither of these solutions is a function of finite grade. Hence it will be instructive to see in what sense these functions may be said to furnish a solution of the differential equation of infinite order since obviously a direct substitution leads to a divergent series.

Applying the operational identity defined by (4.2) to equation (4.3) we have,

$$\begin{aligned} \{2/(1-z^2) + 4xz/(1-z^2) - x^2\} &\to e^x(4/x^3 - 6/x) \\ = (4e^x/2!) \int_0^{\infty} e^{-t}t^2x^{-3}\{2/[1-(1-t/x)^2] \\ &+ 4x(1-t/x)/[1-(1-t/x)^2] - x^2\}\, dt \\ - (6e^x/3!) \int_0^{\infty} e^{-t}t^3x^{-4}\{2/[1-(1-t/x)^2] \\ &+ 4x(1-t/x)/[1-(1-t/x)^2] - x^2\}\, dt \\ = \int_0^{\infty} (te^{x-t}/x^4)\{2x^2 + 4(x-t)x^2 + x^2(t^2 - 2tx)\}\, dt = 0 . \end{aligned}$$

We thus see that $u_1(x)$ is a solution of equation (4.3) in the sense that if it be substituted in the left hand member of the equation, a divergent series is obtained which is summable to zero by the method of Borel. A similar interpretation applies to the statement that $u_2(x)$ is a solution of equation (4.3).

Example 3. Let us now consider the following equation:

$$\{1/(z-1) + 1/(z-2) + 1/(z-3) + 2x\} \rightarrow u(x) = 0 \ . \tag{4.5}$$

We derive a particular value of $Y(t)$ to be,

$$Y(t) = [(t-1)(t-2)(t-3)]^{\frac{1}{2}} \ ,$$

corresponding to which we have the function, $H_1(t) = 2Y(t)$.

For the purpose of discussing this equation we introduce a circuit which was first studied by L. Pochhammer and which may be described as follows:*

If we designate by L_a a circuit in a positive direction around a and by L_a^{-1} a circuit in the opposite sense, then it will be clear that if z traces the path $L_{ab} = L_a L_b L_a^{-1} L_b^{-1}$ a function of the form $Y(t) = (t-a)^{\lambda}(t-b)^{\mu} y(t) e^{xt}$ will return to its initial value.

The integral of this function, if λ and μ are fractions greater than -1, will have the value,

$$u(x) = (1 - e^{2\pi i\mu})(1 - e^{2\pi i\lambda}) \int_a^b (t-a)^{\lambda}(t-b)^{\mu} y(t) e^{xt} dt \ . \tag{4.6}$$

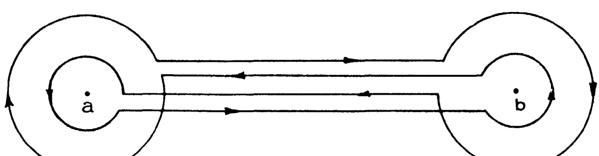

FIGURE 2

The function $u(x)$ can be given a useful formulization in the following manner:

Let us write,

$$u(x) = (e^{\pi i\mu} - e^{-\pi i\mu})(e^{\pi i\lambda} - e^{-\pi i\lambda}) \int_a^b (a-t)^{\lambda}(b-t)^{\mu} y(t)\, dt \ ,$$

$$= -4 \sin \pi\mu \sin \pi\lambda \int_a^b (a-t)^{\lambda}(b-t)^{\mu} y(t)\, dt \ .$$

Noting the identity, $\sin \pi a = \pi/[\Gamma(a)\Gamma(1-a)]$, and using the abbreviation, $K = -4\pi^2\lambda\mu/\Gamma(1-\mu)\Gamma(1-\lambda)$, $u(x)$ may be written,

$$u(x) = K \int_a^b \frac{(a-t)^{\lambda}(b-t)^{\mu}}{\Gamma(\lambda+1)\Gamma(\mu+1)} y(t)\, dt \ ,$$

*See L. Pochhammer: Über ein Integral mit doppelten Umlauf. *Mathematische Annalen*, vol. 35 (1890), pp. 470-494.

$$u(x) = K \int_c^b \frac{(b-t)^{\mu}(a-t)^{\lambda}}{\Gamma(\mu+1)\Gamma(\lambda+1)} y(t)\,dt$$

$$- K \int_c^a \frac{(a-t)^{\lambda}(b-t)^{\mu}}{\Gamma(\lambda+1)\Gamma(\mu+1)} y(t)\,dt \ ,$$

$$= K \{ {}_cD_b^{-(\mu+1)} (a-b)^{\lambda} y(b)/\Gamma(\lambda+1)$$

$$- {}_cD_b^{-(\lambda+1)} (b-a)^{\mu} y(a)/\Gamma(\mu+1) \} \ . \qquad (4.7)$$

Returning now to the problem, it is immediately seen that the following three particular solutions of (4.5) can be written down by properly specializing (4.6).

Hence three particular solutions of the equation are,

$$u_1(x) = \int_1^2 [(t-1)(t-2)(t-3)]^{\frac{1}{2}} e^{xt} dt \ ,$$

$$u_2(x) = \int_1^3 [(t-1)(t-2)(t-3)]^{\frac{1}{2}} e^{xt} dt \ ,$$

$$u_3(x) = \int_2^3 [(t-1)(t-2)(t-3)]^{\frac{1}{2}} e^{xt} dt \ .$$

But these solutions are obviously not independent since $u_2(x) = u_1(x) + u_3(x)$. That they do in fact furnish two independent solutions of the equation is proved by direct substitution and observation of the operational identity $\varphi(z) \to e^{ax} = e^{ax}\varphi(a)$. For example, substituting $u_1(x)$ in the left hand member of the equation we get,

$$\int_1^2 [1/(t-1) + 1/(t-2) + 1/(t-3)$$

$$+ 2x][(t-1)(t-2)(t-3)]^{\frac{1}{2}} e^{xt} dt \ .$$

One integration by parts of the last term yields,

$$\int_1^2 \{1/(t-1) + 1/(t-2) + 1/(t-3) - (3t^2 - 12t$$

$$+ 11)/(t-1)(t-2)(t-3)\}[(t-1)(t-2)(t-3)]^{\frac{1}{2}} e^{xt} dt \ ,$$

which is identically zero.

A third independent solution of the equation is given by the integral,

$$u(x) = \int_{-\infty}^1 [(t-1)(t-2)(t-3)]^{\frac{1}{2}} e^{xt} dt \ , \qquad R(x) > 0 \ .$$

Example 4. Let us consider the following difference equation:

$$x(x+1)u(x+2) - 2x(x+2)u(x+1)$$

$$+ (x+2)(x+1)u(x) = 0 \ . \qquad (4.8)$$

Employing the symbolic identity $e^{az} \to u(x) = u(x+a)$, we may write this equation in the form,

$$\{x^2(e^{2z} - 2e^z + 1) + x(e^{2z} - 4e^z + 3) + 2\} \to u(x) = 0 \ .$$

The adjoint generatrix is then derived to be,

$$(e^t-1)^2 Y''(t) + 3(e^{2t}-1)Y'(t) + 2(e^{2t} + e^t + 1)Y(t) = 0 \ ,$$

with solutions,

$$Y_1(t) = e^t/(e^t-1)^3 \ , \ Y_2(t) = e^t/(e^t-1)^2 \ .$$

Particular solutions of the original equation are immediately obtained from the following integrals:

$$u_1(x) = \int_L \{e^{(x+1)t}/(e^t-1)^3\}\, dt \ ,$$

$$u_2(x) = \int_L \{e^{(x+1)t}/(e^t-1)^2\}\, dt \ ,$$

where the path of integration is about any of the points $t = 2n\pi i$, $n = 0, 1, 2, \cdots$.

These solutions are shown to be,

$$u_1(x) = \pi i\, e^{2n\pi i x}(x^2-x) \ , u_2(x) = 2\pi i\, e^{2n\pi i x} x \ .$$

Since linear combinations of these functions are solutions and since sums with respect to n are solutions, we readily derive the following more general solutions:

$$U_1(x) = \pi(x)x^2 \ , \quad U_2(x) = \pi(x)x \ ,$$

where $\pi(x)$ is any function of unit period, $\pi(x+1) = \pi(x)$.

We can readily show that $\{U_1(x), U_2(x)\}$ forms a complete set of solutions as follows:

Let us write, $Y(t) = Y_1(t) + \lambda Y_2(t)$, and hence compute,

$$H_1(t) = - e^t\{(2\lambda-1)e^t + 2(1-\lambda)\}/(e^t-1)^2 \ ,$$

$$H_2(t) = e^t\{\lambda e^t + 1 - \lambda\}/(e^t-1) \ .$$

If $R(x)$ exceeds 2, it is clear that $\lim_{t=-\infty} H_1(t)e^{xt} = \lim_{t=-\infty} H_2(t)e^{xt} = 0$. We now seek a point $t = a$ in the finite plane such that $H_1(a) = H_2(a) = 0$. If we designate e^a by m, we find that m must satisfy the equations: $m = 1-1/\lambda$, $(2\lambda-1)m + 2(1-\lambda) = 0$. Eliminating m from these equations, we obtain $(\lambda-1)/\lambda = 0$, which yields $a = -\infty$ and $a = 2n\pi i$, $n = 0, 1, 2, \cdots$. Since the first point is not in the finite plane and the other points have been used in obtaining $U_1(x)$ and $U_2(x)$, no path L other than those previously employed can be found to yield a third solution of the equation. Hence $\{U_1(x), U_2(x)\}$ forms a complete set of solutions.

Example 5. Let us now consider the following three equations, the last one of which (C) we have previously discussed in example 2.

$$(A)\quad \{(30z^2-6)+(12z^3-12z)x+(z^2-1)^2x^2\}\to u(x)=0\ ,$$

$$(B)\quad \{(6z^2-4z-2)/[(z-1)^2(z+1)]$$
$$+x(6z-2)/(z-1)+(z+1)x^2\}\to u(x)=0\ ,$$

$$(C)\quad \{2/(1-z^2)+4xz/(1-z^2)-x^2\}\to u(x)=0\ .$$

It will be found upon examination that *these equations share the same adjoint generatrix,*

$$(t^2-1)^2\,Y''-4t(t^2-1)Y'+(6t^2+2)Y=0\ ,$$

two independent solutions of which are $Y_1=t(t^2-1)$ and $Y_2=1-t^2$.

Considering equation (A) we compute the values of $H_i^{(j)}(t)$ and thus obtain the following expressions:

$$a_1H_1^{(1)}+a_2H_1^{(2)}=(t^2-1)^2[a_1(5t^2+1)-6ta_2]\ ,$$
$$a_1H_2^{(1)}+a_2H_2^{(2)}=(t^2-1)^3[a_1t-a_2]\ ,$$

both of which obviously reduce to zero for $t=\pm 1$.

Hence differential equation (A) has four solutions, two of which are given by the integrals,

$$u_1(x)=\int_{-1}^{1}(t^2-1)(t-1)e^{xt}dt$$
$$=(4/x^3-6/x^4)e^x+(4/x^2+8/x^3+6/x^4)e^{-x}\ ,$$
$$u_2(x)=\int_{-1}^{1}(t^2-1)(t+1)e^{xt}dt$$
$$=(-4/x^2+8/x^3-6/x^4)e^x+(4/x^3+6/x^4)e^{-x}\ .$$

The other two solutions are obtained by integrals to infinity and are easily found to be,

$$u_3(x)=\int_{-\infty}^{1}(t^2-1)(t-1)e^{xt}dt=e^x(4/x^3-6/x^4)\ ,$$

$$u_4(x)=\int_{-\infty}^{-1}(t^2-1)(t+1)e^{xt}dt=-e^x(4/x^3+6/x^4)\ .$$

The distinguishing difference between the two sets of solutions $\{u_1(x),\,u_2(x)\}$ and $\{u_3(x),\,u_4(x)\}$ is found in the fact that the former are of finite grade while the latter have poles at the origin.

Proceeding to equation (B) we obtain from the values of $H_i^{(j)}(t)$ the expressions,

$$a_1 H_1^{(1)} + a_2 H_1^{(2)} = (2t^3 + t^2 + 1) a_1 - (3t^2 + 2t - 1) a_2 \ ,$$

$$a_1 H_2^{(1)} + a_2 H_2^{(2)} = (t+1)(t^2-1)(a_1 t - a^2) \ .$$

Upon explicitly substituting the values $t = \pm 1$ in these expressions we see that they reduce to zero provided $a_1 = a_2$, that is for $Y = (t^2-1)(t-1)$. Hence equation (B) has one non-singular solution, $u_1(x)$, and the two singular solutions $u_3(x)$ and $u_4(x)$.

In the same way we find for equation (C) the expressions,

$$a_1 H_1^{(1)} + a_2 H_1^{(2)} = -(t^2 + 1) a_1 + 2ta_2 \ ,$$

$$a_1 H_2^{(1)} + a_2 H_2^{(2)} = (t^2 - 1)(-ta_1 + a_2) \ ,$$

which cannot be simultaneously reduced to zero by the values $t = \pm 1$. Hence equation (C) has no non-singular solutions and only the two singular solutions $u_3(x)$ and $u_4(x)$.

We may note that the essential difference between these three equations is to be found in the rank of the following matrix:

$$A = \left\| \begin{matrix} A_2(a) Y_1'(a), & A_2(a) Y_2'(a) \\ A_2(a) Y_1\ (a), & A_2(a) Y_2\ (a) \\ A_2(b) Y_1'(b), & A_2(b) Y_2'(b) \\ A_2(b) Y_1\ (b), & A_2(b) Y_2\ (b) \end{matrix} \right\| .$$

One may easily verify that for $a = 1$, $b = -1$, the rank of A is zero for equation (A), one for equation (B), and two for equation (C). It may also be shown by explicit substitution that the function $u_1(x)$ is a solution of the first two equations, but is in no sense a solution of the third.

PROBLEMS

1. Solve the equation

$$(x+2)(2x+1)\,u(x+2) - 4(x+1)^2\,u(x+1) + x(2x+3)\,u(x) = 0 \ .$$

2. Show that the following equation:

$$(-4x^2 - 2x + 6)\,u(x+2) + (8x^2 + 12x - 14)\,u(x+1) + (-4x^2 - 10x)\,u(x) = 0$$

has as special solutions two quadratic polynomials.

3. Show that the equation

$$u(x) = x\,u(x-r)$$

has as its general solution the function

$$u(x) = \pi(x/r)\,r^{x/r}\,\Gamma(1 + x/r) \ ,$$

where $\pi(z)$ is a function of unit period. (Pennell).

4. Show that $u_1(x) = \Gamma(x)$ and $u_2(x) = 1/\Gamma(x)$ are particular solutions of the equation

$$(x+1)^2(x-1)\,u(x+2) - (x^2+x+1)(x^2+x-1)\,u(x+1) + x^2(x+2)\,u(x) = 0\ . \quad \text{(Wallenberg and Guldberg).}$$

Solve the following equations:

5. $(3x^2-2x-1)\,u(x+2) - (15x^2-4x-4)\,u(x+1) + (12x^2+16x)\,u(x) = 0\ .$

6. $(x^2-5x+4)\,u(x+2) - (3x^2-13x+8)\,u(x+1) + (2x^2-6x)\,u(x) = 0\ .$

7. $x^2\,u(x+2) - (4x^2+2x+1)\,u(x+1) + (3x^2+6x+3)\,u(x) = 0\ .$

8. Show that $u_1(x) = m^x$, $u_2(x) = ax^2+bx+c$ are particular solutions of the equation:

$$\begin{aligned}&\{(m-1)\,a x^2 + [b(m-1)-2a]\,x + c(m-1) - a - b\}\,u(x+2)\\ &-\{a(m^2-1)\,x^2 + [b(m^2-1)-4a]\,x + (m^2-1)c - 4a - 2b\}\,u(x+1)\\ &+\{(m^2-m)\,a x^2 + [(m^2-m)b + (2m^2-4m)a]\,x + (m^2-4m)a\\ &\qquad + (m^2-2m)b + (m^2-m)c\}\,u(x) = 0\ .\end{aligned}$$

9. If in the equation

$$A(x)\,u(x+2) + B(x)\,u(x+1) + C(x)\,u(x) = 0$$

the functions $A(x)$, $B(x)$, $C(x)$ are quadratic polynomials, determine their coefficients so that $u_1(x) = a\,x^2+bx+c$ and $u_2(x) = \alpha\,x^2+\beta\,x+\gamma$ are particular solutions

10. Express the integral equation

$$\int_0^x [t^2 + 3t(t-x) - (t-x)^2]\,u(t)\,dt = 0$$

in the form

$$(x^2/z - 5x/z^2 + 6/z^3) \to u(x) = 0\ .$$

From this obtain the general solution $u(x) = c\,x^\lambda$, $\lambda = \sqrt{3}$.

5. *The Homogeneous Equation — Case of Degree One* ($p=1$) *with Polynomial Exponent.* Proceeding now to more general considerations it will be convenient first to discuss the case where the generatrix is an equation of first order.

If we let $p = 1$, then equations (2.4) and (2.5) become,

$$A_0(z)X(z) + A_1(z)X'(z) = 0\ ,$$

$$A_0(z)Y(z) - \frac{d}{dz}\{A_1(z)Y(z)\} = 0\ ,$$

with the explicit solutions:

$$X(z) = e^{-\int^z \{A_0(z)/A_1(z)\}dz}, \quad Y(z) = e^{\int^z \{A_0(z)/A_1(z)\}dz}/A_1(z) .$$

Moreover we have

$$H_1(z) = A_1(z)Y(z) = e^{\int^z \{A_0(z)/A_1(z)\}dz} .$$

If now we assume the following expansion:

$$A_0(z)/A_1(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$
$$+ \sum_i [a_{i1}/(z-z_1)^i + a_{i2}/(z-z_2)^i + a_{i3}/(z-z_3)^i$$
$$+ \cdots + a_{im}/(z-z_m)^i] ,$$

then we may write $H_1(z)e^{xz}$ in the form,

$$H_1(z)e^{xz} = e^{\varphi(z)}e^{\psi(z)}(z-z_1)^{a_{11}}(z-z_2)^{a_{12}}(z-z_3)^{a_{13}}\cdots(z-z_m)^{a_{1m}} ,$$

where we employ the abbreviations,

$$\varphi(z) = (a_0 + x)z + a_1 z^2/2 + a_2 z^3/3 + \cdots ,$$
$$\psi(z) = \sum_{i=2} [a_{i1}/(1-i)(z-z_1)^{i-1} + a_{i2}/(1-i)(z-z_2)^{i-1}$$
$$+ \cdots + a_{im}/(1-i)(z-z_m)^{i-1}] .$$

For convenience we shall refer to the function $\varphi(z)$ as *the exponent of* $H_1(z)$ and shall limit ourselves in this section to the case where $\varphi(z)$ is a polynomial: $a_s \neq 0$, $a_n = 0$ for $n > s$.

If we replace z by $r\,e^{\theta i}$ and $a_s/(s+1)$ by $R\,e^{a i}$, then we may write,

$$a_s z^{s+1}/(s+1) = Rr^{s+1}e^{(s+1)\theta i + a i}$$
$$= Rr^{s+1}\{\cos[(s+1)\vartheta + a] + i\sin[(s+1)\vartheta + a]\} .$$

Let us set $\cos\{(s+1)\vartheta + a\} = 0$, from which we get,

$$\vartheta_n = [(2n-1)\pi/2 - a]/(s+1), \quad n = 1, 2, \cdots, 2s+2 .$$

For values of ϑ such that $\vartheta_{2n-1} < \vartheta < \vartheta_{2n}$, we see that the real part of $a_s z^{s+1}/(s+1)$ will be negative. Hence if we choose a path to infinity lying within one of these intervals, it is clear that $H_1(z)$ will approach zero. Evaluating the integral $\int e^{xt}Y(t)dt$ over a path which does not enclose any of the points $z_1, z_2, \cdots, z_m$ and which approaches infinity in two of the above intervals we shall have a formal integral of the equation. *In this manner we obtain* $s+1$ *independent solutions of the equation.*

If $z_1, z_2, \cdots, z_m$ are simple zeros, that is to say if $a_{ij} = 0$, $i > 1$, and if $a_{11}, a_{12}, \cdots, a_{1m}$ are fractions, then we can obtain $m - 1$ solutions of the equation by means of Pochhammer integrals. In case any

value of α_{ij} is a negative integer this factor is excluded from the Pochhammer circuits and a Cauchy integral is substituted. If all the α_{ij} are negative integers no infinite branch is necessary to obtain the totality of solutions. If any two α_{ij} are positive integers, for example α_{11}, α_{12}, then modified Pochhammer circuits,

$$\int_{z_2}^{z_1} Y(z)\, e^{xz} dz \ ,$$

can be employed. If only one α_{ij} is a positive integer then the path must be taken with one infinite branch terminating in z_j.

In considering $\psi(z)$ let us assume that there exists an integer S_1 such that $a_{S_1 1} \neq 0$, $a_{n1} = 0$, $n > S_1$. If we represent $z - z$ by $r\, e^{\theta i}$ and $a_{S_1 1}/(1-S_1)$ by $R\, e^{ai}$, it is clear that we shall have,

$$a_{S_1 1}/[(1-S_1)\,(z-z_1)^{S_1-1}] = r\, R^{1-S_1} e^{(1-S_1)\theta i + ai} \ .$$

By means of an argument which does not differ from the one previously employed we find that the real part of

$$a_{S_1 1}/[(1-S_1)\,(z-z_1)^{S_1-1}]$$

will be negative for values of ϑ within the sectors: $\vartheta_{2n-1} < \vartheta < \vartheta_{2n}$ where we define ϑ_n to be,

$$\vartheta_n = [(2n-1)\pi/2 - a]/(1-S_1) \ , \ n = 1, 2, \cdots, 2S_1 - 2 \ .$$

Hence if a path be chosen which does not include any of the values $z_2, \cdots, z_m$ and which consists of a loop emerging from z_1 in one of the sectors defined above and returning to z_1 in another of the sectors, then the integral taken along this path furnishes a formal solution of the equation. It is thus clear that we shall obtain $S_1 - 1$ integrals. Considering the other values $z_2, z_3, \cdots z_m$ in similar fashion we thus obtain $S_1 + S_2 + \cdots + S_m - m$ integrals .

Combining these solutions with those previously obtained we thus obtain a total of $s + S_1 + S_2 + \cdots + S_m$ formal integrals.

An example is supplied by the equation:

$$\{1/(z-1)^2 + [1/2(z+1) \\ - 1/(z+2) + 1/2(z+3)]x\} \rightarrow u(x) = 0 \ ,$$

where the operators are written for convenience in their closed form, but are to be regarded as equivalent to their power series expansion within the unit circle.

It follows readily that we have:

$$A_0/A_1 = (z+1)\,(z+2)\,(z+3)/(z-1)^2 \\ = 8 + z + 26/(z-1) + 24/(z-1)^2 \ ,$$

from which we compute,

$$H_1(z)\,e^{xz} = e^{(8+x)z+z^2/2}\,e^{-24/(z-1)}\,(z-1)^{26} \ ,$$

$$Y(z)\,e^{xz} = e^{(8+x)z+z^2/2}\,e^{-24/(z-1)}\,(z-1)^{26}(z+1)\,(z+2)\,(z+3) \ .$$

It will be immediately perceived that there exist three formal solutions which can be obtained by integrating $Y(z)\,e^{xz}$ over the three paths indicated in figure 3. The axes of X and Y are respectively the axes of reals and imaginaries.

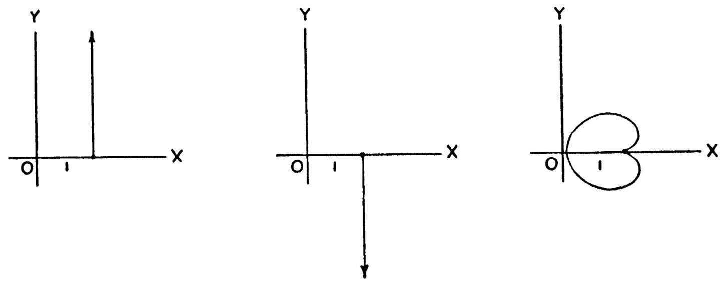

FIGURE 3

6. The Homogeneous Equation — Case of Degree One ($p = 1$) *with Transcendental Exponent.* An extension of the argument of the preceding section is possible if $\varphi(z)$, the exponent of $H_1(z)$, is an entire function of the form,

$$\varphi(z) = e^{g(z)}h(z) + k(z) \ , \qquad (6.1)$$

where $g(z)$, $h(z)$, and $k(z)$ are polynomials.

Let us assume that the degrees of these polynomials are p, q, and r respectively and that the coefficients of the highest powers of z in each are $G\,e^{gi}$, $H\,e^{hi}$, and $K\,e^{ki}$.

For sufficiently large values of $z = R\,e^{\theta i}$ we should then have the asymptotic value,

$$\varphi(z) \backsim HR^q e^{GR^p\{\cos(p\theta+g) + i\sin(p\theta+g)\} + (q\theta+h)i} + KR^r e^{(r\theta+k)i} \ ,$$

$$\backsim HR^q[\cos\{q\vartheta + h + GR^p\sin(p\vartheta+g)\} + i\sin\{q\vartheta + h$$

$$+ GR^r\sin(p\vartheta+g)\}] \times e^{GR^p\cos(p\theta+g)} + KR^r\{\cos(r\vartheta+k)$$

$$+ i\sin(r\vartheta+k)\} \ .$$

Let us define a family of curves in polar coordinates by means of the equation:

$$q\vartheta + h + GR^p\sin(p\vartheta+g) = (2n-1)\pi/2,\ n = \begin{cases} 1, & 2, & 3, \cdots \\ -1, & -2, & -3, \cdots \end{cases} .$$

If we use the abbreviation, $\lambda = p\vartheta + g$, this equation may be written,

$$R^p = \{(2n-1)\pi p - 2q(\lambda - g) - 2ph\}/2Gp \sin\lambda \ . \tag{6.2}$$

It is evident that the curve represented by (6.2) has infinite branches in the direction $\lambda = n\pi$, $n = 0, \pm 1, \pm 2, \cdots, \pm p$ and these branches divide the plane into regions within which the real part of the coefficient of $e^{GR^p \cos(p\theta + g)}$ is of one sign. Let us designate by A_1 any region within which the real part is positive and by A_2 a region within which the real part is negative.

If a second partition of the plane be made by means of the lines $p\vartheta + g = (2n-1)\pi/2$, $n = 1, 2, \cdots, 2p$, then $\cos(p\vartheta + g)$ will be of one sign in each region so constructed. Let us designate by B_1 a region within which the sign is positive and by B_2 a region within which the sign is negative.

If still a third partition of the plane be made by means of the lines $r\vartheta + k = (2n-1)\pi/2$, $n = 1, 2, \cdots, 2r$, we see that the real part of $KR^r\{\cos(r\vartheta + k) + i\sin(r\vartheta + k)\}$ is positive in one set of regions and negative in the other. Let us call such regions C_1 and C_2 respectively.

Let us designate by the symbol $L_{(ab)}$ a path between two points a and b, by ${}_AL_{(ab)}$ a path from the point a to the point b which lies wholly with the region A, and by ${}_{AB}L_{(ab)}$ a path between a and b which lies wholly within a region common to A and B.

Returning now to a consideration of the problem of evaluating the integral,

$$u(x) = \int_L e^{xt} Y(t)\, dt \ ,$$

it is clear that a formal solution of the differential equation is given by means of this integral provided the path of integration avoids the singular points of $Y(t)$ and consists of the following branches:

$$L = {}_{A_2B_1}L_{(\infty a)} + L_{(ab)} + {}_{B_2C_2}L_{(b\infty)} \ . \tag{6.3}$$

There may in particular exist an infinite number of such circuits and hence an infinite number of formal solutions.

Two examples will illustrate these principles.

Example 1. Let us first consider the classical equation,

$$u(x+1) - xu(x) = 0 \ ,$$

which defines in particular the gamma function, $u_0(x) = \Gamma(x)$.

Symbolically this equation can be written in the form of a differential equation of infinite order,

$$(e^z - x) \to u(x) = 0 \ ,$$

from which a simple computation yields,

$$\varphi(z) = -e^z + xz \ , \quad \psi(z) = 0 \ .$$

Referring to the notation of (6.1) we have $p = r = 1$, $q = 0$; $G = H = 1$, $K = x$; $g = k = 0$, $h = \pi$. Equation (6.2) becomes,

$$R = \{(2n - 3)\pi \csc \vartheta\}/2 \ .$$

The partition of the plane is made by a series of lines parallel to the axis of reals and spaced at intervals of π, the first at a distance $\pi/2$ from the axis of reals. The second and third partitions are coincident, being merely a division of the plane into two parts by the axis of imaginaries.

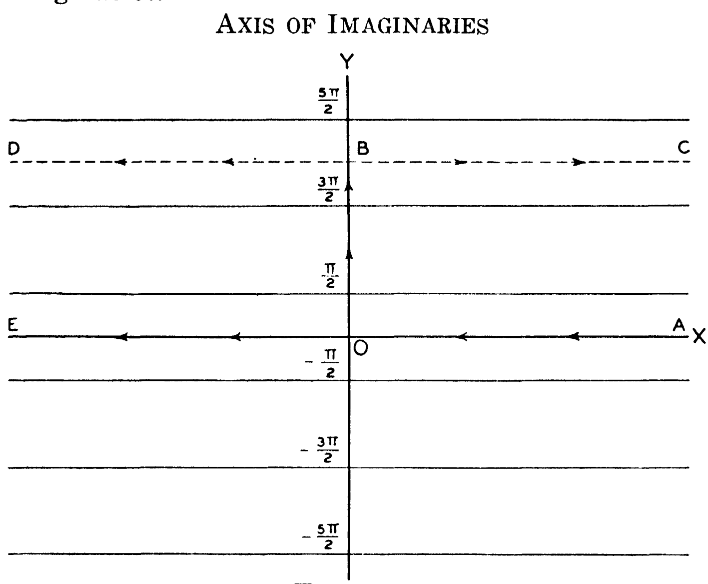

FIGURE 4

One path L can be composed of the segments OA and OE. We thus obtain,

$$u(x) = \int_0^\infty e^{-e^z + xz} dz + \int_{-\infty}^0 e^{-e^z + xz} dz \ .$$

Changing variables by means of the equation $t = e^z$ we get,

$$u(x) = \int_0^\infty e^{-t} t^{x-1} dt \ ,$$

which is the familiar integral representation of $\Gamma(x)$ for the real part of $x > 0$.

It is of interest to choose for a path of integration the segments $AO + OB + BC$ at the extremities of which $e^{\varphi(z)}$ vanishes. We thus obtain,

$$u(x) = \int_{\infty}^{0} e^{-e^z+xz}dz + \int_0^{2\pi} e^{-e^{yi}+xyi}i\,dy + e^{2\pi xi} \int_0^{\infty} e^{-e^z+xz}dz \ ,$$

$$= (e^{2\pi ix}-1) \int_1^{\infty} e^{-t}t^{x-1}dt + \int_0^{2\pi} e^{-e^{yi}+xyi}i\,dy \ . \qquad (6.4)$$

The second integral may be evaluated as follows:
Integrating by parts we find,

$$\int_0^{2\pi} e^{-e^{yi}+xyi}i\,dy = e^{-1}(1 - e^{2\pi ix}) + (x-1) \int_0^{2\pi} e^{-e^{yi}+(x-1)yi}i\,dy \ ,$$

$$= e^{-1}(1- e^{2\pi ix})\{1 + (x-1) + (x-1)(x-2) + \cdots$$
$$+ (x-1)(x-2)\cdots(x-n) + \cdots\} \ .$$

This series is obviously divergent but may be summed by Borel's method. Thus employing the Borel integral

$$\int_0^{\infty} e^{-t}t^n dt = n!$$

we get,

$$\int_0^{2\pi} e^{-e^{yi}+xyi}i\,dy = e^{-1}(1 - e^{2\pi ix}) \int_0^{\infty} e^{-t}\{1 + (x-1)t$$
$$+ (x-1)(x-2)t^2/2! + \cdots$$
$$+ (x-1)(x-2)\cdots(x-n)t^n/n! + \cdots\}\,dt$$
$$= e^{-1}(1 - e^{2\pi ix}) \int_0^{\infty} e^{-t}(1+t)^{x-1}dt = (1 - e^{2\pi ix}) \int_1^{\infty} e^{-t}t^{x-1}dt \ .$$

Substituting this value in equation (6.4) we find that $u(x)$ is identically zero for this path. Employing still a third path ($AOBD$) and noting the results just obtained, we compute,

$$u(x) = \int_{\infty}^{0} e^{-e^z+xz}dz + (1-e^{2\pi ix}) \int_1^{\infty} e^{-t}t^{x-1}dt + e^{2\pi ix} \int_0^{-\infty} e^{-e^z+xz}dz \ ,$$

$$= -\int_1^{\infty} e^{-t}t^{x-1}dt + (1-e^{2\pi ix})\int_1^{\infty} e^{-t}t^{x-1}dt - e^{2\pi ix}\int_0^{1} e^{-t}t^{x-1}dt \ ,$$

$$= - e^{2\pi ix} \int_0^{\infty} e^{-t}t^{x-1}dt \ .$$

Since the coefficient of the integral is a function of unit period and since the employment of paths parallel to BD in higher admissible strips serves to multiply the exponent of e by n, we are able to infer

that the most general solution of the original equation for values of x in the right half plane is

$$u(x) = \pi(x) \int_0^\infty e^{-t} t^{x-1} dt \ ,$$

where $\pi(x)$ is any function of unit periodicity. The solution of the equation for values of x in the left half plane for the poles at the negative integers is obtained from a well known generalization of this integral.*

Example 2. A more complex example is furnished by the equation,

$$\{4iz^3 - (1+2z^2)e^{z^2} + x\} \to u(x) = 0 \ ,$$

from which we have,

$$\varphi(z) = -ze^{z^2} + iz^4 + xz \ , \ \psi(z) = 0 \ .$$

Referring to the notation of (6.1) we find,

$$p = 2, q = 1, r = 4; G = H = K = 1; g = 0, h = \pi, k = \pi/2 \ .$$

Equation (6.2) then becomes,

$$R^2 = \{(2n-3)\pi/2 - \vartheta\}/\sin 2\vartheta \ .$$

If we replace $\sin 2\vartheta$ by $2 \sin\vartheta \cos\vartheta$ and let $x = R\cos\vartheta$, $y = R \sin\vartheta$, this may be written $2xy = (2n-3)\pi/2 - \vartheta$. Taking the tangent of each side and noting $\tan\vartheta = y/x$, we reduce this equation to

$$y = x \tan 2xy \ . \qquad (6.5)$$

Introducing the parameter $\mu = xy$, we obtain equation (6.5) in parametric form,

$$x = \pm(\mu \cot 2\mu)^{\frac{1}{2}} \ , \quad y = \pm(\mu \tan 2\mu)^{\frac{1}{2}} \ .$$

The graphical representation of this function is given in figure 5, the branches being numbered (1) for identification. From the fact that the derivative,

$$dx/dy = x[1 - 2(x^2+y^2)]/y[1 + 2(x^2+y^2]$$
$$= \cot 2\mu(\sin 4\mu - 4\mu)/(\sin 4\mu + 4\mu)$$

vanishes when $\mu = 0$, we find one maximum and one minimum at the points of tangency with the circle of radius $\frac{1}{2}$. The branches shown in the figure correspond to a range of μ from 0 to $5\pi/4$. Other similar branches, not shown, exist for larger values of the parameter.

A second partition of the plane is given by the equation $\vartheta = (2n-1)\pi/4$. These lines are marked (2) in the figure. A third par-

*See for example, Whittaker and Watson: *Modern Analysis*, Cambridge (1920), 3rd edition, p. 243.

tition is made by means of the lines, marked (3), given by $\vartheta = (n-1)\pi/4$.

Referring to equation (6.3) we see that two formal solutions of the equation exist provided the path of integration L is chosen to be either AOB or AOC.

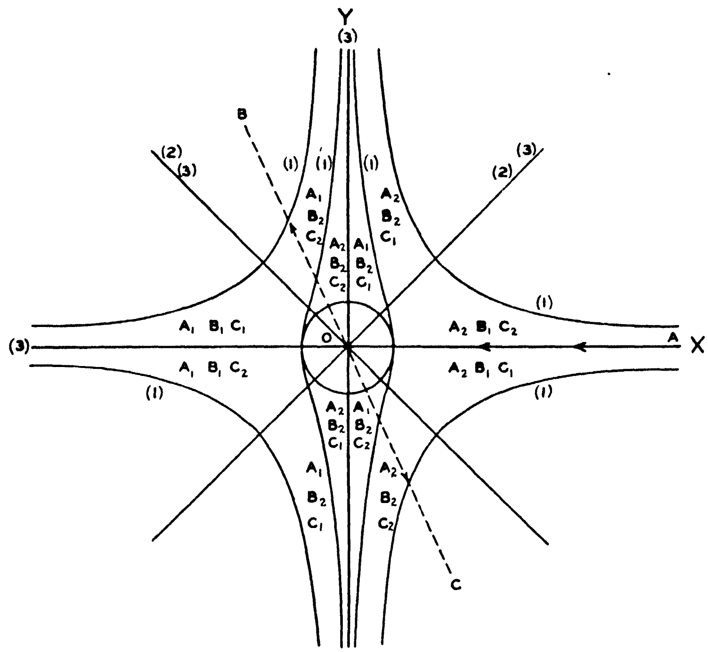

FIGURE 5

7. Perron's Theorem. We have seen from the examples of the preceding sections that the various solutions of the homogeneous equation

$$F(x, z) \to u(x) = 0 \ , \tag{7.1}$$

which appear in the form

$$u(x) = \int_L e^{xt}\, Y(t)\, dt \ , \tag{7.2}$$

where $Y(t)$ is a solution of the equation

$$H_0[Y(t)] = 0 \ , \tag{7.3}$$

may be conveniently classified according to the characteristics of the paths L of the integral. These paths of integration may be either

(1) Pochhammer circuits around two zeros of $A_p(t)$;

(2) Cauchy circuits around single zeros of $A_p(t)$;

(3) Laurent circuits (see section 7, chapter 2) around single zeros of $A_p(t)$;

(4) Circuits not included in the first three classes.

The first three types of circuits may be conveniently referred to as *regular* from their relationship to the regular singular points of equation (7.3); the other circuits are accordingly *irregular.*

If in equation (7.3) the function $A_p(t)$ has n zeros (multiplicity counted, but in no case exceeds p), and if the other coefficients, $A_i(t)$, satisfy the conditions of regularity,* then it is easily shown that possible paths for L are in general the following:

(1) $n - p$ Pochhammer circuits;

(2) s Cauchy circuits;

(3) $p - s$ Laurent circuits:

The regular singular case of (7.1) was studied by O. Perron [see *Bibliography*: Perron (3)], who stated the following theorem:

Theorem 3. If in the equation

$$A_0(t)X(t) + A_1(t)X'(t) + A_2(t)X''(t) + \cdots + A_p(t)X^{(p)}(t) = 0 , \qquad (7.4)$$

all the functions $A_i(t)$ are analytic and satisfy the conditions of regularity in the circle $|t| \leqq q$, and if within the domain considered

*We recall the following definition: A differential equation of the form

$$(x-a)^n y^{(n)}(x) + (x-a)^{n-1} P_1(x)\, y^{(n-1)}(x) + \cdots + (x-a)\, P_{n-1}(x)\, y'(x) + P_n(x)\, y(x) = 0 ,$$

where $P_1(x)$, $P_2(x)$, $\cdots$, $P_n(x)$ are analytic in the neighborhood of $x = a$, is said to have *a regular singular point* at $x = a$. Such an equation may be called a *Fuchsian equation* since the character of its solution was first indicated by L. Fuchs in 1866. The theorem in question states that, for a linear homogeneous differential equation of nth order to have n independent integrals which possess no singularities other than poles and branch points in the neighborhood of the point a, it is necessary and sufficient that the coefficients be of the form indicated above. The solutions of the equation are then of the form

$$y(x) = (x-a)^\lambda[g_1(x) + g_2(x)\log(x-a) + g_3(x)\log^2(x-a) + \cdots + g_{p-1}(x)\log^p(x-a)] ,$$

where $g_1(x)$, $g_2(x)$, $\cdots$, $g_{p-1}(x)$ are functions analytic at the point a.

The value λ is a root of the *indicial equation* (fundamental characteristic equation)

$$f(\lambda) = \lambda(\lambda-1)\cdots(\lambda-n+1) + P_1(a)\lambda(\lambda-1)\cdots(\lambda-n+2) + \cdots + P_{n-1}(a)\lambda + P_n(a) = 0 .$$

If the roots of this equation are distinct and if the difference between the real parts of any of them is not an integer, then the logarithmic singularity is absent from the solutions.

there exist $n \geqq p$ zeros (multiplicity considered and in all instances assumed less than or equal to p) of $A_p(t)$, then there will exist $n-p+s$ linearly independent integrals of (7.1) *the grades of which do not exceed q, where s is the number of linearly independent integrals of* (7.4) *analytic within the circle $|t| \leqq q$.**

Proof: By assumption the n zeros of $A_p(t)$ are regular singular points of (7.4), so for any one of them, for example, $t = a$, there will exist p linearly independent integrals of (7.4) of the form

$$X(t) = (t-a)^{\lambda} P(t) \ ,$$

where λ is a root of the indicial equation (see footnote to this section)

$$\lambda(\lambda-1) \cdots (\lambda-p+1) b_p + \lambda(\lambda-1) \cdots (\lambda-p+2) b_{p-1} + \cdots + b_0 = 0 \ , \qquad (7.5)$$

and $P(t)$ is analytic at $t = a$ (or has at most a logarithmic singularity).

Similarly the adjoint equation (7.3) will have a set of solutions of the form

$$Y(t) = (t-a)^{\mu} Q(t) \ ,$$

where $Q(t)$ is analytic at the point $t = a$ (or has at most a logarithmic singularity) and μ is a root of the equation

$$(\mu+1)(\mu+2) \cdots (\mu+p) b_p - (\mu+1)(\mu+2) \cdots (\mu+p-1) b_{p-1} + \cdots + (-1)^p b_0 = 0 \ . \qquad (7.6)$$

It is obvious upon substitution that the relationship between the roots of (7.5) and (7.6) is given by the equation

$$\lambda = -\mu - 1 \ .$$

Hence if $Y(t)$ has a pole of any order at $t = a$, the corresponding solution of (7.4) will be analytic at $t = a$.

Let us first consider the case where the real part of μ is negative, but where μ is not equal to a negative integer. Then the integral (7.2) will be a solution of (7.1) provided L is a Laurent circuit about the point $t = a$. In assuming the existence of a Laurent circuit we make use of the following theorem due to A. Liapounov:†

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*If $n - p$ is negative, then s integrals satisfying the conditions of the theorem will exist provided s is the number of linearly independent integrals of (7.4) analytic within the circle.

†Problème général de la stabilité du mouvement. *Annales de Toulous*, vol. 9 (series 2), (1907), pp. 203-474. (Originally published in 1893 in the *Communications of the Math. Soc. of Kharhov*).

If the coefficients $P_i(x)$ in the equation

$$P_0(x)y(x) + P_1(x)y'(x) + \cdots + P_{n-1}(x)y^{(n-1)}(x) + y^{(n)}(x) = 0$$

are bounded in the interval $(0, \infty)$, *then there exists a value* z *such that, if* $y(x)$ *is any solution of the equation, the following functions*:

$$y\, e^{zx}\, , y'\, e^{zx}\, , \cdots , y^{(n-1)}\, e^{zx} \;,$$

all tend to zero as $x \to \infty$.

If we assume that $n > p$, it is clear that there will be in general n such integrals, but of these only p are linearly independent. We now remove the restriction that the real part of μ is negative. If the real part of μ is positive or zero, then the Laurent circuit about $t = a$ is replaced by a line integral from $t = a$ to $t = \infty$. The path of integration avoids the singular points of the differential equation and approaches infinity in a sector which makes the integral converge.

The remaining $n - p$ integrals may now be found by combining the singular points in pairs to form Pochhammer circuits.

If μ is a negative integer, then the Laurent circuit is replaced by a Cauchy circuit about the singular point. If there are s such values of μ, and if the corresponding integrals have no other singularity than the pole, then s of the p Laurent circuits are replaced by Cauchy circuits. But if $Y(t)$ has no singularities other than poles then the corresponding adjoints, that is to say, the corresponding solutions of (7.4) will be analytic.

If $n < p$, then all the solutions of (7.1) will be derived from Laurent circuits, unless s of the characteristic values μ are negative integers, in which case s of the Laurent circuits are replaced by s Cauchy circuits.

We now examine the character of the solutions and we may readily show by the methods of chapter 5 (see problem 5, section 2, chapter 5) that the following are true:

If in the integral (7.2) L is a Cauchy circuit about the point $t = a$, then the grade of $u(x)$ is $|a|$; if L is a Laurent circuit about the two points $t = a$ and $t = b$, then the grade of $u(x)$ is equal to the larger of the two numbers $|a|$ and $|b|$; if L is a Laurent circuit about the point $t = a$, then the grade of $u(x)$ is infinite.

Combining these results we see that we have established the theorem.

8. The Non-homogeneous Equation of First Degree $(p = 1)$. Having in previous sections discovered conditions under which formal solutions of the homogeneous equation exist we now turn to the problem of solving the non-homogeneous equation. Before proceeding

to more general considerations we shall first examine the equation of first degree,

$$\sum_{n=0}^{\infty} (a_n + b_n x) u^{(n)}(x) = f(x) \cdot \tag{8.1}$$

We shall assume in the beginning that the functions,

$$A(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } B(z) = \sum_{n=0}^{\infty} b_n z^n ,$$

are analytic in a circle about the origin of radius not less than q, that we have $b_0 \neq 0$, and that $f(x)$ is an infinitely differentiable function but not necessarily of finite grade.

Equations (2.4) and (2.5) thus become,

$$A(z)X(z) + B(z)X'(z) = 0 ,$$

$$A(z)Y(z) - \frac{d}{dz}\{B(z)Y(z)\} = 0 .$$

The solutions of these equations which satisfy the conditions:

$$X(0) = 1 , \quad X(z)Y(z) = 1/B(z) ,$$

are clearly,

$$X(z) = e^{-\int_0^z \{A(z)/B(z)\}dz}, \quad Y(z) = e^{\int_0^z \{A(z)/B(z)\}dz} /B(z) .$$

Referring to equation (2.23) we see that the resolvent generatrix becomes,

$$X(x,z) = e^{-xz}X(z)U(x) + e^{-xz}X(z) \int_a^z e^{xt}Y(t)\,dt , \tag{8.2}$$

where $U(x)$ is a solution of the homogeneous equation and a is any point such that

$$\lim_{z=a} \{H(x,z) \equiv e^{xz} e^{\int_0^z \{A(z)/B(z)\}dz}\} = 0 .$$

As an example consider the equation,

$$u(x+1) - xu(x) = f(x) , \tag{8.3}$$

which in the notation of this paper becomes,

$$(e^z - x) \rightarrow u(x) = f(x) .$$

The values of $X(z)$ and $Y(z)$ are determined from the equations,

$$X'(z) - e^z X(z) = 0 ,$$

$$Y'(z) + e^z Y(z) = 0 ,$$

$$X(0) = 1 , \quad X(z)Y(z) = -1 ,$$

whence we find: $X(z) = e^{e^z}/e$, $Y(z) = - e \cdot e^{-e^z}$.

The general solution of the homogeneous equation has been shown in example 1 of section 6 to be, $u(x) = \pi(x)\Gamma(x)$, where $\pi(x)$ is any function of unit periodicity.

Obviously the point a may be either $+\infty$ or $-\infty$. Choosing the latter and introducing the new variables $u = e^t$, $p = e^z$, we have,

$$X(x,z) = e^{-xz}X(z)\pi(x)\Gamma(x) - p^{-x}\int_0^p u^{x-1}e^{p-u}du \; .$$

The second integral is recognized as the incomplete gamma function which has the expansion,

$$\gamma(p) = \int_0^p u^{x-1}e^{p-u}du = p^x/x + p^{x+1}/x(x+1) + p^{x+2}/x(x+1)(x+2) + \cdots \; .$$

Since we have the operational identities,

$$p^n \to f(x) = f(x+n) \quad \text{and} \quad e^{-xz}X(z) \to f(x) = \text{a constant} \; ,$$

the solution of equation (8.3) becomes,

$$u(x) = \pi(x)\Gamma(x) - \{f(x)/x + f(x+1)/x(x+1) + f(x+2)/x(x+1)(x+2) + \cdots\} \; .^* \tag{8.4}$$

If a second expansion of the incomplete gamma function be employed, we obtain,

$$p^{-x}\gamma(p) = e^p p^{-x}\Gamma(x) - \{1/p + (x-1)/p^2 + (x-1)(x-2)/p^3 + \cdots\} \; .$$

Operating upon $f(x)$ with this new operator we get,

$$u(x) = \pi(x)\Gamma(x) - \{f(0) + f(1)/1! + f(2)/2! + \cdots\}\Gamma(x) + \{f(x-1) + (x-1)f(x-2) + (x-1)(x-2)f(x-3) + \cdots\} \; .^\dagger \tag{8.5}$$

This solution is usually though not always divergent. An example is furnished by the equation,

$$u(x+1) - xu(x) = \Gamma(x+1)e^{x+1} \; .$$

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*This is a well known result due to H. Mellin: Zur Theorie der linearen Differenzengleichungen erster Ordnung. *Acta Mathematica*, vol. 15 (1891), pp. 317-384.

†Since we have $p^{-x}e^p \to f(x) = p^{-x} \to e^p \to f(x) = p^{-x} \to \{1 + p + p^2/2! + \cdots\} \to f(x) = p^{-x} \to \{f(x) + f(x+1) + f(x+2)/2! + \cdots\} = f(0) + f(1) + f(2)/2! + \cdots$.

If we merge the divergent coefficient of the second $\Gamma(x)$ in (10.5) with the arbitrary function $\pi(x)$ we find,

$$f(x-1) + (x-1)f(x-2) + (x-1)(x-2)f(x-3) + \cdots$$
$$= \Gamma(x)e^x + \Gamma(x)e^{x-1} + \Gamma(x)e^{x-2} + \cdots$$
$$= \Gamma(x)e^x e/(e-1) \ ,$$

and hence obtain,

$$u(x) = \pi(x)\Gamma(x) + \Gamma(x)e^{x+1}/(e-1) \ .$$

Borel summability will also serve to rescue a solution in many divergent cases. If as before we merge the constant coefficient of $\Gamma(x)$ with the arbitrary function of unit periodicity, $\pi(x)$, and apply the Borel integral to (8.5) we obtain,

$$u(x) = \pi(x)\Gamma(x) + \int_0^\infty e^{-t}\{f(x-1) + (x-1)f(x-2)t/1! + (x-1)(x-2)f(x-3)t^2/2! + \cdots\}\,dt \ .$$

An example is furnished by the equation,

$$u(x+1) - xu(x) = 1 \ ,$$

for which we obtain,

$$u(x) = \pi(x)\Gamma(x) + \int_0^\infty e^{-t}(1+t)^{x-1}dt \ .$$

9. The Non-homogeneous Equation — Expansion of the Resolvent in Powers of z. Three different forms for the resolvent generatrix can be given if the expansion is made: (1) in powers of z, (2) in inverse factorials in x, (3) in negative powers of x. The first of these expansions has been extensively studied by Hilb and his results will now be attained through the methods of the present chapter. Hilb's actual method of solution by means of the theory of bilinear forms is given in section 13.

Neglecting for the moment the contribution made by the homogeneous equation let us consider the operator,

$$X_0(x,z) = e^{-xz}\,\Phi(z) \ ,$$

where we employ the abbreviation

$$\Phi(z) = X(z) \int_a^z e^{xt}Y(t)\,dt \ .$$

The expansion of this operator in powers of z is obviously

$$X_0(x,z) = \varphi_0 + \varphi_1 z + \varphi_2 z^2/2! + \cdots + \varphi_m z^m/m! + \cdots \ , \quad (9.1)$$

where we abbreviate,

$$\psi_m(x) = \frac{d^m}{dz^m} \left|\{e^{-xz}\,\Phi(z)\}\right|_{z=0}$$

$$= \Phi^{(m)}(0) - mx\,\Phi^{(m-1)}(0) + m(m-1)x^2\,\Phi^{(m-2)}(0)/2! - \cdots \pm x^m\Phi(0) \ .$$

Explicit calculation then gives,

$$\psi_0(x) = X(0)\{-\int_0^a e^{xt}Y(t)\,dt\} = X(0)u_0(x) \ ,$$

$$\psi_1(x) = \Phi'(0) - x\Phi(0)$$
$$= \{X'(0) - xX(0)\}\,u_0(x) + X(0)Y(0) \ ,$$

$$\psi_2(x) = \Phi''(0) - 2x\Phi'(0) + x^2\Phi(0)$$
$$= \{X''(0) - 2xX'(0) + x^2X(0)\}\,u_0(x)$$
$$+ 2Y(0)\{X'(0) - xX(0)\} + X(0)\{Y'(0) + xY(0)\} \ ,$$

If we adopt the notation,

$$[X]^m = X^{(m)}(0) - mxX^{(m-1)}(0) + m(m-1)x^2X^{(m-2)}(0)/2! - \cdots \pm x^mX(0) \ , \tag{9.2}$$

$$\{Y\}^m = Y^{(m)}(0) + mxY^{(m-1)}(0) + m(m-1)x^2Y^{(m-2)}(0)/2! + \cdots + x^mY(0) \ ,$$

the general term may be conveniently written,

$$\psi_m(x) = u_0(x)\,[X]^m + m\{Y\}^0[X]^{m-1} + m(m-1)\{Y\}[X]^{m-2}/2! + \cdots + \{Y\}^{m-1}[X]^0 \ . \tag{9.3}$$

If we place in these symbols the explicit values of $X(z)$ and $Y(z)$ as calculated in the preceding section and represent by $A^{(i)}$ and $B^{(i)}$ the numerical values $A^{(i)}(0)$ and $B^{(i)}(0)$, this solution may be exhibited in terms of the coefficients of the original equation. We thus obtain to four terms the expansion,

$$\begin{aligned} X_0(x,z) = u_0(x) &+ \{1 - (A + Bx)u_0(x)\}z/B + \{-A - B' \\ &- Bx + (A^2 - BA' + AB' + 2xAB \\ &+ x^2B^2)u_0(x)\}z^2/B^2 2! \qquad (9.4) \\ &+ \{A^2 - 2BA' + 3AB' - BB'' + 2B'^2 \\ &+ (BB' + 2AB)x + B^2x^2 \end{aligned}$$

$$+ [3AA'B - 3A^2B' + 2BB'A' - B^2A''$$
$$+ ABB'' - 2AB'^2 - A^3$$
$$- 3x(A^2B - B^2A' + ABB') - 3AB^2x^2$$
$$- B^3x^3]u_0(x)\}z^3/B^3 3! + \cdots ,$$

where we use the abbreviation

$$u_0(x) = - \int_0^a e^{xt} Y(t)\,dt \ .$$

The complete operator is obtained by adding $U(x)e^{-xz}X(z)$ to this expansion.

As an example consider the equation,

$$u(x+1) - xu(x) = x^2 \ . \tag{9.5}$$

Since we have

$$A(z) = e^z , \quad B(z) = -1, \quad U(x) = \pi(x)\Gamma(x) \ ,$$

$$u_0(x) = - \int_{-\infty}^0 e^{xt+1-e^t} dt = - \int_0^1 u^{x-1}e^{1-u} du \ ,$$

the operator becomes

$$X(x,z) = \pi(x)\Gamma(x)e^{-xz}X(z) - \int_0^1 u^{x-1}e^{1-u}du\,\{1 + (1-x)z$$
$$+ (2 - 2x + x^2)z^2/2 + (5 - 6x + 3x^2 - x^3)z^3/6 + \cdots\}$$
$$- \{z + (1-x)z^2/2 + (3 - 2x + x^2)z^3/6 + \cdots\} \ .$$

Applying this operator to $f(x) = x^2$ and merging the constant $e^{-xz}X(z) \to x^2$ with $\pi(x)$ we obtain for the solution of (9.5) the function,

$$u(x) = \pi(x)\Gamma(x) - 1 - x - 2 \int_0^1 u^{x-1}e^{1-u} du \ .$$

We shall refer to equation (9.1) as Hilb's solution of the differential equation since the ψ_m may be identified with those obtained by this writer in a different manner. Consideration of the convergence properties of the operator will be postponed to another section.

10. Expansion of the Resolvent in a Series of Inverse Factorials. A second form of expansion for the operator of the last section may be obtained as follows:

Let us integrate by parts the integral in the operator,

$$X_0(x,z) = e^{-xz}X(z) \int_a^z e^{xt} Y(t)\,dt \ .$$

Recalling that $\lim_{t=a} e^{xt}Y(t) = 0$, we thus obtain,

$$X_0(x,z) = X(z)Y(z)/x - \{e^{-xz}X(z)/x\} \int_a^z e^{xt}Y'(t)\,dt \; .$$

Writing the integrand in the form $e^{(x+1)t}Y'(t)e^{-t}$ and extending the original assumption as to the value of a to include the limit, $\lim_{t=a} e^{xt}Y'(t) = 0$, we obtain the following formula by means of a second integration by parts:

$$X_0(x,z) = X(z)Y(z)/x - X(z)Y'(z)e^{-z}/x(x+1)$$
$$+ \{e^{-xz}X(z)/x(x+1)\} \int_a^z e^{(x+1)t} \frac{d}{dt}\{Y'(t)e^{-t}\}\,dt \; .$$

Employing the abbreviation,

$$w_n(z) = \{\frac{d}{dz} w_{n-1}(z)\}e^{-z} \; , \quad w_0(z) = Y(z) \; , \qquad (10.1)$$

and assuming that we have,

$$\lim_{t=a} e^{(x+n)t} w_n(t) = 0 \; , \qquad (10.2)$$

for all positive values of n, we are able to continue the process outlined above and thus derive the following factorial expansion:

$$X_0(x,z) = X(z)\{w_0(z)/x - w_1(z)e^z/x(x+1)$$
$$+ w_2(z)e^{2z}/x(x+1)(x+2)$$
$$- w_3(z)e^{3z}/x(x+1)(x+2)(x+3) + \cdots \; . \qquad (10.3)$$

To obtain the complete operator we must, of course, add to this the operator $e^{-xz}X(z)U(x)$ where $U(x)$ is a solution of the homogeneous equation.

It is of some interest in particular applications to obtain the condition under which (10.1) is a finite series. An examination of the function $w_n(z)$ shows that we have,

$$w_1(z) = e^{-z}Y'(z) = e^{-z}D \rightarrow Y(z) \; ,$$

where D represents the operator d/dz,

$$w_2(z) = e^{-2z}(D-1)D \rightarrow Y \; ,$$

$$w_3(z) = e^{-3z}(D-2)(D-1)D \rightarrow Y \; ,$$

and in general,

$$w_n(z) = e^{-nz}(D-n+1)\cdots(D-1)D \rightarrow Y \; .$$

If we note from explicit values that

$$(D \rightarrow Y)/Y = \{A(z) - B'(z)\}/B(z) = R(z) \; ,$$

we observe that we may write $w_n(z)/Y(z) = e^{-nz}R_n(z)$, where $R_n(z)$ is expressible in terms of $R(z)$ and its n—1 derivatives. For example,

$$\{(D-1)D \to Y\}/Y = (RY' + R'Y - RY)/Y = R^2 + R' - R \;,$$

$$\{(D-2)(D-1)D \to Y\}/Y$$
$$= R^3 + 3R'R - 3R^2 - 3R' + R'' + 2R \;.$$

It is clear that these expansions furnish a criterion for the termination of the factorial expansion of $X(x,z)$ since we need merely assume that the coefficients $A(z)$ and $B(z)$ satisfy the equation,

$$(D - n + 1) \cdots (D-1)D \to Y(z) = 0 \;.$$

Thus we reach the conclusion that *the expansion of the operator* (10.3) *will terminate with the* nth *term provided* $A(z)$ *and* $B(z)$ *satisfy the equation,*

$$(D - n + 1) \cdots (D-1)D \to Y(z) = 0 \;, \qquad (10.4)$$

which expresses a differential relationship between them.

The explicit derivation of the equations corresponding to the first two cases will be of interest, the first case in particular applying to the theory of singular operators.

The condition $D \to Y = 0$ is equivalent to $A(z) - B'(z) = 0$. Hence the equation,

$$\{\varphi'(z) + \varphi(z)x\} \to u(x) = f(x) \;, \qquad (10.5)$$

where $\varphi(z)$ is an infinite operator with constant coefficients, has for its resolvent generatrix the function,

$$X_0(x,z) = 1/[\varphi(z)x] \;.$$

The condition: $(D - 1)D \to Y = 0$ leads to the equation,

$$R' - R + R^2 = 0 \;.$$

The solution of this equation is,

$$R(z) = \lambda e^z/(1 + \lambda e^z) \;,$$

where λ is an arbitrary parameter. From this we obtain the relationship,

$$A(z) = B(z)\{\lambda e^z/(1 + \lambda e^z)\} + B'(z) \;,$$

and hence we may conclude that the equation,

$$\{\varphi(z)\lambda e^z/(1 + \lambda e^z) + \varphi'(z) + x\varphi(z)\} \to u(x) = f(x) \;,$$

has for its resolvent generatrix the function,

$$X_0(x,z) = 1/[B(z)x] - \lambda e^z/[B(z)(1 + \lambda e^z)x(x+1)] \;.$$

11. Expansion of the Resolvent in Negative Powers of x. A third expansion of the resolvent generatrix is possible if we assume that a value a exists such that

$$\lim_{t=a} e^{-xt}Y^{(n)}(t) = 0$$

for all values of n.

Under this assumption we may apply the integration by parts formula to the resolvent:

$$X_0(x,z) = e^{-xz}X(z) \int_a^z e^{xt}Y(t)\,dt$$

and thus obtain,

$$\begin{aligned} X_0(x,z) &= X(z)Y(z)/x - \{e^{-xz}X(z)/x\} \int_a^z e^{xt}Y'(t)\,dt \ , \\ &= X(z)\{Y(z)/x - Y'(z)/x^2 \\ &\qquad + \cdots + (-1)^{n-1}Y^{(n-1)}(z)/x^n \\ &\qquad + (-1)^n\{e^{-xz}/x^n\} \int_a^z e^{xt}Y^{(n)}(t)\,dt \ , \\ &= X(z)\{Y(z)/x - Y'(z)/x^2 \\ &\qquad + Y''(z)/x^3 + \cdots\} \ . \end{aligned} \tag{11.1}$$

An interesting special case is obtained when $Y(z)$ is a polynomial of degree m since the resolvent generatrix then consists of a finite number of terms,

$$\begin{aligned} X_0(x,z) &= X(z)\{Y(z)/x - Y'(z)/x^2 + Y''(z)/x^3 \\ &\qquad - \cdots \pm Y^{(m)}(z)/x^{m+1}\} \ . \end{aligned} \tag{11.2}$$

The special form of this operator immediately suggests a simple way of obtaining the solutions of the corresponding homogeneous equation. If we assume that $A(z)$ is of the form

$$A(z) = P'(z)B(z)/P(z) + B'(z) \ ,$$

where $P(z)$ is any polynomial, we obtain $Y(z)$ in the desired form, namely $Y(z) = P(z)$.

Since $X(z) = 1/[Y(z)B(z)]$ has poles at the zeros of $Y(z)B(z)$ we can apply the method of Cauchy circuits as explained in theorem 1 of chapter 6. Hence designating by $X_0^{(i)}(x,z)$ a Laurent expansion of $X_0(x,z)$ regarded as a function of z within one of the annular regions bounded by concentric circles through successive zeros of $Y(z)B(z)$ and by $X_0(x,z)$ the expansion of the operator in the region about the origion, we obtain a solution of the homogeneous equation from the function,

$$u(x) = \{X_0^{(i)}(x,z) - X_0(x,z)\} \rightarrow f(x) \ , \tag{11.3}$$

where $f(x)$ is arbitrary to within the limits of existence of the right hand member.

An example of this method is furnished by the *Bessel equation,*

$$\{x^2z^2 + xz + (x^2 - n^2)\} \rightarrow y(x) = 0 \ ,$$

for the special case where n is half an odd integer.

By means of the transformation $y = ux^n$, this equation becomes,

$$\{xz^2 + (2n+1)z + x\} \rightarrow u(x) = 0 \ . \qquad (11.4)$$

We then find $Y(z) = (1 + z^2)^{n-\frac{1}{2}}$ which is a polynomial when n is half an odd integer. For $n = \frac{1}{2}$ we obtain for $X_0(x,z)$ the function,

$$X_0(x,z) = 1/[x(1+z^2)] = (1/2x)\{1/(1+iz) = 1/(1-iz)\} \ .$$

Since we have two zeros on the unit circle in this case we must slightly modify the method outlined above by considering each term of the expression in brackets separately. We specialize the function in (11.3) by setting $f(x) = 1$. Then since we have,

$$1/(1 + iz) \rightarrow 1 = (-i/z + 1/z^2 + i/z^3 - 1/z^4 - \cdots) \rightarrow 1$$
$$= 1 - \cos x - i \sin x \ ,$$

$$1/(1 + iz) \rightarrow 1 = (1 - iz - z^2 + iz^3 + \cdots) \rightarrow 1 = 1 \ ,$$

we obtain as one solution of (11.4) the function,

$$u_1(x) = (-\cos x - i \sin x)/(2x) \ .$$

Similarly since we have $1/(1-iz) \rightarrow 1 = 1 - \cos x + i \sin x$ for one expansion and 1 for the other we get as a second solution,

$$u_2(x) = (-\cos x + i \sin x)/(2x) \ .$$

Linear combinations of these obviously lead to the solutions in customary form,

$$v_1(x) = A\{\cos x\}/x \ , \quad v_2(x) = B\{\sin x\}/x \ .$$

For $n = 3/2$ we have the operator,

$$X_0(x,z) = 1/x(1+z^2) - 2z/x^2(1+z^2)^2 + 2/x^3(1+z^2)^2 \ ,$$
$$= S/(1+iz)^2 + T/(1+iz) + s/(1-iz)^2 + t/(1-iz) \ ,$$

where we abbreviate,

$$S = -i/2x^2 + 1/2x^3 \ , \qquad T = 1/2x + 1/2x^3 \ ,$$
$$s = i/2x^2 + 1/2x^3 \ , \qquad t = 1/2x + 1/2x^3 \ .$$

Proceeding as before and regarding the operators separately, we obtain,

$$u_1(x) = -\cos x/x^3 - \sin x/x^2 + i(\cos x/x^2 - \sin x/x^3) \ ,$$

$$u_2(x) = -\cos x/x^3 - \sin x/x^2 + i(-\cos x/x^2 + \sin x/x^3) \ .$$

As before linear combinations of these yield the solutions,

$$v_1(x) = A(\cos x/x^3 + \sin x/x^2) \ , v_2(x) = B(\cos x/x^2 - \sin x/x^3) \ .$$

12. Extension to Include Solutions for the General Case. It will now be desirable to formulate the results exhibited in the last three sections for the differential equation of first degree ($p = 1$) so that they will include the general case. In order to achieve this end we return to the statements of theorems 1 and 2 from which we shall derive the following theorem:

Theorem 4. If there exists a point a in the complex plane for which we have,

$$\lim_{t=a} H_j[Y_i](t)e^{xt} = 0 \ , \qquad j = i, i+1, \cdots, p \ ;$$

$$\lim_{t=a} I_k[Y_i](t)e^{xt} = 0 \ , \qquad k = 1, 2, \cdots, i-1 \ ,$$

then the resolvent (2.23) *can be expanded in the following series:*

$$\text{(A)} \qquad X(x,z) = U(x)e^{-xz}X(z) + \sum_{i=1}^{\infty} \psi_m(x)z^m/m! \ , \quad (12.1)$$

where we abbreviate

$$\psi_m(x) = u_m(x) - m\,x\,u_{m-1}(x) + m(m-1)x^2u_{m-2}(x)/2! \\ - \cdots \pm x^m u_0(x) \ , \qquad m \leqq p-1 \ ,$$

$$\psi_m(x) = \sum_{i=1}^{p} \{u_{i-1}\}(x)\,[X_i]^m + m\{Y_i\}^0[X_i]^{m-1} \\ + m(m-1)\{Y_i\}[X_i]^{m-2}/2! + \cdots + \{Y_i\}^{m-1}[X_i]^0 \ , \\ m > p-1 \ ,$$

in which we write,

$$u_i(x) = -\int_a^0 e^{xt}Y_i(t)\,dt \ ,$$

$$[X_i]^m = X_i^{(m)}(0) - mxX_i^{(m-1)}(0) \\ + m(m-1)x^2X_i^{(m-2)}(0)/2! - \cdots \pm x^mX_i(0) \ ,$$

$$\{Y_i\}^m = Y_i^{(m)}(0) + mxY_i^{(m-1)}(0) \\ + m(m-1)x^2Y_i^{(m-2)}(0)/2! + \cdots + x^mY_i(0) \ .$$

If the additional conditions are fulfilled,

$$\lim_{t=a} e^{xt} Y_i^{(m)}(t) = 0 \ , \qquad m = 0, 1, 2, \cdots$$

then the resolvent has also the expansions,

$$\begin{aligned} X(x,z) &= U(x) e^{-xz} X(z) \\ \text{(B)} \qquad & + \{ (-1)^{p-1} e^{z(p-1)}/x(x+1) \cdots (x+p-1) \} \\ & \times \{ w_{p-1}(z,z) - w_p(z,z) e^z/(x+p) \\ & \quad + w_{p+1}(z,z) e^{2z}/(x+p)(x+p+1) - \cdots \} \ , \end{aligned} \tag{12.2}$$

where we abbreviate,

$$\begin{aligned} w_n(z,t) &= \left\{ \frac{\partial}{\partial t} w_{n-1}(z,t) \right\} e^{-t}, \ w_0(z,t) = W(z,t) \\ &= \sum_{i=1}^{p} X_i(z) Y_i(t) \ . \end{aligned} \tag{12.3}$$

$$\begin{aligned} X(x,z) &= U(x) e^{-xz} X(z) + \{ (-1)^{p-1} x^p \} \\ \text{(C)} \qquad & \times \{ W_{p-1}(z,z) - W_p(z,z)/x + W_{p+1}(z,z)/x^2 - \cdots \} \ , \end{aligned} \tag{12.4}$$

in which we use the notation,

$$\begin{aligned} W_n(z,t) &= \frac{\partial}{\partial t} W_{n-1}(z,t) \ , \\ W_0(z,t) &= W(z,t) = \sum_{i=1}^{p} X_i(z) Y_i(t) \ . \end{aligned} \tag{12.5}$$

Proof. The derivation of the formulas contained in this theorem proceeds from the formal expansion of the operator (2.23). If

$$X_i(z) e^{-xz} \text{ and } \int_a^z e^{xt} Y_i(t)\, dt$$

be expanded in power series in z and account taken of the defining conditions (2.9) and (2.14) we obtain (12.1). (A) is thus the power series solution of the generatrix equation (2.2).

Assuming next that a point exists such that $\lim_{t=a} e^{xt} Y_i^{(m)}(t) = 0$, we can perform the following integration by parts:

$$\begin{aligned} \int_a^z e^{xt} W(z,t)\, dt &= e^{xz} W(z,z)/x - \int_a^z e^{(x+1)t} \{\partial W(z,t)/\partial t\} e^{-t} dt/x \\ &= e^{xz} W(z,z)/x - e^{xz} e^z \{e^{-t} \partial W(z,t)/\partial t\}_{t=z}/x(x+1) \\ &\quad + \int_a^z e^{(x+2)t} \frac{\partial}{\partial t} \{[\partial W(z,t)/\partial t] e^{-t}\} e^{-t} dt/x(x+1) \ , \end{aligned}$$

$$\int_a^z e^{xt} W(z,t)\,dt = e^{xz}\{w_0(z,z)/x - w_1(z,z)e^z/x(x+1)$$

$$+ w_2(z,z)e^{2z}/x(x+1)(x+2) - \cdots\} ,$$

where we use the abbreviations (12.3).

From (2.8) however, we observe that the first $p - 2$ terms are identically zero and we thus obtain expression (12.2).

Finally the operator (C) is derived by means of a similar integration by parts and the employment of (2.8) to eliminate the first $p - 2$ terms.

It will be of interest to consider an example:

$$\{2(1 + z^2 + z^4 + z^6 + \cdots)$$

$$+ 4x(z + z^3 + z^5 + \cdots) - x^2\} \to u(x) = f(x) .$$

The homogeneous case has already been dsicussed in example 2, section 4, where the solutions:

$$[U_1(x) = e^x(4/x^3 - 6/x^4) , \; U_2(x) = - e^{-x}(4/x^3 + 6/x^4)] ,$$

were found. From the resolvent,

$$2X + 4zX' + (z^2 - 1)X'' = 0 ,$$

and the adjoint resolvent,

$$[2/(1-z^2)]Y - (d/dz)[4zY/(1-z^2)] - Y'' = 0 ,$$

with account taken of conditions (2.9) and (2.14), we derive the fundamental sets of solutions:

$$X_1(z) = 1/(1-z^2) , \qquad X_2(z) = z/(1-z^2) ,$$

$$Y_1(z) = z^3 - z , \qquad Y_2(z) = -z^2 + 1 .$$

Making use of (12.1) and neglecting the contribution of the first term of the operator which is easily constructed from the values of $U_1(x)$ and $U_2(x)$ given above, we can write,

$$X_0(x,z) = \int_{-\infty}^{z} e^{xt}\{(t^2-1)(z-t)/(1-z^2)\}dt , \qquad R(x) > 0 ,$$

$$=\{-1/x^2 + 6/x^4 + z^2/x^2 - 4z/x^3\}/(1-z^2) . \quad (12.6)$$

This is obviously expansion (C) of theorem 4; expansion (A) is easily made through multiplication of the expression in braces by the series $1 + z^2 + z^4 + \cdots$.

In order to obtain series (B) we resort to the following calculation:

$$w_0(z,t) = (t^2-1)(z-t)/(1-z^2) \quad ;$$

$$w_0(z,z) = 0 \ , \ w_0'(z,z) = 1 \ ,$$

$$w_0''(z,z) = -4z/(1-z^2) \ ,$$

$$w_0'''(z,z) = -6/(1-z^2) \ , \ w_0^{(n)}(z,z) = 0 \ , \ n > 3 \ .$$

The derivatives are here taken with respect to t and then z is substituted for t in the resulting functions.

From these values we next compute:

$$w_1(z,z) = e^{-z} \ , \ w_2(z,z) = e^{-2z}(z^2 - 4z - 1)/(1-z^2) \ ,$$

$$w_3(z,z) = e^{-3z}(-2z^2 + 12z - 4)/(1-z^2) \ ,$$

$$w_4(z,z) = e^{-4z}(6z^2 - 44z + 30)/(1-z^2) \ ,$$

$$w_5(z,z) = e^{-5z}(-24z^2 + 200z - 186)/(1-z^2) \ , \cdots \ .$$

Adopting the notation: $x^{(-n)} = 1/[x(x+1)(x+2)\cdots(x+n-1)]$, we then obtain the resolvent operator in the form

$$\begin{aligned} X_0(x,z) = \{ & (-1+z^2)x^{(-2)} + (-1-4z+z^2)x^{(-3)} \\ & + (4-12z+2z^2)x^{(-4)} + (30-44z+6z^2)x^{(-5)} \\ & + (186-200z+24z^2)x^{(-6)} + \cdots \}/(1-z^2) \ . \quad (12.7) \end{aligned}$$

This expression is easily verified by substituting in (12.6) the following identity:

$$1/x^{n+1} = \sum_{s=0}^{\infty} (-1)^s {}_{s+n-1}C_s B_s^{(s+n)} / [x(x+1)(x+2)\cdots(x+s+n)] \ ,^*$$

where $B_s^{(m)}$, the Bernoulli number of order m and degree s, is obtained from the series:

$$\{t/(e^t - 1)\}^m = \sum_{s=0}^{\infty} t^s B_s^{(m)}/s! \ , \qquad |t| < 2\pi \ .$$

For values interesting to us here $B_s^{(m)}$ is found explicitly to be,†

$$B_0^{(m)} = 1 \ , \quad B_1^{(m)} = -m/2 \ ,$$

$$B_2^{(m)} = m(3m-1)/12 \ , \quad B_3^{(m)} = -m^2(m-1)/8 \ ,$$

$$B_4^{(m)} = m(15m^3 - 30m^2 + 5m + 2)/240 \ ,$$

$$B_5^{(m)} = -m^2(m-1)(3m^2 - 7m - 2)/96 \ .$$

Substituting the explicit expansions of $1/x^2$, $1/x^3$, and $1/x^4$ in (12.6) we obtain series (12.7).

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*See N. E. Nörlund: *Differenzenrechnung*, Berlin, (1924), p. 243.

†Nörlund: *loc. cit.*, p. 146.

PROBLEMS

1. Solve the equation

$$u(x+1) - x\,u(x) = \Gamma(x+1) \ . \qquad \text{(Boole)}$$

2. Discuss the equation

$$(6 + x^2)\,u - 2x\,u' + 6u'' - 2x\,u^{(3)} + 6u^{(4)} - 2x\,u^{(5)} + 6u^{(6)} - 2x\,u^{(7)} + \cdots = f(x) \ .$$

Note that a particular set of solutions of the specialized resolvent (2.4) is given by $P_2(z) = \frac{1}{2}(3z^2-1)$, $Q_2(z) = \frac{1}{2}\,P_2(z)\,\log[(z+1)/(z-1)] - 3z/2$.

3. Solve the equation

$$(a\cos z + x) \to u(x) = f(x) \ .$$

4. Obtain the solution of the equation in problem 2 when $f(x) = x^3$.

5. Employing the method of the illustrative problem of section 11, compute the explicit values for the Bessel functions of orders $n = 5/2$ and $n = 7/2$.

13. Other Methods of Inversion — Limited Bilinear Forms — Appell Polynomials. In this section we shall consider two methods of solving equation (1.1). The first of these, the method of limited bilinear forms, is due to E. Hilb (see section 6, chapter 1), who has thus furnished an excellent example of the practical application of the theorems relating to these forms. Because of the complexities of the general problem the discussion will be limited to the case $p = 1$ and the reader is referred to Hilb's paper for other details.

The second method, that of Appell polynomials, is due to S. Pincherle who applied it to the case $p = 0$, that is to say, the case of differential equations of infinite order with constant coefficients. The extension to the general case was effected by I. M. Sheffer. A brief account, together with a bibliography, of Appell polynomials is given in section 6, chapter 1.

The Method of Limited Bilinear Forms.

We shall consider the solution of equation (1.1) for the case $p = 1$, that is, of the equation

$$\sum_{n=0}^{\infty} (a_n + b_n\,x)\,u^{(n)}(x) = f(x) \ , \tag{13.1}$$

where the functions

$$A(z) = \sum_{n=0}^{\infty} a_n\,z^n \ , \quad B(z) = \sum_{n=0}^{\infty} b_n\,z^n$$

are analytic in a circle about the origin of radius greater than q, and where $b_0 \neq 0$.

If equation (13.1) is differentiated an infinite number of times, an infinite system of linear equations is obtained, which is a special case of the general system discussed in section 13 of chapter 4. Referring to (13.5) of chapter 4, we see that the matrix of the coefficients of the unknowns $u^{(n)}(x)$ is given by

$$\| a_{n-k} + (k-1)\, b_{n-k+1} + b_{n-k} x \| , \qquad (13.2)$$

where a_m and b_m are zero for $m < 0$.

If the kth equation of the system is divided by $[k-1]\, q^{k-1}$, where $[k-1] = k-1$ for $k \geqq 2$, $[1-1] = 1$, and if $u^{(n)}(x)$ is replaced by $q^n\, v^{(n)}(x)$, then (13.2) becomes

$$A = \| a_{nk} \| = \left\| \frac{[a_{n-k} + (k-1)\, b_{n-k+1} + b_{n-k}\, x]\; q^{n-1}}{[k-1]\; q^{k-1}} \right\| .$$

The problem resolves itself essentially into three parts: (1) to show that the bilinear form

$$(A) = \sum_{n,k=1}^{\infty} a_{nk}\, x_n\, y_k$$

is limited; (2) to find a matrix Φ which satisfies the equation

$$\Phi \cdot A = A \cdot \Phi = I \; ; \qquad (13.3)$$

and (3) to show that the bilinear form

$$(\Phi) = \sum_{k,n=1}^{\infty} \varphi_{nk}\, x_k\, y_n$$

is limited.

(1) *To show that* (A) *is limited.*

This first problem is easily disposed of by means of theorem 11, chapter 3. Thus replacing $n-k$ by m we consider

$$|(A)| \leqq \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} | a_m + (k-1)\, b_{m+1} + b_m\, x | \left[\frac{q^m}{k-1} \right] | x_m | \, | y_{m+k} | \, .$$

Since $[k-1] \geqq 1$, we obtain the inequality

$$|(A)| \leqq \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \{ | a_m | + | b_{m+1} | + | b_m | \, | x | \}\, q^m \, | x_m | \, | y_{m+k} | \, .$$

But since $\sum_{m=0}^{\infty} a_m\, z^m$ and $\sum_{m=0}^{\infty} b_m\, z^m$ converge absolutely within the circle of radius q, theorem 11, chapter 3 applies and the matrix A is limited.

(2) *To find the matrix* Φ .

If we designate by Φ the matrix

$$\Phi = \left\| \frac{\varphi_{nk}\, q^{n-1}\, [n-1]}{q^{k-1}} \right\| ,$$

then we must determine φ_{nk} so as to satisfy equation (13.3).

If the forward inverse is equal to the backward inverse, we are led to the equation

$$\| \sum_{m=1}^{\infty} \varphi_{mk} [a_{n-m} + (m-1)\, b_{n-m+1} + b_{n-m}\, x]\; q^{n-k} \| = \| \delta_{nk} \| . \tag{13.4}$$

It will now be observed that the function

$$\varphi_k(z) = \sum_{m=1}^{\infty} \varphi_{mk}\, z^{m-1}$$

satisfies the differential equation

$$L[\varphi_k(z)] \equiv [A(z) + x\, B(z)]\, \varphi_k(z) + B(z) \frac{d}{dz} \varphi_k(z) = z^{k-1} . \tag{13.5}$$

Conversely, the coefficients of the series development of each integral of (13.5) satisfy the matrix equation (13.4).

It is also clear that the singular points of (13.5) are the values of z for which $B(z) = 0$. Moreover, for $z < | z_1 |$, where z_1 is the first zero (i. e., with smallest modulus) of $B(z)$, there will exist an in- infinite number of solutions of (13.5) and hence there will exist an infinite number of inverse matrices. In order to get a single inverse we must limit z to the annulus between $z = | z_1 |$ and $z = | z_2 |$, where z_2 is the first zero of $B(z)$ of modulus greater than $|z_1|$. Then the series

$$\varphi_k(z) = \sum_{m=1}^{\infty} \varphi_{mk}\, q^{m-1}\, [m-1]$$

converges only if the integral of (13.5), i. e.,

$$\varphi_k(z) = e^{-xz} M(z) [\int_0^z N_k(x,z)\, dz + \gamma_k] , \tag{13.6}$$

where we abbreviate

$$M(z) = \exp [- \int_0^z [A(z)/B(z)]\, dz ,$$

$$N_k(x,z) = z^{k-i}\, e^{xz}/[B(z) M(z)] ,$$

is regular at the point $z = z_1$.

Let us now write

$$[A(z)/B(z)] = \lambda + c_1(z-z_1) + c_2(z-z_1)^2 + \cdots]/(z-z_1) \ .$$

It then follows that, for values of $\lambda > 0$, the expression inside the brackets of (13.6) will be

$$(z-z_1)^\lambda P(z-z_1) + K \ ,$$

where $P(z-z_1)$ is a series in positive powers of $(z-z_1)$ such that $P(0) \neq 0$, and K is an arbitrary constant.

From these considerations the value of γ_k in (13.6) can be easily calculated. If λ is a positive integer, one sees that $\gamma_k = 0$. If this is not the case, then let z make a circuit C about the points 0 and z_1 and identify the new integral $\varphi_k'(z)$ with $\varphi_k(z)$. We thus get

$$\varphi_k^*(z) = e^{-2\pi i\lambda} e^{-xz} M(z) [\int_c N_k(x,z)\, dz$$

$$+ e^{2\pi i\lambda} \int_0^z N_k(x,z)\, dz + \gamma_k] = \varphi_k(z) \ .$$

and it follows by comparison that

$$\gamma_k = \frac{1}{e^{2\pi i\lambda} - 1} \int_c N_k(x,z)\, dz \ .$$

(3) *To show that* (Φ) *is limited.*

We must show finally that the bilinear form

$$(\Phi) = \sum_{n,k=1}^{\infty} \varphi_{nk} \frac{q^{n-1}}{q^{k-1}} [n-1]\, x_k\, y_n$$

is limited, where x_k and y_n are subject to the limitations

$$\sum_{k=1}^{\infty} |\, x_k\,|^2 \leqq 1 \ , \quad \sum_{n=1}^{\infty} |\, y_n\,|^2 \leqq 1 \ .$$

First, applying the Schwarz inequality (see section 9, chapter 3), we have

$$|\sum_{k=1}^{\infty} \varphi_{1k}\, q^{1-k}\, x_k\,| \leqq [\sum_{k=1}^{\infty} |\,\varphi_{1k}\,|^2\, q^{2-2k} \sum_{k=1}^{\infty} |\, x_k\,|^2]^{\frac{1}{2}} \leqq [\sum_{k=1}^{\infty} |\,\varphi_{1k}\,|^2\, q^{2-2k}]^{\frac{1}{2}}$$

But $\varphi_{1k} = \varphi_k(0) = \gamma_k$ and we have

$$\gamma_k = O\left(\int_c |\, z^{k-1}\,|\, dz\right) = O(q^k/k) \ ,$$

where we employ the customary notation that $g(k) = O[h(k)]$ implies that $|g(k)/h(k)|$ is limited as $k \to \infty$.

Hence we see that

$$|\sum_{k=1}^{\infty} \varphi_{1k}\, q^{1-k}\, x_k| \leqq M(\sum_{k=1}^{\infty} 1/k^2)^{\frac{1}{2}} = M\,\pi\, 6^{-\frac{1}{2}} . \qquad (13.7)$$

We next consider a form (Φ'), which is identical with (Φ) except for the deletion of the term in y_1. From the expansion of $\varphi_k(k)$ and (13.5) we see that

$$\sum_{n=2}^{\infty} \varphi_{nk}\, z^{n-2}(n-1) = \frac{d}{dz}\varphi_k(z) = -\frac{A(z)\varphi_k(z) + z^{k-1}}{B(z)} - x\,\varphi_k(z) .$$

Since $d\,\varphi_k(z)/dz$ is regular at the point $z = z_1$, the right hand member must also be regular at this point. We now write

$$\frac{d}{dz}\varphi_k(z) = h_k(z) + g_k(z)$$

where we define

$$h_k(z) = z^{k-1}B_1(z) + (z^{k-1} - z_1^{k-1})/[(z-z_1)B_1(z_1)] ,$$

$$g_k(z) = -A(z)\beta(z)\varphi_k(z) - x\,\varphi_k(z) - a(z)\varphi_k(z)/B_1(z)$$
$$- A(z_1)[\varphi_k(z) - \varphi_k(z_1)]/[B_1(z_1)(z-z_1)] ,$$

in which we use the abbreviations

$$B_1(z) = B(z)/(z-z_1) , \quad a(z) = [A(z) - A(z_1)]/(z-z_1) ,$$
$$\beta(z) = [B_1(z_1) - B_1(z)]/[(z-z_1)B_1(z_1)B_1(z)] .*$$

Since both $h_k(z)$ and $g_k(z)$ are analytic about $z = 0$ they can be expanded into the following series

$$h_k(z) = \sum_{n=k+1}^{\infty} \beta_{n-k}\, z^{n-2} + \sum_{n=2}^{k} z^{n-2}\, z_1^{k-n}/B_1(z_1) ,$$

$$g_k(z) = \sum_{n=2}^{\infty} g_{kn}\, z^{n-2} .$$

It follows from (13.6) that if $0 \leqq \vartheta \leqq 2\pi$ and if $1/|q\,e^{i\theta} - z_1|$ remains bounded, then $\varphi_k(q\,e^{i\theta}) = O(q^k/k)$ and consequently

$$g_k(q\,e^{i\theta}) = O[q^{k-1}/(k-1)] .$$

*In deriving this identity use was made of the fact that if $z = z_1$ in (13.5), then $A(z_1)\,\phi_k(z_1) = z_1^{k-1}$, since $B(z_1) = 0$.

We know, moreover, that*

$$\int_0^{2\pi} | g_k(q\, e^{i\theta}) |^2 \, d\vartheta = 2\pi \sum_{n=2}^{\infty} | g_{kn} |^2 q^{2(n-2)} \; . \tag{13.8}$$

From this and the fact that

$$| g_k(q\, e^{i\theta}) | \leqq M\, q^{k-1}/(k-1) \; ,$$

we deduce that

$$\sum_{n=2}^{\infty} | g_{kn} |^2 q^{2(n-2)} \leqq M^2 q^{2(k-1)}/(k-1)^2 \; .$$

Returning to the main problem, namely, to show that (Φ') is limited, we obtain from the Schwarz inequality

$$| \sum_{k=1}^{\infty} \sum_{n=2}^{\infty} \frac{g_{kn}\, q^{n-2}}{q^{k-2}} x_k\, y_n | \leqq \sum_{k=1}^{\infty} | x_k | q^{2-k} [\sum_{n=2}^{\infty} | g_{kn} |^2 q^{2(n-2)} \sum_{n=2}^{\infty} | y_n |^2]^{\frac{1}{2}}$$

$$\leqq \sum_{k=2}^{\infty} \frac{| x_k |}{q^{k-2}} \frac{M\, q^{k-1}}{k-1} + | x_1 | \, q \, [\sum_{n=2}^{\infty} | g_{kn} |^2 q^{2(n-2)}]^{\frac{1}{2}}$$

$$\leqq (M/q) [\sum_{k=2}^{\infty} | x_k |^2 \sum_{k=2}^{\infty} 1/(k-1)^2]^{\frac{1}{2}} + M' = C_1 \; , \tag{13.9}$$

where C_1 is finite.

In a similar way we can show that

$$| \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} \frac{\beta_{n-k}(n-1)\, q^{n-2}}{q^{k-2}} x_k\, y_n + \sum_{k=1}^{\infty} \sum_{n=2}^{k} \frac{q^{n-2} z_1^{k-n}}{B_1(z_1)\, q^{k-2}} x_k\, y_n |$$

$$\leqq \sum_{n=1}^{\infty} \beta_n\, q^n + \frac{1}{(1-|z_1|/q) B_1(z_1)} = C_2 \; , \tag{13.10}$$

where C_2 is finite.

By combining inequalities (13.7), (13.9) and (13.10), we have proved that (Φ) is a limited bilinear form.

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*Thus if

$$g_k(z) = \sum_{n=2}^{\infty} g_{kn} z^{n-2} = \sum_{n=2}^{\infty} (a_{kn} + b_{kn}\, i)\, z^{n-2}$$

so that

$$g_k(q\, e^{i\theta}) = \sum_{n=2}^{\infty} q^{n-2} [a_{kn} \cos(n-2)\,\theta - b_{kn} \sin(n-2)\,\theta]$$

$$+ i \sum_{n=2}^{\infty} q^{n-2} [b_{kn} \cos(n-2)\,\theta + a_{kn} \sin(n-2)\,\theta] \; ,$$

equation (13.8) is readily deduced from the fact that

$$\int_0^{2\pi} \sin n\theta \sin m\theta \, d\theta = \int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta = \delta_{mn} \; ,$$

where δ_{mn} is the Kronecker symbol, and that

$$\int_0^{2\pi} \sin n\theta \cos m\theta \, d\theta = 0 \; .$$

We can now state Hilb's theorem on the basis of the three results obtained in this section:

Theorem 5. If q is a number such that $|z_1| < q < |z_2|$, where z_1 and z_2 are the two zeros of $B(z)$ of smallest moduli, and if $f(x)$ is a function of grade not greater than q, then there exists one and only one solution, $u(x)$ of equation (13.1), *whose grade is not greater than q, and this solution is given by*

$$u(x) = \sum_{n=1}^{\infty} \varphi_{n1} f^{(n-1)}(x) .$$

The Method of Appell Polynomials.

In section 6, chapter 1, we have already defined Appell polynomials by means of their generatrix to be the polynomials $P_n(x)$ in the expansion

$$A(t)\, e^{xt} = \sum_{n=0}^{\infty} P_n(x)\, t^n . \tag{13.11}$$

Well known examples of such polynomials are the Bernoulli polynomials of mth order, $B_n^{(m)}(x)$,* and the Hermite polynomials, $h_n(x)$.† In the first case we have

$$A(t) = t^m/(e^t - 1)^m , \qquad P_n(x) = B_n^{(m)}(x)/n!$$

and in the second

$$A(t) = e^{-\frac{1}{2}t^2} , \qquad P_n(x) = h_n(x)/n! .$$

The definition given in (13.11) admits of a slight extension which will be useful to us here. Thus, adopting the notation of Sheffer, we shall refer to the polynomials $G_n(x)$ defined by the expression

$$[A_0(t) + x\, A_1(t) + \cdots + x^p A_p(t)]\, e^{xt} = \sum_{n=0}^{\infty} G_n(x)\, t^n \tag{13.12}$$

as *generalized Appell polynomials.*

Identifying the functions $A_r(t)$ with those given in section 2,

$$A_r(t) = \sum_{n=0}^{\infty} a_{nr}\, t^n ,$$

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*For further information concerning the Bernoulli polynomials consult N. E. Nörlund: *Vorlesungen uber Differenzenrechnung*, Berlin (1924), or H. T. Davis: *Tables of the Higher Mathematical Functions*, vol. 2 (1935), pp. 181-272.

†Two forms of the Hermite polynomials are in common use connected by the relation

$$h_n(x) = 2^{-\frac{1}{2}n} H_n(x/\sqrt{2}) .$$

We use the definition

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2} .$$

See example 3, section 1, chapter 12.

we see that the polynomials $G_n(x)$ are explicitly given by the formula

$$G_n(x) = \sum_{m=0}^{\infty} x^m \left[a_{0m} \frac{x^n}{n!} + a_{1m} \frac{x^{n-1}}{(n-1)!} + \cdots + a_{nm} \right].$$

Conversely, the values of the functions $A_r(t)$ can be expressed in terms of the Appell polynomials by means of the relationship

$$A_r(t) = \frac{1}{r!} \sum_{n=0}^{\infty} \{G_n^{(r)}(0) - {}_rC_1\, G_{n-1}^{(r-1)}(0) + {}_rC_2\, G_{n-2}^{(r-2)}(0) - \cdots$$
$$+ (-1)^r\, {}_rC_r\, G_{n-r}(0)\}\, t^n ,$$

where ${}_rC_m$ is the mth binomial coefficient.

Let us adopt the notation

$$F(x,t) = A_0(t) + x\, A_1(t) + \cdots + x^p A_p(t) .$$

Then since the functions are analytic at $t = 0$ we may obviously write

$$G_n(x) = \frac{1}{2\pi i} \int_C \frac{F(x,t)}{t^{n+1}} e^{xt}\, dt ,$$

where C is a path about the origin.

Turning now to the solution of equation (1.1), we shall first consider the case where $p = 0$. In section 3, chapter 6, the solution of the equation

$$A_0(z) \to u(x) = f(x) \tag{13.13}$$

was obtained in the form

$$u(x) = \frac{1}{2\pi i} \int_C \frac{V(t)}{A_0(t)} e^{xt}\, dt , \tag{13.14}$$

where we define

$$V(t) = f_0/t + f_1/t^2 + f_2/t^3 + \cdots , \qquad f_n = f^{(n)}(0) , \tag{13.15}$$

and where C is a path about the origin.

Operating now with $A_0(z)$ upon (13.14) we obtain formally

$$f(x) = \frac{1}{2\pi i} \int_C V(t)\, e^{xt}\, dt = \frac{1}{2\pi i} \int_C \frac{V(t)}{A_0(t)} A_0(t)\, e^{xt}\, dt$$
$$= \sum_{n=0}^{\infty} k_n\, G_n(x) , \tag{13.16}$$

where the coefficients are defined by

$$k_n = \frac{1}{2\pi i} \int_C \frac{V(t)}{A_0(t)} t^n\, dt . \tag{13.17}$$

If, then, in (13.14) the function e^{xt} is expanded as a power series in xt, and if account is taken of (13.17), the following solution of (13.13) is obtained:

$$u(x) = \sum_{n=0}^{\infty} k_n \, x^n/n! \, . \tag{13.18}$$

The convergence of series (13.16) may be readily established provided the coefficients are subject to the limitation

$$\sup \lim_{n=\infty} | \, k_n \, |^{1/n} = R$$

and $A_0(t)$ is analytic within the circle $| \, t \, | = r$, where $r > R$.

If A represents the maximum value of $A_0(t)$ on the circle $| \, t \, | = r$, and if C is the circumference of this circle, then, taking account of Cauchy's inequality [see equation (2.1), chapter 5], we obtain

$$| \, G_n(x) \, | = \left| \frac{1}{2\pi i} \int_c \frac{A_0(t) \, e^{xt}}{t^{n+1}} \, dt \right| \leq e^{XR} \, A/r^n \; , \quad | \, x \, | = X \; .$$

Hence, since $r > R$, we have

$$| \sum_{n=0}^{\infty} k_n \, G^n(x) \, | < M \sum_{n=0}^{\infty} (R/r)^n = M/(1-R/r) \; .$$

Not only does this equality establish the uniform convergence of (13.16), but it also shows that the series represents an entire function.

The method which we have developed above for the elementary case may be generalized in the following manner:

Let us assume that $u(x)$ may be expressed as the integral

$$u(x) = \frac{1}{2\pi i} \int_c U(t) \, e^{xt} \, dt$$

where C is a suitably chosen path about the origin. We shall then have

$$f(x) = F(x,z) \rightarrow u(x) = \frac{1}{2\pi i} \int_c U(t) F(x,t) \, e^{xt} \, dt$$

$$= \frac{1}{2\pi i} \int_a U(t) \sum_{n=0}^{\infty} G_n(x) \; t^n \, dt = \sum_{n=0}^{\infty} k_n \, G_n(x) \; , \tag{13.19}$$

where we define

$$k_n = \frac{1}{2\pi i} \int_c U(t) \, t^n \, dt \; . \tag{13.20}$$

As in the previous case $u(x)$ has the expansion (13.18) in which the coefficients of $G_n(x)$ are defined by (13.20).

The problem of inverting the original equation (1.1) in any given instance is, of course, the determination of the function $U(t)$. Although this determination might be made by inverting (13.20), a more effective method of procedure is the following:

Let us assume that $u(x)$ may also be written in the form

$$u(x) = \frac{1}{2\pi i} \int_c V(t) X(x,t)\, dt$$

where $V(t)$ is the function defined by (13.15).

Operating upon $u(x)$ with $F(x,z)$ we obtain

$$f(x) = F(x,z) \rightarrow u(x) = \frac{1}{2\pi i} \int_c V(t) [F(x,z) \rightarrow X(x,t)]\, dt$$

from which it follows that we must have

$$F(x,z) \rightarrow X(x,t) = e^{xt} . \tag{13.21}$$

But this equation is precisely (2.3) and hence $X(x,z)$ is the resolvent generatrix of (1.1). Hence, since (13.21) is also equal to the original equation (1.1) with the function $f(x)$ replaced by e^{xt}, we shall have

$$X(x,t) = X(x,z) \rightarrow e^{xt} = e^{xz} X_0(x,z) \rightarrow e^{xt} = X_0(x,t)$$

where $X_0(x,t)$ is a particular solution of equation (2.3) in which z has been replaced by t. Let us observe that $X_0(x,t)$ is not uniquely defined since to any particular solution of (2.3) there must be added the general solution of the homogeneous equation (2.4). The restrictions imposed by theorem 1 will be assumed here.

If finally it happens that $X_0(x,t)$ has the following expansion:

$$X_0(x,t) = X_0(t) + X_1(t)x + X_2(t)x^2/2! + \cdots ,$$

then $u(x)$ may be written

$$u(x) = \frac{1}{2\pi i} \int_c V(t) \sum_{n=0}^{\infty} \frac{X_n(t)x^n}{n!}\, dt$$

$$= \sum_{n=0}^{\infty} k_n\, x^n/n! .$$

Since $f(x)$ has the expansion (13.19), it follows that

$$f(x) = \frac{1}{2\pi i} \int_c V(t) \sum_{n=0}^{\infty} X_n(t) G_n(x)\, dt .$$

PROBLEMS

1. Prove that

$$f(x) = \sum_{n=0}^{\infty} k_n G_n(x) ,$$

where $G_n(x)$ are the generalized Appell polynomials given by (13.12), defines an entire function of grade not greater than R, provided the functions $A_r(t)$ are analytic in $|t| < R$, and provided $\sup \lim_{n\to\infty} |k_n|^{1/n} = R$. (Sheffer).

2. Prove that $L(x^n) = P_n(x)$, where we define

$$L(u) = \sum_{n=0}^{\infty} L_n(x) u^{(n)}(x)$$

$$P_n(x) = p_{n0} + p_{n1} x + \cdots + p_{nn} x^n ,$$

$$L_n(x) = (1/n!) [P_n(x) - {}_nC_1\, x P_{n-1}(x) + {}_nC_2\, x^2 P_{n-2}(x) + \cdots + (-1)^n x^n P_0(x)] . \qquad \text{(Sheffer).}$$

3. If the operator $L(u)$ is defined as in problem 2, show that $L(u) - \lambda u = 0$ has a polynomial solution if and only if λ is one of the characteristic numbers $\lambda_n = p_{nn}$. (Sheffer).

14. Validity of the Solutions. We shall now investigate the character of the solutions which have been formally defined in the preceding sections. We have seen that the general resolvent operator for equation (1.1) is the sum of terms of the form

$$X(x,z) = X(z) F(x,z) , \tag{14.1}$$

where we abbreviate

$$F(x,z) = e^{-xz} \int_a^z e^{xt} Y(t)\, dt . \tag{14.2}$$

Let us first observe that

$$X(x,z) \to f(x) = F(x,z) \to [X(z) \to f(x)] , \tag{14.3}$$

a result immediately obtained by forming the Bourlet product $[F \cdot X]$ of section 3, chapter 4. But from theorem 6, chapter 5, we see that the grade of the function $g(x) = X(z) \to f(x)$ is the same as the grade of $f(x)$, provided $X(z)$ satisfies certain mild restrictions. These conditions we shall assume are fulfilled by $X(z)$. Hence we need consider only the function

$$u(x) = F(x,z) \to g(x) . \tag{14.4}$$

There are several cases to be studied, which may be listed as follows:

(1) The function $g(x)$ is of finite grade.

(a_1) The limit of the integral in (14.2) is finite.

(b_1) The limit of the integral is infinite.

(2) The function $g(x)$ is of infinite grade.

(a_2) The limit of the integral in (14.2) is finite.

(b_2) The limit of the integral is infinite.

The case (a_1) has already been covered by theorem 7 of chapter 5, and its first corollary, where it was proved that *the grade of $u(x)$ does not exceed the larger of the two numbers q and $|a|$*, provided *$Y(t)$ is analytic throughout the interior of the circle $\varrho = R$, where R is larger than q and $|a|$.*

The following two theorems pertain to case (b_1):

Theorem 6. If $Y(z)$ and $g(x)$ are functions of finite grade Q and q respectively, then $u(x)$ exists in the region $|x| > Q$ and is in general of infinite grade.

Proof: To establish this theorem we employ formula (12.4), which for simplicity we limit to the case $p = 1$,

$$F(x,z) = Y(z)/x - Y'(z)/x^2 + Y''(z)/x^3 - \cdots . \qquad (14.5)$$

From theorem 6, chapter 5, we know that $Y(z) \to g(x)$ is a function of grade q and hence, from the assumptions regarding $Y(z)$, we have $|Y^{(n)}(z) \to g(x)| \leqq M_n Q^n$, where $\lim_{n=\infty} M_n^{1/n} = 1$. Hence we find a majorant for $F(x,z) \to g(x)$ in the series

$$M_0/|x| + M_1 Q/|x|^2 + M_2 Q^2/|x|^3 + M_3 Q^3/|x|^4 + \cdots ,$$

which, from the fact that $\lim_{n=\infty} M^{1/n} = 1$, is seen to converge for $|x| > Q$.

We thus establish the existence of a function in a region which does not include the origin in its interior. Since, in general, a singular point is included in the region bounded by the circle $\varrho = Q$, the function $u(x)$ is in general of infinite grade.

If we admit the validity of the semi-convergent series of the form

$$r(x)\{a_0 + a_1/x + a_2/x^2 + a_3/x^3 + \cdots\} , \qquad (14.6)$$

as an asymptotic representation of $u(x)$, where the expression in braces satisfies the Poincaré criterion (see section 4, chapter 5), then a further extension of the region of definition of $u(x)$ is possible. This we state in the following theorem:

Theorem 7. If $Y(z)$ is a function of unbounded grade but otherwise satisfies the conditions of theorem 4 for the case $p = 1$ and if $g(x)$ is a function of grade q, then $F(x,z) \to g(x)$, where $F(x,z)$ is defined by (14.5), *yields a semi-convergent series which is the asymptotic representation of the integral*

$$u(x) = F(x,z) \to g(x) = (1/x) \int_0^\infty e^{-t}\{Y(z - t/x) \to g(x)\}\, dt \tag{14.7}$$

in the sense of (14.6), *where* $r(x)$ *is a function of grade not greater than* q.

Proof. Representing by $S_n(x,z)$ the operator

$$S_n(x,z) = \sum_{m=1}^{n} (-1)^{m-1} Y^{(m-1)}(z)/x^m \ ,$$

let us consider the remainder operator,

$$R_n(x,z) = F(x,z) - S_n(x,z)$$

$$= (-1)^n e^{-xz}(1/x)^n \int_{-\infty}^{z} e^{xt} Y^{(n)}(t)\, dt \ .$$

Making use of equation (3.7) of chapter 5, we obtain the inequality,

$$|x|^n \, |R_n(x,z) \to f(x)|$$

$$\leqq \left| g(0) \int_{-\infty}^{0} e^{xt} Y^{(n)}(t)\, dt \right| + \left| P(0) \int_{0}^{q} e^{xt} Y^{(n)}(t)\, dt \right| + |e^{qx} S(x)| \ .$$

By hypothesis $Y^{(n)}(t)$ is bounded by $M_n A^n n!$, $\lim_{n \to \infty} M_n^{1/n} = 1$ and satisfies the assumptions of theorem 4. Hence the inequality may be reduced to

$$|x|^n \, |R_n(x,z) \to f(x)|$$

$$\leqq \{e^{qx} M_n A^n n! + |P(0)| e^{qx} M_n A^n n! |x| + e^{qx} |S(x)| \, |x|\}/|x|$$

$$< e^{qx} T(x) \{M_n A^n n!/|x|\} \ ,$$

where $T(x)$ is a positive function of zero grade which dominates both $|x|$ and $|S(x)|\,|x|$.

We then conclude that the function $r(x)$ of (14.6) is dominated by a function of grade q and hence is a function of grade which does not exceed q.

In order to attain the integral of which (14.6) is the asymptotic representation, we apply Borel's integral (see section 4, chapter 5) to (14.6) and thus obtain,

$$F(x,z) = (1/x) \int_0^\infty e^{-t} Y(z - t/x)\, dt \ .$$

The integral representation of $u(x)$, (14.7), follows as an immediate consequence.

As an example illustrating Theorem 7 consider the equation:

$$[(1 + x) + z + z^2 + z^3 + \cdots] \to u(x) = f(x) \ .$$

Since we have $X(z) = 1 - z$, $Y(z) = 1/(1 - z)$, the solution is given by the operator,

$$u(x) = e^{-xz}(1 - z) \int_{-\infty}^{z} \{e^{xt}/(1 - t)\} dt \to f(x)$$

$$= \{1/x - 1!/(1-z)x^2 + 2!/(1-z)^2 x^3$$

$$- 3!/(1-z)^3 x^4 + \cdots\} \to f(x) \ .$$

If we set $f(x) = 1$ we replace z by zero and thus obtain the asymptotic development of the integral,

$$u(x) = \int_0^{\infty} \{e^{-s}/(x + s)\} ds \ .$$

If, however, we set $f(x) = e^{ax}$, $0 < a < 1$, we get

$$u(x) = e^{ax}\{1/x - 1!/(1 - a)x^2 + 2!/(1 - a)^2 x^3$$

$$- 3!/(1 - a)^3 x^4 + \cdots\} \ .$$

This series is the asymptotic representation in the sense of (14.6), where $g(x) = e^{ax}$, of the formal solution

$$u(x) = e^{ax}(1 - a) \int_0^{\infty} \{e^{-s}/[(1 - a)x + s]\} ds \ .$$

We turn next to a consideration of cases (a_2) and (b_2) above. The situation here presents numerous theoretical difficulties, but a partial solution is furnished by theorem 10, chapter 5, where conditions are stated which are sufficient to secure Borel-summable equivalents for $u(x)$ when $g(x)$ is a function of infinite grade. In general the grade of $u(x)$ is infinite. The reader is referred to theorem 10, chapter 5, for further details.

If the limit of the integral (14.2) is infinite [case (b_2)], this is equivalent to adding

$$g(0) \int_0^{\infty} e^{-xt} Y(-t) dt \qquad (14.8)$$

to the solution obtained in case (a_2). The conditions imposed upon $Y(t)$ by theorem 10, chapter 5, assure the convergence of the integral in (14.8). If, then, $g(0)$ exists, the function (14.8) may be added to the solution obtained in case (a_2). The grade of $u(x)$ is infinite.

Finally let us consider the convergence of the factorial form of the operator (14.1) as given by the expansion (12.2), which for convenience we shall specialize for the case $p = 1$.

Since the nth term of the series is readily seen to be of the form

$$w_n(z)e^{nz} = (D-n+1)\cdots(D-1)D \to Y$$
$$= (-1)^{n-1}(n-1)!\,D(1-D)(1-D/2)$$
$$\cdots[1-D/(n-1)] \to Y(z) \ ,$$

where $D = d/dz$ we are led to a consideration of the operator,

$$A_n(D) = D(1-D)(1-D/2)\cdots[1-D/(n-1)] \ .$$

Product operators of the form,

$$D(1-D/a_1)(1-D/a_2)\cdots(1-D/a_n)\cdots \ ,$$

where $\sum_{n=1}^{\infty} 1/|a_n|$ converges, have been extensively studied by J. F. Ritt,* but it is clear that the condition imposed by him is not satisfied in the case of the operator $\lim_{n\to\infty} A_n(D)$.

We may then proceed as follows:

Introducing convergence factors $e^{D/m}$ into the product A_n we have,

$$A_n(D) = [D(1-D)e^D(1-D/2)e^{D/2}\cdots\{1-D/(n-1)\}e^{D/(n-1)}]$$
$$\times \ e^{-D\{1+1/2+\ldots 1/(n-1)\}} \ . \qquad (14.9)$$

For sufficiently large values of n the product in the brackets may be written, $-e^{CD}/\Gamma(-D) + e^{CD}\,\varepsilon_n(D)$, where C is Euler's constant and $\varepsilon_n(D)$ is a function that tends to zero as $n \to \infty$.† Similarly the coefficient of $-D$ in the exponent of e may be written $\log n + C_n$, where the difference $C_n - C$ becomes vanishingly small with increasing n.‡

We can then write (14.9) in the form,

$$A_n(D) = [-1/\Gamma(-D) + \varepsilon_n(D)]e^{-D\log n}\,e^{D(C-C_n)} \ .$$

Employing theorem 6, chapter 5 and recalling that $1/\Gamma(D)$ is analytic in the entire plane we see that $A_n(D) \to Y(z)$ is a function of grade less than or equal to Q provided $Y(z)$ is of grade Q. If in particular $Y(z) = e^{-Qz}$ where Q is a positive number, we shall have

$$A_n(D) \to Y(z) = \{-1/\Gamma(Q) + \varepsilon_n(-Q)\}e^{-Q(C-C_n)}\,e^{-Qz}\,n^Q \ ,$$

from which we infer that,

$$A_n(D) \to Y(z) = O(n^Q) \ .$$

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*See section 6, chapter 1 and section 10, chapter 6.

†Whittaker and Watson, *Modern Analysis*, 3rd ed. (1920), p. 236.

‡*Ibid.*: p. 235.

More generally let us consider the case where $Y(z)$ is any function of grade Q. Operating upon $Y(z)$ with $1/\Gamma(-D)e^{D(C-C_n)}$ and recalling theorem 6, chapter 5, we obtain a new function $R(z)$, which is also of grade Q. Operating upon $R(z)$ with $e^{-D\log n}$ we get $R(z - \log n)$. Since $R(z)$ is of grade Q it is dominated for large values of the argument by a function of the form $e^{Qz}P(z)$, where $P(z)$ is a suitably chosen positive function of genus 0. Hence we can write $|R(z - \log n)| \leqq n^Q e^{Qz} P(z + \log n)$ for n sufficiently large. But since $P(z)$ is a function of genus zero, $P(z + \log n)$ is dominated by n^δ, where δ is an arbitrarily small positive constant,* and we have $|R(z - \log n)| < n^{Q+\delta} e^{Qz}$.

We are thus able to assert that

$$A_n(D) \rightarrow Y(z) = o(n^{Q+\delta}), \quad \delta > 0 , \qquad (14.10)$$

provided $Y(z)$ is a function of grade Q.

The situation is much more complicated if $Y(z)$ is a function of infinite grade. An important special case, however, is furnished by the series,

$$Y(z) = a_0 + a_1 e^z + a_2 e^{2z} + \cdots + a_n e^{nz} + \cdots , \qquad (14.11)$$

where $a_n = O(1/n!)$, which from the example of section 8 is seen to include the classical operator $u(x+1) - xu(x)$, that is when $a_n = (-1)^n/n!$.

Operating with $A_n(D)$ and noting that $A_n(D) \rightarrow e^{mz} = 0$ if $m \leqq n-1$, and $(-1)^{n-1}\Gamma(m+1)e^{mz}/\Gamma(n)\Gamma(m-n+1)$ if $m \geqq n$, we shall have

$$\begin{aligned} A_n(D) \rightarrow Y(z) &= (-1)^{n-1} n e^{nz}\{a_n + a_{n+1}(n+1)e^z/1! \\ &\qquad + a_{n+2}(n+1)(n+2)e^{2z}/2! + \cdots\} \\ &= (-1)^{n-1} n a_n e^{nz}\{1 + a_{n+1}(n+1)e^z/1!a_n \\ &\qquad + a_{n+2}(n+1)(n+2)e^{2z}/2!\,a_n + \cdots\} . \end{aligned}$$

Under the hypothesis that $a_n = O(1/n!)$ it is clear that the function within the braces is entire and bounded with n and hence that

$$A_n(D) \rightarrow Y(z) = O[e^{nz}/(n-1)!] \quad . \qquad (14.12)$$

We now turn to a consideration of series (12.2) when $p = 1$. Let us first examine the case where (14.10) applies.

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*This follows from the fact that

$$\lim_{n=\infty} P(z + \log n)/n^\delta = \lim_{x=\infty} P(z+x)/e^{x\delta} = 0 .$$

From the operational identity

$$w_n(z)e^{nz} = (-1)^n(n-1)!A_n(D) \to Y(z)$$

we write equation (12.2) in the form,

$$\begin{aligned} X_0(x,z) = X(z)\{Y(z)/x + (A_0 \to Y)/x(x+1) \\ + (A_1 \to Y)1!/x(x+1)(x+2) \qquad (14.13) \\ + (A_2 \to Y)2!/x(x+1)(x+2)(x+3) + \cdots\} . \end{aligned}$$

We now recall (see section 11, chapter 6) the fundamental convergence fact associated with a factorial series

$$F(x) = \sum_{n=0}^{\infty} a_{n+1} n!/x(x+1)\cdots(x+n) ,$$

namely, that this series converges with the exception of the points 0, -1, -2, $\cdots$, for values of x the real part of which exceeds a value, λ, called the *abscissa of convergence*. If the series $A = \sum_{n=0}^{\infty} a_{n+1}$ diverges then $\lambda \geqq 0$ and is determined by the limit,

$$\limsup_{p\to\infty} \log\left|\sum_{n=0}^{p} a_{n+1}\right|/\log p .$$

If the series A converges then $\lambda \leqq 0$ and is determined by the limit,

$$\limsup_{p\to\infty} \log\left|\sum_{n=p+1}^{\infty} a_{n+1}\right|/\log p .$$

The abscissa of convergence of (14.13) is easily obtained in the two cases already discussed: (1) equation (14.10); (2) equation (14.12).

In the first case the series $\sum_{n=0}^{\infty} n^{Q+\delta}$ diverges for $Q+\delta > -1$. Hence employing the abbreviation $q = Q + \delta$, we compute,

$$\sum_{n=0}^{p} n^q = p^{q+1}/(q+1) + \tfrac{1}{2}p^q + qB_1p^{q-1}/2! - \cdots .$$

From this we obtain

$$\lambda_1 = \limsup_{p\to\infty}[\log p^{q+1}/\{(q+1)\log p\} - 1] = q .$$

We are thus able to conclude that the abscissa of convergence for the case where $Y(z)$ is a function of grade Q is not *in general* smaller than $Q + \delta$, although the special case $Y(z) = e^{mz}$, where m is a positive integer, shows that it may be $-\infty$.

In the second case, equation (14.12), the series $\sum_{n=0}^{\infty} e^{nz}/(n-1)!$ obviously converges. Hence from the function,

$$f(p) = \sum_{n=p+1}^{\infty} e^{nz}/(n-1)!$$
$$= \{e^{(p+1)z}/p!\}\{1 + e^{z}/(p+1)$$
$$+ e^{2z}/(p+1)(p+2) + \cdots\} \ ,$$

we compute the abscissa of convergence to be,

$$\lambda_2 = -\limsup_{p=\infty} |\log f(p)|/\log p$$
$$= -\lim_{p=\infty} |(p+1)z - \log p!|/\log p = -\infty \ .$$

We are thus able to state the theorem:

Theorem 8. If $Y(z)$ is a function of grade Q then an abscissa of convergence exists for series (14.13) *which is in general not smaller than $\lambda = Q$; if $Y(z)$ is a function which has an expansion of the form* (14.11) *then the abscissa of convergence is $\lambda = -\infty$.*

15. Operators with Regular Singularities. In the discussions that we have made hitherto of the inversion of the Laplace equation which we shall write conveniently in the form,

$$\{A_0(z) + xA_1(z) + x^2A_2(z) + \cdots + x^pA_p(z)\} \to u(x) = f(x) \ ,$$
$$p > 0 \ , \qquad (15.1)$$

where the $A_i(z)$ are the functions,

$$A_i(z) = \sum_{n=0}^{\infty} a_{ni} z^n \ ,$$

it has been necessary to impose the restriction $A_p(0) \neq 0$.

O. Perron has formally removed this restriction in the following manner:*

If in equation (15.1) we make the transformation,

$$u(x) = e^{ax} v(x) \ ,$$

then, from the fact that $z^n \to u(x) = e^{ax}(z+a)^n \to v(x)$, equation (15.1) becomes,

$$\sum_{m=0}^{p} x^m A_m(z+a) \to v(x) = e^{-ax} f(x) \ . \qquad (15.2)$$

*See *Bibliography*: Perron (3), in particular p. 41. See also Sheffer, (1), p. 351.

Hence in the transformed equation the coefficient of x^p has the value $A_p(a)$ when $z = 0$ and the auxiliary parameter a is then to be so chosen that $A_p(a) = 0$.

The resolutions of the difficulty in this manner, however, is only apparent as may be seen from the following elementary example:

$$(1 - xz) \to u(x) = x \ .$$

If we make use of (15.2) and (2.24) we get

$$u(x) = e^{ax} \int_0^\infty \{(z+a)e^{-xs}/(z+a-s)^2\}\, ds \to xe^{-ax} \ .$$

But if we recall the operational identity,

$$F(x,z) \to xe^{-ax} = e^{-ax}xF(x,-a) + F_z'(x,-a) \ ,$$

and apply it to the present problem, we observe from the singularity of the integrand that the solution cannot be attained in this manner.

The nature of the difficulty is revealed, however, if we set $a = 0$ and observe the expansion

$$X(x,z) = \int_0^\infty \{ze^{-xs}/(z-s)^2\}ds = 1 - e^{-\vartheta}\, \vartheta\, li(e^\vartheta) \ ,$$

$$= 1 - e^{-\vartheta}\, \vartheta\{C + \log\vartheta + \vartheta + \vartheta^2/2\cdot 2! + \vartheta^3/3\cdot 3! + \cdots\} \ ,$$

where $\vartheta = xz$ and $C = .5772157\cdots$ (Euler's constant). The appearance of $\log z$ in the inverse $X(x,z)$ suggests that the resolution of the problem is to be found in an interpretation of the logarithmic operator which was discussed in section 11, chapter 2.

The object of this section, therefore, is to discuss the operational-theoretic problem involved here for the case where the multiplicity of the zero of $A_p(z)$ at $z = 0$ does not exceed p. It will be necessary, also, to make the assumption that the generatrix equation (2.4) has only a regular singularity at $z = 0$.

We shall find it convenient to begin by proving several lemmas:

Lemma 1. The operator

$$A_0(x,z) = \varphi(z) \int_0^\infty e^{-xs}(z-s)^{-\nu}ds \ , \qquad R(x) > 0 \ ,$$

where ν is any constant and $\varphi(z)$ is any function analytic about the origin, is equivalent to the operator

$$B_0(x,z) = z^{-\nu} \to \{\varphi(z)/x)\} \ ,$$

where $z^{-\nu}$ is the fractional integration operator defined by (7.1) *of chapter* 2.

Proof: In order to establish this identity let us consider the solution of the following differential equation of infinite order,

$$\{z^\nu \varrho'(z) + xz^\nu \varrho(z)\} \to u(x) = f(x) \ , \qquad (15.3)$$

where $\varrho(z)$ is a function analytic about the origin and not vanishing there.

Since this is an equation of the singular type excluded from the general theory, we apply the transformation of Perron and obtain the limiting case where $a = 0$. We thus easily find

$$X(z) = \varrho(0)/\varrho(z) \ , \quad Y(z) = 1/[z^\nu \cdot \varrho(0)] \ ,$$

and hence the resolvent operator

$$X_0(x,z) = \{1/\varrho(z)\} \int_0^\infty e^{-xs}(z-s)^{-\nu}\, ds \ . \qquad (15.4)$$

To obtain a second form for this operator we expand the integral formally as a series in $1/z$ and thus find,

$$\begin{aligned} X_0(x,z) = \{1/\varrho(z)\} \int_0^\infty e^{-xz} z^{-\nu}\{1 + \nu s/z \\ + \nu(\nu+1)s^2/z^2 \cdot 2! + \cdots\}\, ds \ , \\ = \{x/\varrho(z)\} z^{-\nu}\{1 + \nu/xz + \nu(\nu+1)/(xz)^2 \\ + \nu(\nu+1)(\nu+2)/(xz)^3 + \cdots\} \ . \end{aligned}$$

When $z^{-\mu}$ is replaced by its operational equivalent,

$$z^{-\mu} = \int_c^x (x-t)^{\mu-1} e^{z(t-x)} dt/\Gamma(\mu) \ ,$$

this series reduces to

$$X_0(x,z) = \{1/\varrho(z)\} \int_c^x e^{z(t-x)}\{(x-t)^{\nu-1}/t \cdot \Gamma(\nu)\} dt \ . \qquad (15.5)$$

In order to effect a further transformation we now compute

$$z^\nu \to u(x) = z^\nu \to \{X_0(x,z) \to f(x)\} \ ,$$

where $X_0(x,z)$ is the operator (15.5). We thus obtain:

$$\begin{aligned} z^\nu \to u(x) = \frac{d}{dx} \int_c^x \frac{(x-t)^{-\nu}}{\Gamma(1-\nu)}\, dt \\ \times \int_c^t e^{z(s-t)} \frac{(t-s)^{\nu-1}}{s\,\Gamma(\nu)}\, ds\, \{1/\varrho(z)\} \to f(x) \ , \\ = \frac{d}{dx} \int_c^x \frac{(x-t)^{-\nu}}{\Gamma(1-\nu)}\, dt \int_c^t \frac{\vartheta(s)}{s} \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}\, ds \ , \end{aligned}$$

$$z^\nu \to u(x) = \frac{d}{dx}\int_c^x \frac{\vartheta(s)}{s}\,ds \int_s^x \frac{(x-t)^{-\nu}}{(t-s)^{1-\nu}} \frac{dt}{\Gamma(\nu)\Gamma(1-\nu)} ,$$

where we use the abbreviation $\vartheta(x) = [1/\varrho(z)] \to f(x)$.

Making the transformation, $(x-t)/(t-s) = (1-y)/y$, we obtain the last equation in the form

$$z^\nu \to u(x) = \frac{d}{dx}\int_c^x \frac{\vartheta(s)}{s}\,ds \int_0^1 \frac{(1-y)^{-\nu}y^{\nu-1}}{\Gamma(\nu)\Gamma(1-\nu)}\,dy ,$$

$$= \vartheta(x)/x . \tag{15.6}$$

From this we deduce that $u(x) = z^{-\nu} \to \{\vartheta(x)/x\}$, and hence that

$$X_0(x,z) = z^{-\nu} \to \{[1/\varrho(z)]/x\} . \tag{15.7}$$

Let us verify this conclusion directly. We first note by use of the Bourlet product (3.1) of chapter 4 that

$$\{\varrho'(z) + x\varrho(z)\} \to z^\nu = z^\nu\varrho'(z) + xz^\nu\varrho(z) . \tag{15.8}$$

Computing likewise the operational product,

$$A(x,z) = \{1/[\varrho(z)\cdot x]\} \to \{\varrho'(z) + x\varrho(z)\} .$$

we get,

$$A(x,z) = (\varrho' + \varrho x)/(\varrho x) + \varrho(-\varrho'/x\varrho^2) = 1 ,$$

which establishes the fact that $1/[\varrho(z)\cdot x]$ is the inverse of

$$\varrho'(z) + x\varrho(z) .$$

Combining this with (15.8), we immediately derive (15.7) as the inverse of the operator $z^\nu\varrho'(z) + xz^\nu\varrho(z)$.

If we finally replace $1/\varrho(z)$ by $\varphi(z)$ in this argument, we see that we have established the lemma.

Lemma 2. The operator

$$A_1(x,z) = \varphi(z)\int_0^\infty e^{xs}\log(z-s)\,(z-s)^{-\nu}\,ds ,$$
$$R(x) > 0 ,$$

is equivalent to the operator,

$$B_1(x,z) = z^{\nu}\log z \to \{\varphi(z)/x\} ,$$

where $\varphi(z)$ *is a function analytic about the origin and* $z^{-\nu}\log z$ *is the operator defined by* (11.1) *in chapter* 2.

Proof: Employing the abbreviation,

$$O = \int_0^\infty e^{-xz}\log(z-s)\,(z-s)^{-\nu}\,ds ,$$

we may write, $O = L + M$, where we define,

$$L = \int_0^\infty e^{-xz} \log z \cdot z^{-\nu}(1-s/z)^{-\nu}\, ds \quad , \text{ and}$$

$$M = z^{-\nu} \int_0^\infty e^{-xz} \log(1-s/z)\,(1-s/z)^{-\nu}\, ds \quad .$$

Limiting our attention to the first element, we shall have

$$L \to f(x) = \int_0^\infty e^{-xs}(1-s/z)^{-\nu}\, ds \to \{z^{-\nu}\log z \to f(x)\} \quad ,$$

$$= \{1/x + \nu/zx^2 + \nu(\nu+1)/z^2x^3 + \cdots\}$$
$$\to \{z^{-\nu}\log z \to f(x)\} \quad ,$$

$$= \int_c^x \{1 + \nu(x-t)/x + \nu(\nu+1)(x-t)^2/x^2 2! + \cdots\}$$
$$\times \{z^{-\nu}\log z \to f(x)|_{x-t}\} dt/x \quad ,$$

$$= \int_c^x (t/x)^{-\nu} dt \int_c^t (t-s)^{\nu-2}\{\psi(\nu-1)$$
$$- \log(t-s)\} f(s)\, ds/[x\Gamma(\nu-1)] \quad ,$$

$$= \int_c^x f(s)\, ds \int_s^x (t/x)^{-\nu}(t-s)^{\nu-2}\{\psi(\nu-1)$$
$$- \log(t-s)\} dt/[x\Gamma(\nu-1)] \quad ,$$

where $\psi(\nu)$ is the psi function defined by $\psi(\nu) = \Gamma'(\nu)/\Gamma(\nu)$.

Employing the transformation $1-y = (t-s)/(x-s)$, we can write this equation as follows:

$$L \to f(x) = \int_c^x \frac{f(s)\,(x-s)^{\nu-1}}{\Gamma(\nu-1)}\, ds \int_0^1 \frac{(1-y)^{\nu-2}}{(1-py)^\nu}\{\psi(\nu-1)$$
$$- \log[(x-s)(1-y)]\}\, dy/x \quad ,$$

$$= \int_c^x \frac{f(s)\,(x-s)^{\nu-1}}{\Gamma(\nu)}\{\psi(\nu-1) - \log(x-s)\}\frac{ds}{s}$$
$$- \int_c^x \frac{f(s)\,(x-s)^{\nu-1}}{\Gamma(\nu-1)}\frac{ds}{x}\int_0^1 \frac{(1-y)^{\nu-2}}{(1-py)^\nu}\log(1-y)\, dy \quad ,$$

where we abbreviate $p = (x-s)/x$.

Similarly for the second element we get

$$M \to f(x) = -\sum(1/m) \int_0^\infty e^{-xs}(s/z)^m(1-s/z)^{-\nu}\, ds$$
$$\to [z^{-\nu} \to f(x)] \quad ,$$

$$M \to f(x) = -\sum(1/m) \left\{ \frac{m!}{x^{m+1}z^{m+1}} + \frac{(m+1)!}{x^{m+2}z^{m+2}}\nu \right.$$
$$\left. + \frac{(m+2)!\,\nu(\nu+1)}{x^{m+3}z^{m+3}\,2!} + \cdots \right\} \to (z^{1-\nu} \to f) \;,$$
$$= -\sum(1/m) \int_c^x \left\{ \frac{(x-t)^m}{x^m} + \frac{\nu(x-t)^{m+1}}{x^{m+1}} + \frac{\nu(\nu+1)}{2!\,x^{m+2}} + \cdots \right\}$$
$$\to \left\{ z^{1-\nu} \to f(x)|_{x=t} \right\} \frac{dt}{x} \;,$$
$$= -\sum(1/m) \int_c^x \frac{(x-t)^m}{x^m} \left(\frac{x}{t}\right)^\nu \frac{dt}{x} \int_c^t \frac{(t-s)^{\nu-2}}{\Gamma(\nu-1)} f(s)\,ds \;,$$
$$= -\sum(1/m) \int_c^x \frac{f(s)}{\Gamma(\nu-1)} \frac{ds}{x} \int_s^x \left(\frac{x-t}{x}\right)^m \left(\frac{x}{t}\right)^\nu$$
$$\times\ (t-s)^{\nu-2}\,dt \;,$$
$$= \int_c^x \frac{f(s)}{\Gamma(\nu-1)} \frac{ds}{x} \int_s^x \log(t/x)\,(x/t)^\nu (t-s)^{\nu-2}\,dt \;.$$

Making the transformation $1 - y = (t-s)/(x-s)$, we reduce the last equation to

$$M \to f(x) = \int_c^x \frac{f(s)}{\Gamma(\nu-1)} (x-s)^{\nu-1} \frac{ds}{x} \int_0^1 \log(1-py) \frac{(1-y)^{\nu-2}}{(1-py)^\nu}\,dy \;.$$

where we abbreviate as before $p = (x-s)/x$.

Now combining L and M we obtain

$$O \to f(x) = (L+M) \to f(x)$$
$$= \int_c^x \frac{f(s)\,(x-s)^{\nu-1}}{\Gamma(\nu)} \{\psi(\nu-1) - \log(x-s)\} \frac{ds}{s}$$
$$+ \int_c^x \frac{f(s)\,(x-s)^{\nu-1}}{\Gamma(\nu-1)} I(p)\,ds/x \;,$$

where we abbreviate,

$$I(p) = \int_0^1 \frac{(1-y)^{\nu-2}}{(1-py)^\nu} \log\frac{(1-py)}{(1-y)}\,dy \;.$$

For the evaluation of $I(p)$ we note that

$$I(p) = -\frac{d}{d\nu} \int_0^1 \frac{(1-y)^{\nu-2}}{(1-py)^\nu}\,dy = -\frac{d}{d\nu} \frac{1}{(1-p)(\nu-1)}$$
$$= (x/s)\{1/(\nu-1)^2\} \;. \tag{15.9}$$

Recalling the functional equation

$$\psi(\nu+1) = \psi(\nu) + 1/\nu \ ,$$

and introducing (15.9) into O, we get

$$O \to f(x) = \int_c^x \frac{f(s)\,(x-s)^{\nu-1}}{\Gamma(\nu)} \{\psi(\nu) - \log(x-s)\} \frac{ds}{s} \ ,$$

$$= z^{-\nu} \log z \to \{f(x)/x\} \ .$$

The lemma follows from this fact, since $\varphi(z)$ is an operator independent of x and hence permutable with the symbol $z^{-\nu} \log z$.

Lemma 3. The operator

$$A_n(x,z) = \varphi(z) \int_0^\infty e^{-xs} \log^n (z-s)\,(z-s)^{-\nu}\, ds \ ,$$

$$R(x) > 0 \ ,$$

is equivalent to the operator

$$B_n(x,z) = z^{-\nu} \log^n z \to [\varphi(z)/x] \ ,$$

where $\varphi(z)$ is analytic about $z = 0$ and $z^{-\nu} \log^n z$ is defined by (11.10) *of chapter* 2.

Proof: The equivalence stated in the lemma can be justified by an analysis which differs from that given above only in the introduction of a greater complexity of detail.

If, however, we note the principle introduced in the derivation of (11.8) of chapter 2, namely that differentiation with respect to the parameter ν is justified when the operator involves $z^{-\nu}$, we derive from the operational identity

$$A_0(x,z) = B_0(x,z) = z^{-\nu} \to [\varphi(z)/x] \ ,$$

the desired result

$$\frac{d^n}{d\nu^n} A_0(x,z) = (-1)^n A_n(x,z) = (-1)^n B_n(x,z) \ ,$$

We are now in a position to prove the following theorem:

Theorem 9. If the differential equation (2.4) *has the origin as a regular singular point and if the numbers of the sequence* $\{\lambda_i\}$, $i = 1, 2, \cdots, p$, *where the λ_i are roots of the indicial equation,*

$$A_0(0) + A_1'(0)\lambda + A_2''(0)\lambda(\lambda-1)/2! + \cdots$$

$$+ A_p^{(n)}(0)\lambda(\lambda-1) \cdots (\lambda-r+1)/n! = 0 \ ,$$

do not differ by integers, then the formal resolvent operator for equation* (15.1) *is given by

$$X_0(x,z) = z^{-\lambda_1-1}\, y_1(z) \to [z^{\lambda_1}x_1(z)/x]$$
$$+ z^{-\lambda_2-1}\, y_2(z) \to [z^{\lambda_2}x_2(z)/x] + \cdots$$
$$+ z^{-\lambda_p-1}\, y_p(z) \to [z^{\lambda_p}\, x_p(z)/x] \ , \qquad (15.10)$$

where the $z^{\lambda_i}\, x_i(z)$ *are a fundamental set of solutions of equation* (2.4), *the* $z^{-\lambda_i-1}\, y_i(z)$ *are their Lagrange adjoints, and* $z^{-\nu}$ *is the fractional operator previously defined. If* r *of the values* $\{\lambda_i\}$ *differ by integers, then in general* r *members of* (15.10) *will be replaced by terms which contain operators of the form*

$$z^{\mu} \log^n z\, y(z) \to [z^{\lambda} \log^m z\, x(z)/x] \ , \ m \textit{ and } n \textit{ integers,}$$

where $z^{-\nu} \log^s z$ *is defined by* (11.10) *of chapter* 2 *and* $y(z)$, $x(z)$ *are functions regular about the origin and do not vanish there.*

Proof: In order to prove this theorem let us first note that the function,

$$W(z,t) = \sum_{i=1}^{p} X_i(z)\, Y_i(t) \ ,$$

which appears in the resolvent operator,

$$X_0(x,z) = e^{-xz} \int_{-\infty}^{z} e^{xt}\, W(z,t)\, dt \ , \quad R(x) > 0 \ , \qquad (15.11)$$

may be written in the form,

$$W(z,t) = \sum_{i=1}^{p} z^{\lambda_i}\, t^{-1-\lambda_i} x_i(z)\, y_i(t) \ , \qquad (15.12)$$

where the λ_i are roots of the indicial equation. We assume that the differences $\lambda_i - \lambda_j$ are not integers. The functions $x_i(z)$ and $y_i(t)$ are regular about zero and do not vanish there.

Making in (15.11) the transformation $t = z - s$, and substituting the right member of (15.12) for $W(z,t)$, we obtain

$$X_0(x,z) = \int_0^{\infty} e^{-xs} \sum_{i=1}^{p} z^{\lambda_i} (z-s)^{-1-\lambda_i}\, x_i(z)\, y_i(z-s)\, ds \ . \qquad (15.13)$$

Finally taking account of lemma 1, we get from this equation,

$$X_0(x,z) = \sum_{i=1}^{p} z^{\lambda_i - 1}\, y_i(z) \to [z^{\lambda_i}\, x_i(z)/x] \ ,$$

which establishes the first statement of the theorem.

If we now remove the restriction that the differences $\lambda_i - \lambda_j$ are not integers and assume instead that r of the characteristic numbers differ by integers, then the fundamental solutions $X_i(z)$ and

$Y_i(t)$ may contain terms of the form $z^\mu \log^n z\, x(z)$ and $t^\nu \log^m t\, y\,(t)$. Hence r terms of $W(z,t)$ will be replaced by a function of the form

$$\vartheta(z,t) = z^\lambda t^{-1-\lambda} \sum \vartheta_{mn}(z,t) \log^m t \cdot \log^n z \ ,$$

where the functions $\vartheta_{mn}(z,t)$ are regular about the point $(0,0)$.

Furthermore the functions $\vartheta_{mn}(z,t)$ have the form $\sum_{r=1}^{r} Z_r(z) T_r(t)$, from which it follows that terms will appear in (15.13) of the form

$$z^\lambda Z_r(z) \log^p z \int_0^\infty e^{-xs} \log^q(z-s)\,(z-s)^{-1-\lambda}\, T_r(z-s)\,ds \ .$$

But by means of lemmas 2 and 3 we see at once that this expression is equivalent to the following:

$$z^{-1-\lambda} \log^q z\, T_r(z) \to [z^\lambda Z_r(z) \log^p z/x] \ .$$

These results combine into the second statement in the theorem.

We shall now apply the foregoing results to obtain the formal inverses of two linear equations. The first is a singular integral equation of Volterra type, the second a difference equation which is used by G. Wallenberg and A. Guldberg (see *Bibliography*) as an illustrative example throughout their treatise. We have previously examined the homogeneous case of the latter example in section 4.

Example 1. The integral equation,

$$\begin{aligned} \int_x^\infty \{ [-\tfrac{1}{2}\alpha\beta + \tfrac{1}{2}(\gamma-\alpha-\beta-1)x]\, e^{x-t} & \\ + [\alpha\beta + (2\alpha+2\beta+2-\gamma)x + 2x^2] e^{2(x-t)} & \\ - [\tfrac{1}{2}\alpha\beta + \tfrac{1}{2}(3\alpha+3\beta+3-\gamma)x & \\ + 3x^2]\, e^{3(x-t)}\}\, u(t)\,dt = f(x) \ , & \quad (15.14) \end{aligned}$$

has as its resolvent equation the hypergeometric differential equation multiplied by the reciprocal of $\pi(z) = (1-z)(2-z)(3-z)$, namely,

$$\frac{1}{\pi(z)} \{ z(1-z) X''(z) + [\gamma - (\alpha+\beta+1)z]\, X'(z) - \alpha\beta X(z) \} = 0 \cdot$$

In order to simplify the discussion we shall consider only a single specialization, namely, $\alpha = \beta = \gamma = 1$, for which we obtain the resolvent equation,

$$\frac{z(1-z)}{\pi(z)} X''(z) + \frac{(1-3z)}{\pi(z)} X'(z) - \frac{1}{\pi(z)} X(z) = 0 \cdot$$

A fundamental set of solutions for this equation is found to be $X_1(z) = 1/(1-z)$, $X_2(z) = \log z/(1-z)$.

Solutions of the adjoint,

$$\frac{d^2}{dt^2}\left\{\frac{t(1-t)}{\pi(t)}Y(t)\right\} - \frac{d}{dt}\left\{\frac{(1-3t)}{\pi(t)}Y(t)\right\} - \frac{1}{\pi(t)}Y(t) = 0 ,$$

are computed from the relations,

$$Y_1(t) = X_2(t)/\left\{W(t)\frac{t(1-t)}{\pi(t)}\right\} ,$$

$$Y_2(t) = -X_1(t)/\left\{W(t)\frac{t(1-t)}{\pi(t)}\right\} ,$$

where $W(t)$ is the Wronskian of the solutions $X_1(t)$, $X_2(t)$.

The values of the solutions of the adjoint equation are explicitly,

$$Y_1(t) = -\log t\,\pi(t) \quad , \quad Y_2(t) = \pi(t) \ .$$

In terms of these functions, the Cauchy function becomes,

$$W(z,t) = \frac{\pi(t)}{1-z}\log (z/t) \ \cdot$$

Hence the formal inverse of (15.14), subject to the assumed specialization, becomes,

$$\begin{aligned} u(x) &= \int_0^\infty e^{-xs}\,W(z, z-s)\,ds \rightarrow f(x) \ , \\ &= \{[\log z/(1-z)]\int_0^\infty e^{-xs}\,\pi(z-s)\,ds \\ &\quad - [1/(1-z)]\int_0^\infty e^{-xs}\,\pi(z-s)\,\log(z-s)\,ds\} \rightarrow f(x) \ . \end{aligned}$$

Now invoking theorem 9, we may write this equation in the form,

$$\begin{aligned} u(x) &= \pi(z) \rightarrow \{[z^{-1}\log z\frac{z}{1-z} \rightarrow f(x)]/x\} \\ &\quad - \pi(z)\,\log z \rightarrow \left\{[\frac{1}{1-z} \rightarrow f(x)]/x\right\} \end{aligned} \qquad (15.15)$$

We now note that

$$z^n/(1-z) \rightarrow g(x) = \int_0^\infty e^{-s}\,g^{(n)}(x+s)\,ds \ ,$$

and we shall employ the abbreviation,

$$g(x) = \int_0^\infty e^{-s}\,f(x+s)\,ds \ . \qquad (15.16)$$

Returning to (15.15) and making use of (15.16) and (15.17), we may write $u(x)$ in the form,

$$u(x) = \pi(z) \to \{[z^{-1} \log z \to g'(x)]/x\}$$
$$- \pi(z) \log z \to \{g)x)/x\} ,$$
$$= \pi(z) \to \{[z^{-1} \log z \to g'(x)]/x\}$$
$$- \pi(z) \to \{z^{-1} \log z \to [g'(x)/x - g(x)/x^2]\} .$$

Making use of formula (11.1), chapter 2, we then obtain

$$u(x) = \pi(z) \to (1/x) \int_c^x [\psi(1) - \log(x-t)] \, g'(t) dt$$
$$- \pi(z) \to \int_c^x [\psi(1) - \log(x-t)] [g'(t)/t - g(t)/t^2] \, dt ,$$
$$= \pi(z) \to \int_c^x [\psi(1) - \log(x-t)] [g'(t)/x$$
$$- g'(t)/t + g(t)/t^2] \, dt ,$$
$$= \pi(z) \to \int_c^x [\psi(1) - \log(x-t)] \, h(x,t) dt ,$$

where we abbreviate $h(x,t) = g'(t)/x - g'(t)/t + g(t)/t^2$.

In order to interpret this last result it will be necessary to employ theorem 4 of chapter 6. By means of it we are at once able to compute,

$$z \to \int_c^x [\psi(1) - \log(x-s)] \, h(x,s) ds$$
$$= \psi(1) h(x,x) - [h(x,c) \log(x-c)]/c$$
$$+ \int_c^x \{\psi(1) h_x'(x,s) - [h_x'(x,s) + h_s'(x,s)] \log(x-s)\} \, ds .$$

The derivatives z^2 and z^3 which also appear in $\pi(z)$ are similarly attained and the formal problem is thus completely resolved.

Example 2. Let us consider the formal solution of the difference equation,

$$x(x+1) u(x+2) - 2x(x+2) u(x+1)$$
$$+ (x+2)(x+1) u(x) = f(x) , \quad (15.17)$$

which, replacing $u(x+n)$ by $e^{nz} \to u(x)$, can be written

$$[x^2(e^{2z}-2e^z+1)+x(e^{2z}-4e^z+3)+2] \to u(x) = f(x) \; . \tag{15.18}$$

Since we have already considered the homogeneous case of this equation in example 4, section 4, we shall content ourselves here with the determination of a particular solution of the non-homogeneous problem.

The generatrix equation,

$$(e^{2z}-2e^z+1)X''(z)+(e^{2z}-4e^z+3)X'(z)+2X(z)=0 \; , \tag{15.19}$$

has $z=0$ as a regular singular point, and from the indicial equation, $\lambda(\lambda-1)-2\lambda+2=0$, we obtain the corresponding indicial numbers $\lambda=1, \lambda=2$.

A set of fundamental solutions of (15.19) is seen to be,

$$X_1(z)=(e^z-1)^2/e^{2z} \; , \quad X_2(z)=(e^z-1)/e^{2z} \; .$$

Computing the corresponding adjoint functions, we get

$$Y_1(t)=e^t/(e^t-1)^3 \; , \quad Y_2(t)=-e^t/(e^t-1)^2 \; ,$$

and from them the Cauchy function,

$$\begin{aligned} W(z,t) &= X_1(z)Y_1(t)+X_2(z)Y_2(t) \; , \\ &= e^{-2z}\, e^t(e^z-1)(e^t-1)^{-3}(e^z-e^t) \; . \end{aligned}$$

Employing this expression, we find the explicit form of the resolvent operator to be,

$$X_0(x,z)=(e^z-1)\int_0^\infty e^{-xs}\{e^{-s}(1-e^{-s})/(e^{z-s}-1)^3\}\, ds \; , \qquad R(x)>0 \; .$$

Since $s-z$ appears as a cubic singularity of the integrand we write the operator in the form,

$$\begin{aligned} X_0(x,z) &= (e^z-1)\int_0^\infty \{e^{-xs}[\varphi(s,z)-(e^{-2z}-e^{-z}) \\ &\quad +(s-z)(e^{-2z}+e^{-z})/2!]/(s-z)^3 ds\} \\ &\quad +(e^z-1)\{(e^{-2z}-e^{-z})\int_0^\infty [e^{-xs}/(s-z)^3]\, ds \\ &\qquad -(e^{-2z}-e^{-z})\int_0^\infty [e^{-xs}/2(s-z)^2] ds \; , \end{aligned}$$

where we abbreviate $\varphi(s,z)=(e^{-2s}-e^{-s})(e^{z-s}-1)^3/(z-s)^3$.

With attention to the identities,

$$e^{az} \to f(x) = f(x+a) \ , \quad \Delta f(x) = f(x+1) - f(x) \ ,$$
$$\Delta^2 f(x) = f(x+2) - 2f(x+1) + f(x) \ ,$$

we get as the formal solution of (15.18) the function,

$$u(x) = \int_0^{\infty} \{e^{-xs}[\varphi(s,z) - (e^{-2z} - e^{-z}) + (s-z)(e^{-2z} + e^{-z})/2]/(s-z)^3\} \, ds \to \Delta f(x)$$
$$+ \int_c^x [\{(x-t)^2 \Delta^2 f(t-2) + (x-t)[f(t-2) - f(t)]\}/2t] \, dt + \Omega(x) \ , \qquad (15.20)$$

where $\Omega(x) = A_0 + A_1 x + A_2 x^2$. The last two terms of $\Omega(x)$ constitute the general solution of the homogeneous case; the constant A_0 is to be so determined that (15.17) is satisfied at the point $x = c$.

Let us evaluate (15.20) for two special functions, namely $f(x) = 1$ and $f(x) = x$.

In the first case we get,

$$u(x) = A_0 + A_1 x + A_2 x^2 \ .$$

The constant A_0 is determined from (15.17) by noting that $u(0) = A_0 = \frac{1}{2}$.

In the second case, where $f(x) = x$, we get

$$u(x) = \int_0^{\infty} e^{-xs}\{-e^s/(1-e^s) + 1/s^2\} \, ds - x \log x + x + \Omega(x).$$

Integrating by parts we then obtain,

$$u(x) = e^{-xs}\{e^s/(e^s-1) - 1/s\}\Big|_0^{\infty} + x \int_0^{\infty} e^{-xs}\{e^s/(e^s-1) - 1/s\} ds - x \log x + x + \Omega(x) \ . \qquad (15.21)$$

We now note the definition,*

$$\psi(x) = \log x + \int_0^{\infty} e^{-xs}\{e^s/(1-e^s) + 1/s\} \, ds \ , \quad R(x) > 0 \ ,$$

where $\psi(x)$ satisfies the equation,

$$\psi(x+1) - \psi(x) = 1/x \ , \quad \psi(1) = -C \text{ (Euler's constant)}.$$

Hence (15.21) becomes,

$$u(x) = -\tfrac{1}{2} - x\,\psi(x) + x + \Omega(x) \ .$$

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*See Davis: *Tables of the Higher Mathematical Functions*, vol. 1 (1933), p. 277.

We must now determine the value of A_0 in $\Omega(x)$, so that the original equation is satisfied. We notice that this requires that $u(0) = 0$. Noting that $\lim_{x=0} x\,\psi(x) = -1$, we obtain for the desired solution the function,

$$u(x) = -1 - x\,\psi(x) + x + A_1 x + A_2 x^2 \ .$$

It is possible, of course, to obtain these particular results from the expansion $X_0(x,z)$ as a power series in z. This may be accomplished without essential difficulty and to four terms may be shown to equal:

$$\begin{aligned} X_0(x,z) = \tfrac{1}{2} &- [x\,\psi(x) + 1 + x/2]\,z + [3x^2\,\psi(x) \\ &+ 2x\,\psi(x) + 2 + 2x + x^2/2]\,z^2/2 - [6x^3\,\psi(x) \\ &+ 9x^2\,\psi(x) + 4x\,\psi(x) + 4 + 6x + 3x^2]z^3/6 + \cdots \ . \end{aligned}$$

PROBLEMS

1. Show that the equation

$$(2x+1)(3x+4)\,u(x+1) - (2x+7)(3x+1)\,u(x) = 2x+1$$

has the solution

$$u(x) = -\tfrac{1}{4}(2x+1)/(3x+1) + \pi(x)[(2x+1)(2x+3)(2x+5)/(3x+1)] \ .$$

In this solution $\pi(x)$ is a function of unit period.

(Markoff and Selivanoff).

2. Express the integral equation

$$\int_0^x [t^2 + 3t(t-x) - (t-x)^2]\,u(t)\,dt = f(x)$$

in the form

$$(x^2/z - 5x/z^2 + 6/z^3) \rightarrow u(x) = f(x) \ .$$

Hence show that the equation has the particular solution

$$u(x) = (\sqrt{3}/6)\{z^{-\lambda_1+2} \rightarrow (z^{\lambda_1}/x \rightarrow f) - z^{-\lambda_2+2} \rightarrow (z^{\lambda_2}/x \rightarrow f) \ ,$$

where we write $\lambda_1 = 3 + \sqrt{3}$, $\lambda_2 = 3 - \sqrt{3}$.

3. Show that if $f(x) = x^\mu$, $\mu > 2$, then the equation of problem 2 has the general solution

$$u(x) = c\,x^\lambda + \frac{\Gamma(\mu+1)}{\Gamma(\mu-2)}\,\frac{x^{\mu-3}}{\mu^2-6\mu+6} \ , \qquad \lambda = \sqrt{3} \ .$$

4. Express the integral equation

$$\int_0^x [t^2 + t(t-x) + (t-x)^2/2!]\,u(t)\,dt = f(x)$$

in the form

$$(x^2/z - 3x/z^2 + 5/z^3) \rightarrow u(x) = f(x) \ .$$

Hence show that if $f(x)$ is a function which vanishes together with its derivatives of first two orders at $x=0$, then the general solution of the integral equation is given by

$$u(x) = (1/x) \int_0^x \sin[\log(x/t)]\, f(t)\, dt \ .$$

5. Derive lemma 2 of this section from lemma 1 by invoking the principle introduced in the derivation of (11.8) of chapter 2.

6. Solve the equation

$$(1 + xz + x^2 z^2) \rightarrow u(x) = x^3 \ .$$

7. Find the solution of the equation

$$(4 - 3xz + x^2 z^2) \rightarrow u(x) = x^2 \ .$$

8. The ordinary differential equation

$$(a_0 + a_1 xz + a_2 x^2 z^2/2! + \cdots + a_n x^n z^n/n!) \rightarrow u(x) = f(x)$$

is called *Euler's equation* (see next chapter). Discuss its general solution by the methods of this section.

9. Discuss the solution of the equation

$$x \Delta u(x) + \lambda u(x) = f(x) \ .$$

CHAPTER IX

THE GENERALIZED EULER DIFFERENTIAL EQUATION OF INFINITE ORDER

1. The Functional Equation. The following rather general type of functional equation,*

$$\int_0^x \frac{u(t)}{(x-t)^\nu}\,dt = x^{1-\nu}\sum_{i=0}^{m} c_i\, u(\mu_i x) + f(x) \ ,$$

$$0 \leqq \nu < 1 \ , \qquad (1.1)$$

where $f(x)$ is a known function of the form $x^{1-\nu}\,g(x)$, $g(x)$ being analytic about $x = 0$, and where the c_i and μ_i are any set of constants, leads to the integration of what is referred to in the literature of differential equations as the *Euler equation:*†

$$F(x) = a_0 u(x) + a_1 x u'(x)$$

$$+ \frac{a_2}{2!}\,x^2 u''(x) + \frac{a_3}{3!}\,x^3 u'''(x) + \cdots \ , \qquad (1.2)$$

where the a_n are bounded as $n \to \infty$ and $F(x)$ is analytic about $x = 0$.

It will be seen that (1.1) generalizes the functional equation,

$$\int_0^x u(t)\,dt = \frac{x}{6}\,[u(0) + 4u(\tfrac{1}{2}x) + u(x)] \ , \qquad (1.3)$$

and also includes as a special case Abel's well known integral equation,

$$\int_0^x [u(t)/(x-t)^\nu]\,dt = f(x) \ . \qquad (1.4)$$

In order to exhibit the connection between (1.1) and (1.2), we assume that $u(t)$ and $u(\mu_i x)$ can be developed into the following Taylor's series:

$$u(t) = u(x) + (t-x)\,u'(x) + \frac{(t-x)^2}{2!}\,u''(x) + \cdots \ ,$$

$$u(\mu_i x) = u(x) + (\mu_i - 1)\,x u'(x) + \frac{(\mu_i-1)^2}{2!}\,x^2\,u''(x) + \cdots \ .$$

*This equation is related to a functional equation studied by P. J. Browne: *Annales de Toulouse*, vol. 4 (3), pp. 63-198. See also C. Popovici: *Comptes Rendus*, vol. 158, pp. 1866-1869. Also, *American Mathematical Monthly*: Question 34 (*1917*, 134, 341; *1920*, 114, 301, 405, 460; *1921*, 19) and problem 3076 (*1924*, 254).

†For the finite case see: E. L. Ince: *Ordinary Differential Equations*, London (1927), pp. 141-144; pp. 534-536.

When these values have been substituted in equation (1.1), we obtain a differential equation of Euler type where $F(x) = x^{\nu-1} f(x)$ and

$$a_n = (-1)^n[1/(n-\nu+1) - \sum_{i=0}^{m} c_i(1-\mu_i)^n] \ .$$

In equation (1.3) these values of a_n are seen to reduce to

$$(-1)^n[1/(n+1) - 1/6 - 1/(3\cdot 2^{n-1})] \ , \quad a_0 = 0 \ ,$$

and for Abel's equation we have,

$$a_n = (-1)^n/(n+1-\nu) \ , \tag{1.5}$$

It now happens rather curiously that all the formal aspects of the solution of equation (1.2) are preserved in the discussion of a considerably more general equation which we shall call the *generalized Euler differential equation of infinite order.* This equation we shall write in the following form:

$$a_0u(x) + (a_1 + b_1x)u'(x) + (a_2 + b_2x + c_2x^2/2!)u''(x)$$
$$+ (a_3 + b_3x + c_3x^2/2! + d_3x^3/3!)u'''(x) + \cdots = g(x) \ . \tag{1.6}$$

It will be convenient to abbreviate this equation as follows:

$$G(x,z) \to u(x) = g(x) \ , \tag{1.7}$$

where $G(x,z)$ is the generatrix function,

$$a_0 + (a_1 + b_1x)z + (a_2 + b_2x + c_2x^2/2!)z^2 + \cdots \ .$$

2. The Homogeneous Case of the Generalized Euler Equation. It will be desirable for us first to examine the homogeneous case of equation (1.6),

$$G(x,z) \to u(x) = 0 \ . \tag{2.1}$$

Let us first assume that the solution can be expanded in a series of the form,

$$u(x) = x^\lambda \varphi(x) = x^\lambda(\varphi_0 + \varphi_1/x$$
$$+ \varphi_2/x^2 + \cdots + \varphi_n/x^n + \cdots) \ . \tag{2.2}$$

It will be convenient to adopt the following notation:

$$f_0(\lambda) = a_0 + b_1\lambda + c_2\lambda(\lambda-1)/2! + d_3\lambda(\lambda-1)(\lambda-2)/3! + \cdots \ ,$$
$$f_1(\lambda) = a_1\lambda + b_2\lambda(\lambda-1) + c_3\lambda(\lambda-1)(\lambda-2)/2!$$
$$+ d_4\lambda(\lambda-1)(\lambda-2)(\lambda-3)/3! + \cdots \ ,$$
$$f_2(\lambda) = a_2\lambda(\lambda-1) + b_3\lambda(\lambda-1)(\lambda-2)$$
$$+ c_4\lambda(\lambda-1)(\lambda-2)(\lambda-3)/2! + \cdots \ , \tag{2.3}$$

or, if we employ the abbreviation,

$$\lambda^{(n)} = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1) ,$$

$$f_n(\lambda) = a_n\lambda^{(n)} + b_{n+1}\lambda^{(n+1)} + c_{n+2}\lambda^{(n+2)}/2! + \cdots .$$

We can then state the following theorem:

Theorem 1. If the equation,

$$f_0(\lambda) = 0 , \tag{2.4}$$

has a set of roots, $\lambda_1, \lambda_2, \cdots, \lambda_r$, which do not differ from one another by integers, and if for each one of these roots the infinite matrix of quantities $\|f_i(\lambda-m)\|$, $i = 0, 1, 2, \cdots$, $m = 1, 2, \cdots$, exists, then equation (2.1) *possesses r formal solutions of the form,*

$$u(x) = x^{\lambda_i}(\varphi_0 + \varphi_1/x + \varphi_2/x^2 + \cdots + \varphi_n/x^n + \cdots) , \tag{2.5}$$

where the φ_n are explicitly calculated from the formula $\varphi_n = \varphi_n(\lambda_i)$, in which we write,

$$\varphi_n(\lambda) = \\ \frac{-\varphi_0 \begin{vmatrix} f_0(\lambda-1) & 0 & 0 & \cdots & 0 & f_1(\lambda) \\ f_1(\lambda-1) & f_0(\lambda-2) & 0 & \cdots & 0 & f_2(\lambda) \\ f_2(\lambda-1) & f_1(\lambda-2) & f_0(\lambda-3) & \cdots & 0 & f_3(\lambda) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{n-1}(\lambda-1) & f_{n-2}(\lambda-2) & f_{n-3}(\lambda-3) & \cdots & f_1(\lambda-n+1) & f_n(\lambda) \end{vmatrix}}{f_0(\lambda-1)\, f_0(\lambda-2)\, f_0(\lambda-3)\cdots f_0(\lambda-n)} . \tag{2.6}$$

Proof: The proof of this theorem is obtained from an explicit substitution of (2.2) in (2.1) and the subsequent use of the following identity:

$$\lambda^{(n)} - {}_nC_1 m\lambda^{(n-1)} + {}_nC_2 m(m+1)\lambda^{(n-2)} \\ - {}_nC_3 m(m+1)(m+2)\lambda^{(n-3)} + \cdots \\ = (\lambda-m)(\lambda-m-1)(\lambda-m-2)\cdots(\lambda-m-n+1) , \tag{2.7}$$

where the ${}_nC_i$ are the binomial coefficients.*

*This is referred to by G. Chrystal in his *Algebra*, vol. 2 (1889), p. 9 as *Vandermonde's theorem*, although it was probably employed before the time of C. A. Vandermonde (1735-1796) who stated it in his memoir: Sur des irrationnelles des différens ordres avec une application au cercle. *Histoire de l'Académie Royale des Sciences* (1772), pt. I, Paris (1775) pp. 489-498.

This identity is easily established as follows: Taking the nth derivative of the function $x^\lambda x^{-m}$ and setting $x = 1$ we get by the rule of Leibnitz:

$$\lim_{x=1} \frac{d^n}{dx^n} (x^\lambda x^{-m}) = \lambda^{(n)} - {}_nC_1 m\lambda^{(n-1)} + {}_nC_2 m(m+1)\lambda^{(n-2)} - \cdots .$$

But an expression identical with this is obtained as the limit,

$$\lim_{x=1} \frac{d^n}{dx^n} x^{\lambda-m} = (\lambda-m)(\lambda-m-1)(\lambda-m-2)\cdots(\lambda-m-n+1) .$$

We now substitute (2.2) in (2.1) and arrange coefficients in the following manner:

$$G(x,z) \to u(x) \equiv \tag{2.8}$$

$$\begin{aligned}
&x^\lambda (a_0 + b_1\lambda + c_2\lambda^{(2)}/2! + d_3\lambda^{(3)}/3! + \cdots)\,\varphi(x) \\
&\qquad + (b_1 + 2c_2\lambda/2! + 3d_3\lambda^{(2)}/3! + \cdots)\,x\,\varphi'(x) \\
&\qquad\qquad + (c_2/2! + 3d_3\lambda/3! + \cdots)\,x^2\,\varphi''(x) \\
&\qquad\qquad\qquad + (d_3/3! + \cdots)\,x^3\,\varphi'''(x) \\
&\qquad\qquad\qquad + \cdots\cdots\cdots \\
&\quad + (a_1\lambda + b_2\lambda^{(2)} + c_3\lambda^{(3)}/2! + d_4\lambda^{(4)}/3! + \cdots)\,\varphi(x)/x \\
&\quad + (a_1 + 2b_2\lambda + 3c_3\lambda^{(2)}/2! + 4d_4\lambda^{(3)}/3! + \cdots)\,\varphi'(x) \\
&\qquad + (b_2 + 3c_3\lambda/2! + 6d_4\lambda^{(2)}/3! + \cdots)\,x\,\varphi''(x) \\
&\qquad\qquad + (c_3/2! + 4d_4\lambda/3! + \cdots)\,x^2\,\varphi'''(x) \\
&\qquad\qquad\qquad + (d_4/3! + \cdots)\,x^3\,\varphi^{(4)}(x) \\
&\qquad\qquad\qquad + \cdots\cdots\cdots \\
&+ (a_2\lambda^{(2)} + b_3\lambda^{(3)} + c_4{}^{(4)}/2! + d_5\lambda^{(5)}/3! + \cdots)\,\varphi(x)/x^2 \\
&+ (2a_2\lambda + 3b_3\lambda^{(2)} + 4c_4\lambda^{(3)}/2! + 5d_5\lambda^{(4)}/3! + \cdots)\,\varphi'(x)/x \\
&+ (a_2 + 3b_3\lambda + 6c_4\lambda^{(2)}/2! + 10d_5\lambda^{(3)}/3! + \cdots)\,\varphi''(x) \\
&\qquad + (b_3 + 4c_4\lambda/2! + 10d_5\lambda^{(2)}/3! + \cdots)\,x\,\varphi'''(x) \\
&\qquad\qquad + (c_4/2! + 5d_5\lambda/3! + \cdots)\,x^2\,\varphi^{(4)}(x) \\
&\qquad\qquad\qquad + (d_5/3! + \cdots)\,x^3\,\varphi^{(5)}(x) \\
&\qquad\qquad\qquad + \cdots\cdots\cdots \\
&+ (a_3\lambda^{(3)} + b_4\lambda^{(4)} + c_5\lambda^{(5)}/2! + d_6\lambda^{(6)}/3! + \cdots)\,\varphi(x)/x^3 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Noting the fact that the mth derivative of $\varphi(x)$ is,

$$\varphi^{(m)}(x) = (-1)^m[m!\varphi_1/x^{m+1} + (m+1)!\varphi_2/1!\,x^{m+2}$$
$$+ (m+2)!\varphi_3/2!\,x^{m+3} + \cdots], \quad m > 0 ,$$

we sum the columns of the above array for the coefficients of x^{-m} and find that the multipliers of a_i, b_i, c_i $\cdots$ are sums of the form given in the left member of (2.7). Transforming these by means of the identity we obtain for the expansion of (2.8) the following series:

$$G(x,z) \to u(x) = x^\lambda\{f_0(\lambda)\varphi_0 + [f_1(\lambda)\varphi_0 + f_0(\lambda-1)\varphi_1]/x$$
$$+ \cdots + [f_n(\lambda)\varphi_0 + f_{n-1}(\lambda-1)\varphi_1 + f_{n-2}(\lambda-2)\varphi_2$$
$$+ \cdots + f_0(\lambda-n)\varphi_n]/x^n + \cdots .$$

Equating the coefficients of x^{-n} to zero,

$$f_n(\lambda)\varphi_0 + f_{n-1}(\lambda-1)\varphi_1 + f_{n-2}(\lambda-2)\varphi_2 + \cdots + f_0(\lambda-n)\varphi_n = 0 ,$$
$$n = 0, 1, 2, \cdots , \qquad (2.9)$$

and solving for φ_n, $n > 0$, in terms of φ_0, we are led to the equation,

$$f_0(\lambda) = 0 ,$$

and the determinant (2.6). These results are expressed in the statement of the theorem.

As in the analogous theory of the Fuchsian equation of finite order (the linear differential equation with regular singular points)* as it relates to expansions about the origin, we see from (2.6) that the formal solution (2.5) cannot be attained if two of the characteristic numbers differ by an integer, $\lambda_i - \lambda_j = m$, since in this case $f_0(\lambda_i - m)$ would vanish.

To avoid this difficulty we assume the existence of a solution of the form,

$$u(x) = x^\lambda \log x(\psi_0 + \psi_1/x + \psi_2/x^2 + \cdots)$$
$$+ x^\mu(\varphi_0 + \varphi_1/x + \varphi_2/x^2 + \cdots) .$$

Substituting this function in equation (2.1) we find after a tedious calculation similar in detail to the one given above the following identity:

*See E. L. Ince: *Ordinary Differential Equations*, pp. 365-375.

$$\begin{aligned}G(x,z) \to u(x) &\equiv x^\lambda \log x\{f_0(\lambda)\psi_0 + [f_1(\lambda)\psi_0 + f_0(\lambda-1)\psi_1]/x \\ &\quad + [f_2(\lambda)\psi_0 + f_1(\lambda-1)\psi_1 + f_0(\lambda-2)\psi_2]/x^2 + \cdots\} \\ &\quad + x^\lambda\{f_0'(\lambda)\psi_0 + [f_1'(\lambda)\psi_0 + f_0'(\lambda-1)\psi_1]/x \\ &\quad + [f_2'(\lambda)\psi_0 + f_1'(\lambda-1)\psi_1 + f_0'(\lambda-2)\psi_2]/x^2 + \cdots\} \\ &\quad + x^\mu\{f_0(\mu)\varphi_0 + [f_1(\mu)\varphi_0 + f_0(\mu-1)\varphi_1]/x \\ &\quad + [f_2(\mu)\varphi_0 + f_1(\mu-1)\varphi_1 + f_0(\mu-2)\varphi_2]/x^2 + \cdots\} .\end{aligned}$$

If λ be the smaller of the two characteristic numbers λ and $\mu = \lambda + p$, we can compute ψ_i in terms of ψ_0 by the algorithm stated in (2.6). And since $\mu = \lambda + p$ is also a root of $f_0(\lambda) = 0$, we get for the determination of φ_i the following equations:

$$\begin{aligned}&f_1(\lambda+p)\varphi_0 + f_0(\lambda+p-1)\varphi_1 = 0 , \\ &f_2(\lambda+p)\varphi_0 + f_1(\lambda+p-1)\varphi_1 + f_0(\lambda+p-2)\varphi_2 = 0 , \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &f_p(\lambda+p)\varphi_0 + f_{p-1}(\lambda+p-1)\varphi_1 + \cdots \\ &\qquad\qquad + [f_0(\lambda)\varphi_p + f_0'(\lambda)\psi_0] = 0 , \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{aligned} \tag{2.10}$$

If λ is not a double root of $f_0(\lambda) = 0$, this system can be solved uniquely for φ_i in terms of φ_0 and ψ_0. If λ is a double root, however, or if three roots exist which differ by integers, λ, $\mu = \lambda + m$, $\nu = \lambda + m + n$, we must then assume a solution of the form,

$$\begin{aligned}u(x) &= x^\lambda \log^2 x(\vartheta_0 + \vartheta_1/x + \vartheta_2/x^2 + \cdots) \\ &\quad + x^\mu \log x(\psi_0 + \psi_1/x + \psi_2/x^2 + \cdots) \\ &\quad + x^\nu(\varphi_0 + \varphi_1/x + \varphi_2/x^2 + \cdots) .\end{aligned}$$

If this function is substituted in equation (2.1) we get

$$\begin{aligned}G(x,z) \to u(x) &\equiv x^\lambda \log^2 x\{f_1(\lambda)\vartheta_0 + [f_1(\lambda)\vartheta_1 + f_0(\lambda-1)\vartheta_0/x \\ &\quad + \cdots\} + x^{\lambda+m}\log x\{f_0(\lambda+m)\psi_0 + [f_1(\lambda+m)\psi_0 \\ &\quad + f_0(\lambda+m-1)\psi_1]/x + \cdots + [f_m(\lambda+m)\psi_0 \\ &\quad + f_{m-1}(\lambda+m-1)\psi_1 + \cdots + f_1(\lambda+1)\psi_{m-1} + f_0'(\lambda)\vartheta_0]/x^m \\ &\quad + \cdots\} + x^{\lambda+m+n}\{f_0(\lambda+m+n)\varphi_0 + [f_1(\lambda+m+n)\varphi_0 \\ &\quad + f_0(\lambda+m+n-1)\varphi_1]/x + \cdots + [f_n(\lambda+m+n)\varphi_0 \\ &\quad + \cdots + f_1(\lambda+m+1)\varphi_{n-1} + f_0'(\lambda+n)\psi_0]/x^m + \cdots \\ &\quad + [f_{n+m}(\lambda+m+n)\varphi_0 + \cdots + f_1(\lambda+1)\varphi_{n+m-1} \\ &\quad + f_0'(\lambda+n)\psi_0 + \cdots + f_1'(\lambda+1)\psi_{n-1} \\ &\quad + f_0''(\lambda)\vartheta_0]/x^{n+m} + \cdots\} = 0 .\end{aligned}$$

It is clear that the ϑ_i can be computed in terms of ϑ_0 by the algorithm of theorem 1, the ψ_i can be computed in terms of ϑ_0 and ψ_0, and the φ_i in terms of ϑ_0, ψ_0, and φ_0.

This conclusion can be stated with easily established generality as follows:

Theorem 2. If the equation,

$$f_0(\lambda) = 0 \ ,$$

has one root of multiplicity r, or r single roots differing by integers: $\lambda_1 = \lambda$, $\lambda_2 = \lambda + m_1, \cdots, \lambda_r = \lambda + m_{r-1}$, *where the* m_i *are integers, then equation* (2.1) *has r formal solutions of the form,*

$$u_1(x) = x^{\lambda}\, \varphi_{11}(x) \ ,$$

$$u_2(x) = x^{\lambda} \log x\, \varphi_{21}(x) + x^{\lambda+m_1}\, \varphi_{22}(x) \ ,$$

$$\cdots\cdots\cdots\cdots$$

$$u_r(x) = x^{\lambda} \log^r x\ \varphi_{r1}(x) + x^{\lambda+m_1} \log^{r-1} x\, \varphi_{r2}(x) + \cdots$$
$$+ x^{\lambda+m_{r-1}} \log x\, \varphi_{rr-1}(x) + x^{\lambda+m_{r-1}} \varphi_{rr}(x) \ ,$$

where the $\varphi_{ij}(x)$ *are expansions about infinity.*

$$\varphi_{ij}(x) = \varphi_{ij}^{(0)} + \varphi_{ij}^{(1)}/x + \varphi_{ij}^{(2)}/x^2 + \cdots \ .$$

It will be convenient in illustration to consider the hypergeometric equation which is a special case of the generalized Euler equation,

$$(x - x^2)u''(x) + [\gamma - (\alpha+\beta+1)x]u'(x) - \alpha\,\beta\, u(x) = 0 \ . \tag{2.11}$$

Since in our notation

$$a_0 = -\alpha\,\beta \ , \ a_1 = \gamma \ , \ a_2 = 0 \ , \ b_1 = -(\alpha+\beta+1) \ , \ b_2 = 1 \ , \ c_2 = -2 \ ,$$

we shall have,

$$f_0(\lambda) = -(\lambda+\alpha)(\lambda+\beta) \ ; \ f_1(\lambda) = (\gamma-1)\lambda + \lambda^2 \ ; \ f_2(\lambda) = 0 \ .$$

Hence we obtain for the root $\lambda = -\alpha$,

$$\varphi_{n+1}/\varphi_n = - f_1(\lambda-n)/f_0(\lambda-n-1)$$
$$= (n+\alpha)(n+1+\alpha-\gamma)/[(n+1)(n+1+\alpha-\beta)] \ ,$$

from which we get the solution,

$$u_1(x) = x^{\alpha}\{1 + \frac{\alpha(1+\alpha-\gamma)}{(1+\alpha-\beta)x}$$
$$+ \frac{\alpha(\alpha+1)(\alpha-\gamma+1)(\alpha-\gamma-2)}{(\alpha-\beta+1)(\alpha-\beta+2)2!\,x^2} + \cdots\} \ , \tag{2.12}$$

which, in the customary notation of the hypergeometric function,* is

$$u_1(x) = x^{-a} F(a, a - \gamma + 1; a - \beta + 1; 1/x) \ .$$

In a similar way the second solution is obtained for the characteristic root, $\lambda = -\beta$, and we find,

$$u_2(x) = x^{-\beta} F(\beta, \beta - \gamma + 1; \beta - a + 1; 1/x) \ .$$

If $\beta - a = p$ is a positive integer or zero we must employ the result of theorem 2 since the denominator of the coefficient of x^{-a-p} in (2.12) is zero. The solution of the hypergeometric equation will then become,

$$u_1(x) = x^{-\beta} \log x \, F(\beta, \beta - \gamma + 1; \beta - a + 1; 1/x)\, \psi_0$$
$$+ x^{-\beta-p}(\varphi_0 + \varphi_1/x + \varphi_2/x^2 + \cdots) \ ,$$

where we have,

$$\psi_0 = (\beta - 1)(\nu - \beta)/p \ ,$$

$$\varphi_{n+1}/\varphi_n = (n+a)(n+1+a-\gamma)/[(n+1)(n+1-p)] \ ,$$
$$n \leqq p-2 \ ,$$

$$\varphi_p = 0 \ , \ \varphi_{p+m+1} = (\beta+m+1-\gamma)(\beta+m)\varphi_{p+m}/[(m+1)(m+1+p)]$$
$$+ [(\gamma - 1 - 2\beta - 2m)(m+1)(m+1+p)$$
$$+ (2m+2+p)(m+\beta)(m+1+\beta-\gamma)]\psi_m$$
$$\div [(m+1)^2(m+1+p)^2] \ , \qquad m \geqq 0 \ ,$$

in which ψ_m is the coefficient of x^{-m} in the expansion of

$$F(\beta, \beta - \gamma + 1; \beta - a + 1; 1/x) \ .$$

A second solution can be obtained by setting $\varphi_p =$ constant $\neq 0$, or what is equivalent, by adding

$$u_2(x) = x^{-\beta} F(\beta, \beta - \gamma + 1; p+1; 1/x)$$

to this function.

If p is a negative integer or zero, a similar expansion is obtained by interchanging the rôle of β and a.

These solutions will be found to accord with those obtained in another manner by E. Lindelöf.†

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*For ready reference we recall the definition:

$$F(\alpha, \beta; \gamma; x) =$$
$$1 + \frac{\alpha\beta}{1! \cdot \gamma} x + \frac{\alpha(\alpha+1)(\beta(\beta+1)}{2! \cdot \gamma(\gamma+1)} x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{3! \cdot \gamma(\gamma+1)(\gamma+2)} x^3 + \cdots$$

†Sur l'integration de l'équation différentielle de Kummer. *Acta Societatis Scientiarum Fennicae*, vol. 19 (1893), pp. 3-31; in particular, pp. 16-17.

The distinguishing feature of these solutions, it should be especially noted, is found in the fact that we have attained directly the solution of the equation for the regular singular point $x = \infty$. The solution about the origin belongs to the theory developed in the next chapter.

3. Excursus on the Factorial Series $f(\lambda)$. In the last section we exhibited the fundamental rôle played in our theory by the function defined by the factorial series

$$f(\lambda) = a_0 + a_1\lambda + a_2\lambda(\lambda-1)/2! + a_3\lambda(\lambda-1)(\lambda-2)/3! + \cdots \tag{3.1}$$

We shall, therefore, set forth briefly a few facts about this series which we shall find important in a later section.* This series has a considerable history the modern epoch of which began with Abel's investigation of the convergence of the binomial series, that is to say, the special case, $a_n = a^n$.†. The series is frequently referred to as Newton's series because his interpolation formula is another special case.

We have already shown in section 11, chapter 6, that the region of convergence of (3.1) coincides with the region of convergence of a Dirichlet's series and hence the theorems of that section apply to $f(\lambda)$. If the coefficients $\{a_i\}$ are real numbers then the region of convergence of the series (3.1) is a half plane bounded on the left by a *line of convergence*. The point (λ_0) where this line crosses the axis of reals is called the *abscissa of convergence* and is defined analytically by the following limits:

$$\lambda_0 = \{\limsup_{n\to\infty} \log |\sum_{i=0}^{n-1} (-1)^i a_i|/\log n\} - 1 ,$$

$$\lambda_0^* = \{\limsup_{n\to\infty} \log |\sum_{i=n}^{\infty} (-1)^i a_i|/\log n\} - 1 . \tag{3.2}$$

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*The author is indebted to an admirable account of this series by N. E. Nörlund: *Vorlesungen über Differenzenrechnung*, Berlin (1924) pp. 222-240; also: Sur les formules d'interpolation de Stirling et de Newton. *Annales de l'École Normale Supérieure*, vol. 39 (3rd ser.) (1922), pp. 343-403, vol. 40 (3rd ser.) (1923), pp. 35-54. The reader is also referred to the following: J. L. W. V. Jensen: Om Räkkers Konvergens. *Tidsskrift for Math.*, vol. 2 (5th ser.) (1884), pp. 69-72; J. Bendixson: Sur une extension à l'infini de la formule d'interpolation de Gauss. *Acta Mathematica*, vol. 9 (1886), pp. 1-34; E. Landau: Über die Grundlagen der Theorie der Fakultätenreihen. *Sitzungsberichte der Akad. München (math.-phys.)*, vol. 36 (1906), pp. 151-218; S. Pincherle: Alcune spigolature nel campo delle funzioni determinanti. *Atti del IV Congresso dei Matematici*, vol. 2 (1908), pp. 44-48; also: Quelques remarques sur les fonctions déterminantes. *Acta Mathematica*, vol. 36 (1912), pp. 269-280.

†N. H. Abel: *Werke*, vol. 1 (1881), pp .219-250.

The first limit is to be used if $\lambda_0 + 1$ is positive or zero and the second if $\lambda_0^* + 1$ is negative.

The theorem can be illustrated by the binomial theorem, where $a_i = a^i$. If $|a| < 1$, the abscissa of convergence is negative and hence we compute

$$\sum_{i=n}^{\infty} (-1)^i a^i = (-1)^n a^n/(1+a) \ ,$$

$$\lambda_0^* = \{\limsup_{n=\infty} [\log a^n/(1+a)]/\log n\} - 1 = \infty \ .$$

The series thus converges for all values of λ.

If $|a| > 1$, the abscissa of convergence is positive and we compute

$$\sum_{i=0}^{n-1} (-1)^i a^i = [1 + (-1)^{n-1} a^n]/(1+a) \ ,$$

$$\lambda_0 = \{\limsup_{n=\infty} (n \log a/\log n)\} - 1 = \infty \ .$$

The series thus diverges for all values of λ.

Similarly for the case where $a = -1$, we find that $\lambda_0 = 0$, and hence the series converges for all values of λ such that $R(\lambda) > 0$. Also for $a = 1$, $\lambda_0 = 0 - 1$, and the series converges for all values of λ such that $R(\lambda) > -1$.

Factorial series of the type under discussion have one rather unfortunate pecularity, namely, that they include the development of zero. Hence the expansion of a function in such a series is not, in general, unique.

For example, the series

$$F_1(\lambda) = \sum_{n=0}^{\infty} (-1)^n {}_\lambda C_n \ ,$$

where ${}_\lambda C_n$ is the nth binomial coefficient, converges identically to zero in the half plane, $R(\lambda) > 0$.

Similarly the function

$$F_{r+1}(\lambda) = {}_\lambda C_r F_r(\lambda - r) = \sum_{n=r}^{\infty} (-1)^{r+n} {}_n C_r \, {}_\lambda C_n$$

converges identically to zero in the half plane $R(\lambda) > r$.

Thus the null development

$$U(\lambda) = c_1 F_1(\lambda) + c_2 F_2(\lambda) + \cdots + c_n F_n(\lambda) \ ,$$

where $c_1, c_2, \cdots, c_n$ are arbitrary constants, can be added to $f(\lambda)$ provided the abscissa of convergence exceeds $n - 1$.

It will thus be clear that the expansion (3.1) is unique only when the abscissa of convergence is negative or zero.

The following two expansions are useful integral representations of Newton series:

$$(A)\quad \int_0^a (t/a)^\lambda \varphi(t)\,dt = \int_0^a (t/a)^\lambda (a_0 + a_1 t + a_2 t^2 + \cdots)\,dt$$
$$= \varphi_1(a) - \lambda\varphi_2(a)/a + \lambda(\lambda-1)\varphi_3(a)/a^2 \qquad (3.4)$$
$$- \lambda(\lambda-1)(\lambda-2)\varphi_4(a)/a^3 + \cdots,$$

where we use the abbreviation:

$$\varphi_n(a) = \int_0^a \varphi(t)\,(a-t)^{n-1}\,dt/(n-1)!\ ,$$

and assume that

$$\lim_{t=0} t^{\lambda-n+1}\varphi_n(t) = 0$$

for all values of n.

$$(B)\quad \int_0^\infty t^\lambda e^{-t}\,\vartheta(t)\,dt/\Gamma(\lambda+1)$$
$$= \vartheta_0 - \vartheta_1\lambda + \vartheta_2\lambda(\lambda-1)/2!$$
$$- \vartheta_3\lambda(\lambda-1)(\lambda-2)/3! + \cdots, \qquad (3.5)$$

where we write:

$$\vartheta_n = \int_0^\infty e^{-t}\,\vartheta(t)\,L_n(t)\,dt\ ,$$

in which $L_n(t)$ is the nth *Laguerre polynomial*:

$$L_n(t) = \sum_{m=0}^{n} (-1)^m\,{}_nC_m\,t^m/m!\ ,$$

and $\vartheta(t)$ is a function expansible in terms of these polynomials.*

Expansion (3.4) is obtained by means of a suitable integration by parts; expansion (3.5), however, is not so obviously derived but can be obtained as follows:

$$\int_0^\infty t^\lambda e^{-t} L_n(t)\,dt = \int_0^\infty t^\lambda e^{-t}\{\sum_{m=0}^{n}(-1)^m\,{}_nC_m\,t^m/m!\}\,dt\ ,$$
$$= \Gamma(\lambda+1)\{{}_nC_0 - {}_nC_1(\lambda+1)$$
$$+ {}_nC_2(\lambda+1)(\lambda+2)/2! - \cdots\}\ .$$

But we have the identity†

*See R. Courant and D. Hilbert: *Methoden der Mathematische Physik*, vol. 1, Berlin (1924), pp. 77-79. These authors define the Laguerre polynomials to be $n!\,L_n(t)$. See also J. Shohat: *Théorie générale des polynomes orthogonaux de Tchebichef*, Mémorial des Sciences Mathématiques, fasicule 66, Paris (1934), 69 pp.

†See I. J. Schwatt: *An Introduction to the Operations with Series.* Philadelphia, (1924), p. 48, prob. iv.

$$\sum_{m=0}^{n} (-1)^m {}_nC_m(\lambda+1) \cdots (\lambda+m)/m!$$
$$= (-1)^n \lambda(\lambda-1) \cdots (\lambda-n+1)/n! \ ,$$

and hence we can write

$$\int_0^\infty t^\lambda e^{-t} L_n(t)\, dt = (-1)^n \Gamma(\lambda+1)\lambda(\lambda-1) \cdots (\lambda-n+1)/n! \ .$$

We thus see that

$$\int_0^\infty t^\lambda e^{-t} \sum_{n=0}^{\infty} \vartheta_n L_n(t)\, dt/\Gamma(\lambda+1) = \sum_{n=0}^{\infty} (-1)^n \vartheta_n \lambda^{(n)} \ , \quad \lambda^{(0)} = 1 \ .$$

But we also know that if $\vartheta(t)$ is a properly defined function it can be expanded in the form:*

$$\vartheta(t) = \sum_{n=0}^{\infty} \vartheta_n L_n(t) \ . \tag{3.6}$$

Multiplying by $e^{-t} L_n(t)$, integrating from 0 to ∞, and recalling the orthogonality properties of the Laguerre polynomials, we have

$$\int_0^\infty e^{-t} \vartheta(t) L_n(t)\, dt = \vartheta_n \int_0^\infty e^{-t} L_n^2(t)\, dt = \vartheta_n \ .$$

4. The Non-homogeneous Case of the Generalized Euler Equation. Proceeding now to the formal solution of equation (1.6) we seek the resolvent generatrix, $Y_0(x,z)$, which satisfies the equation,

$$Y_0(x,z) \to G(x,z) = 1 \ . \tag{4.1}$$

For this purpose we differentiate (1.6) an infinite number of times, or casting this statement in the language of operators, we form

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*The following conditions for the convergence of (3.6) have been specified by J. V. Uspensky: On the Development of Arbitrary Functions in Series. *Annals of Mathematics*, vol. 28 (2nd ser.) (1926-27), pp. 593-619, in particular, p. 618:

(1) That the integral $\int_a^\infty e^{-t}[\theta(t)]^2\, dt$ exists for a certain value of a.

(2) That the integral $\int_0^b t^{-1/4}|\theta(t)|\, dt$ exists for a certain value of b.

(3) That $\theta(t)$ is of limited variation in a certain interval $t - d$, $t + d$ and absolutely integrable in any finite interval.

Under these conditions the series (3.6) converges and has for its sum $\frac{1}{2}\{\theta(t+0) + \theta(t-0)\}$.

See also: G. Szegö: Beiträge zur Theorie der Laguerreschen Polynome. I. Entwicklungssätze. *Math. Zeitschrift*, vol. 25 (1926), pp. 87-115; E. Hille: *Proc. of National Academy of Sciences*, vol. 12 (1926), pp. 261-269, 348-352.

the operational products, $z^n \to G(x,z)$, $n = 0, 1, 2, \cdots$, and equate them to z^n. We thus obtain a set of equations of the form:

$$\begin{aligned} a_{11}u_1 + a_{12}u_2 + a_{13}u_3 + \cdots + a_{1n}u_n + \cdots &= g_1 \,, \\ a_{22}u_2 + a_{23}u_3 + \cdots + a_{2n}u_n + \cdots &= g_2 \,, \qquad (4.2) \\ a_{33}u_3 + \cdots + a_{3n}u_n + \cdots &= g_n \,, \\ \cdots\cdots\cdots\cdots\cdots & \end{aligned}$$

where we have in particular, $g_1 = 1$, $g_2 = z$, $g_3 = z^2, \cdots, g_n = z^{n-1}$, and where the matrix of the coefficients, $\|a_{ij}\|$, is explicitly given as follows:

$$\|a_{ij}\| =$$

$$\left\| \begin{array}{lllll} a_0 \,, & a_1 + b_1x \,, & a_2 + b_2x + c_2x^2/2! \,, & a_3 + b_3x + c_3x^2/2! + d_3x^3/3! & ,\cdots \\ 0 \,, & a_0 + b_1 \,, & a_1 + b_2 + (b_1 + c_2)x \,, & a_2 + b_3 + (b_2+c_3)x + (c_2+d_3)x^2/2! & ,\cdots \\ 0 \,, & 0 \,, & a_0 + 2b_1 + c_2 \,, & a_1 + 2b_2 + c_3 + (b_1 + 2c_2+d_3)x & ,\cdots \\ 0 \,, & 0 \,, & 0 \,, & a_0 + 3b_1 + 3c_2 + d_3 & ,\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right\| .$$

As we have learned from sections 6 and 7, chapter 3, the solution of system (4.2) can be explicitly obtained either by the method of the Liouville-Neumann series or by the method of segments. In the first case, the expansion is explicitly the following:

$$u_i = g_i/a_{ii} - \sum_{j=i+1}^{\infty} a_{ij}\, g_j/a_{ii}a_{jj} \\ + \sum_{j=i+1}^{\infty} \sum_{k=j+1}^{\infty} a_{ij}\, a_{jk}\, g_k/a_{ii}\, a_{jj}\, a_{kk} - \cdots .$$

In the second case, the solution appears in the more convenient form,

$$u_i = g_i/a_{ii} - D_1^{(i)}\, g_{i+1}/a_{ii}\, a_{i+1,i+1} + D_2^{(i)}\, g_{i+1}/a_{ii}\, a_{i+1,i+1}\, a_{i+2,i+2} - \cdots ,$$
$$i = 1, 2, 3, \cdots , \qquad (4.3)$$

where we abbreviate,

$$D_0 = 1,\; D_1^{(i)} = a_{i,i+1} \,,\; D_2^{(i)} = \begin{vmatrix} a_{i,i+1} & , a_{i,i+2} \\ a_{i+1,i+1} & , a_{i+1,i+2} \end{vmatrix} ,$$

$$\cdots,\; D_n^{(i)} = \begin{vmatrix} a_{i,i+1} \,, & a_{i,i+2} \cdots \,, & a_{i,i+n} \\ a_{i+1,i+1} \,, & a_{i+1,i+2} \cdots \,, & a_{i+1,i+n} \\ 0 \,, & a_{i+2,i+2} \cdots \,, & a_{i+2,i+n} \\ \cdots & \cdots & \cdots \\ 0 \,, & 0 \quad \cdots \,, & a_{i+n-1,i+n} \end{vmatrix} , \cdots .$$

According to the theorems of sections 6 and 7, chapter 3, both formal solutions converge and represent a solution of the original

system provided the g_i are bounded and the coefficients are subject to the restriction,

$$\sum_{j=i+1}^{\infty} |a_{ij}| < |a_{ii}| \quad , \quad i = 1, 2, 3, \cdots .$$

Making the proper specializations in (4.3), we are then able to state the following theorem:

Theorem 3. If there exists a region S of the variable x for which the following inequalities hold:

$$|A_n(a_0)| > \sum_{i=1}^{\infty} |A_n(a_i) + A_n(b_i)x + A_n(c_i)x^2/2!$$
$$+ A_n(d_i)x^3/3! + \cdots | \quad , \quad n = 0, 1, 2, \cdots ,$$

where we use the abbreviation,

$$A_n(p_m) = p_m + nq_{n+1} + n(n-1)r_{n+2}/2!$$
$$+ n(n-1)(n-2)s_{n+3}/3! + \cdots ,$$

then there exists within the region S a resolvent operator for equation (1.6), *convergent in the region* $|z| < 1$, *which is given explicitly by the following formula:*

$$Y_0(x,z) = 1/\Delta(0) - D_1(x)z/\Delta(1)$$
$$+ D_2(x)z^2/\Delta(2) - \cdots , \qquad (4.4)$$

where we abbreviate,

$$\Delta(0) = a_0 \ , \quad \Delta(1) = a_0(a_0 + b_1) \ , \quad \Delta(2) = a_0(a_0 + b_1)$$
$$\times (a_0 + 2b_1 + c_2), \cdots , \Delta(n) = \{a_0 + nb_1 + n(n-1)c_2/2!$$
$$+ n(n-1)(n-2)d_3/3! + \cdots \} \Delta(n-1), \cdots ;$$
$$D_1(x) = a_1 + b_1 x \ ,$$
$$D_2(x) = \begin{vmatrix} a_1 + b_1 x \ , & a_2 + b_2 x + c_2 x^2/2! \\ a_0 + b_1 \ , & a_1 + b_2 + (b_1 + c_2)x \end{vmatrix} ,$$

and $D_n(x)$ *is the* nth *principal minor of the determinant formed by omitting the first column of the matrix* $\|a_{ij}\|$.

5. *The Inversion of Abel's Integral.* In order to illustrate the application of the theorem of the last section we shall consider here the solution of Abel's integral equation,

$$\int_0^x [u(t)/(x-t)^a]\, dt = f(x) \ , \qquad 0 \leqq a < 1 \ . \qquad (5.1)$$

This equation we have already solved by means of fractional derivatives (see section 6, chapter 6). The following method, however, is essentially novel,

In order to bring (5.1) under our theory we first write it in the form (see section 1),

$$x^{a-1}(x)=u(x)/(1-a)\ -u'(x)x/(2-a)\ +u''(x)x^2/(3-a)2!$$
$$-u'''(x)x^3/(4-a)3!+\cdots. \qquad (5.2)$$

The matrix of the coefficients from which the resolvent is to be formed is readily computed to be,

$$\left|\begin{array}{lllll}
\dfrac{1}{(1-a)}, & \dfrac{-x}{(2-a)}, & \dfrac{x^2}{2!(3-a)}, & \dfrac{-x^3}{3!(4-a)}, & \cdots \\
0, & \dfrac{1}{(1-a)(2-a)}, & \dfrac{-x}{(2-a)(3-a)}, & \dfrac{x^2}{2!(3-a)(4-a)}, & \cdots \\
0, & 0, & \dfrac{2!}{(1-a)(2-a)(3-a)}, & \dfrac{-2!x}{(2-a)(3-a)(4-a)}, & \cdots \\
0, & 0, & 0, & \dfrac{3!}{(1-a)(2-a)(3-a)(4-a)}, & \cdots \\
0, & 0, & 0, & 0, & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right|$$

Computing the principal minors, $\Delta(n)$, we get, $\Delta(0)=1/(1-a)$, $\Delta(1)=1!/(1-a)^2(2-a)$, $\Delta(2)=1!\,2!/(1-a)^3(2-a)^2(3-a),\cdots$, $\Delta(n)=1!\ 2!\ 3!\ \cdots\ n!/(1-a)^{n+1}(2-a)^n(3-a)^{n-1}\cdots(n+1-a)$, $n=0,1,2,\cdots$.

Similarly the principal minors of the determinant reduced by omitting the first column are,

$$D_1=\frac{-x}{(2-a)},$$

$$D_2=\frac{-a\,x^2}{(1-a)(2-a)^2(3-a)\cdot 2!},$$

$$D_3=\frac{-1!\cdot 2!\,a(1+a)x^3}{(1-a)^2(2-a)^3(3-a)^2(4-a)\cdot 3!},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$D_n=\frac{-1!\cdot 2!\cdot 3!\cdots(n-1)!\,a(1+a)(2+a)\cdots(n-2+a)x^n}{(1-a)^{n-1}(2-a)^n(3-a)^{n-1}(4-a)^{n-2}\cdots(n+1-a)\cdot n!},\cdots.$$

Hence the general operator becomes,

$$Y_0(xz) = \sum_{n=0}^{\infty} (-1)^n D_n z^n/\Delta(n) = (1-a)[1+(1-a)xz$$
$$-a(1-a)(xz)^2/(2!)^2 + a(a+1)(1-a)(xz)^3/(3!)^2$$
$$-a(a+1)(a+2)(1-a)(xz)^4/(4!)^2+\cdots] .$$

From this resolvent operator the customary form of the solution of Abel's equation can be obtained in the following manner:

$$Y_0(xz) = (1-a)z \to [x - a\,x^2\,z/2! + a(a+1)x^3z^2/3!\cdot 2! - \cdots ,$$
$$= \{(1-a)/\Gamma(1-a)\cdot\Gamma(a)\}\,z \to \{x\,\Gamma(1-a)\Gamma(a)/\Gamma(2)$$
$$-\Gamma(1-a)\Gamma(a+1)xz/\Gamma(3)$$
$$+\Gamma(1-a)\Gamma(a+2)(xz)^2/2!\cdot\Gamma(4) - \cdots\} ,$$
$$= \{1/\Gamma(1-a)\cdot\Gamma(a)\}\,z \to$$
$$\int_0^1 \sum_{m=0}^{\infty} s^{m+a-1}(1-s)^{a-1}(-1)^m\, x^{m+1}z^m\, ds/m!$$
$$= \{1/\Gamma(1-a)\cdot\Gamma(a)\}\,z \to \int_0^1 (1-s)^{1-a}\, s^{a-1}\, e^{-xzs}\, x\, ds .$$

By means of the change of variables: $x - t = sx$, this becomes

$$Y_0(xz) = \{1/\Gamma(1-a)\cdot\Gamma(a)\}\,z \to \int_0^x \{e^{(t-x)z}\, t^{1-a}/(x-t)^{1-a}\}\, dt .$$

If we now replace $1/\Gamma(1-a)\cdot\Gamma(a)$ by $\sin a\pi/\pi$ and operate upon $x^{a-1}f(x)$, recalling the identity $e^{(t-x)z} \to \varphi(x) = \varphi(t)$, we get,

$$Y_0(xz) \to x^{a-1}f(x) = (\sin a\pi/\pi)z \to \int_0^x \{f(t)/(x-t)^{1-a}\}\, dt ,$$

which is the well known inversion of the integral.

In order to prove that this solution is unique we refer to theorem 1 and compute the function,

$$f_0(\lambda) = 1/(1-a) - \lambda/(2-a) + \lambda^{(2)}/(3-a)\cdot 2!$$
$$-\lambda^{(3)}/(4-a)\cdot 3! + \cdots . \qquad (5.3)$$

Making the following specialization of equation (3.4):

$$\varphi(t) = (1-t)^{-a} ,$$

$$\varphi_n(1) = \int_0^1 (1-t)^{n-a-1}\, dt/(n-1)! = 1/(n-a)(n-1)! ,$$

we can write,

$$f_0(\lambda) = \int_0^1 t^\lambda (1-t)^{-a}\, dt = \Gamma(\lambda+1)\Gamma(1-a)/\Gamma(\lambda-a+2) \ , \qquad \lambda > -1 \ .$$

This function possesses zeros only when $\lambda - a + 2 = -m$, where m is an integer or zero, that is to say for $\lambda = a - M$, where M is an integer greater than one. But since a is less than 1 we should then have $\lambda < -1$. For this range, however, series (5.3) does not converge and hence no solution exists for the homogeneous equation.

6. *Inversion by the Method of Bourlet.* It will be illuminating to effect the inversion of the Euler equation (1.2) by the method of Bourlet which we have developed in section 10 of chapter 4. The generatrix is obviously the function,

$$G(xz) = a_0 + a_1 xz + a_2 x^2 z^2/2! + a_3 x^3 z^3/3! + \cdots \ , \tag{6.1}$$

which, from the condition imposed that the a_n are bounded as $n \to \infty$, is seen to be an entire function in xz of genus 1 or 0.

Therefore we may write,

$$G(xz) = e^{cxz} \prod_{i=0}^{\infty} (1 - xz/\beta_i) \ , \tag{6.2}$$

where the β_i are the zeros of the entire function $G(y)$.

As we have previously explained (see section 10, chapter 4) the case where the number of zeros is infinite has never been completely discussed from the Bourlet point of view. In the event that the number of zeros is finite, however, the analysis becomes relatively simple and we can write (6.2) in the form,

$$G(xz) = e^{cxz} P(xz) \ ,$$

where $P(xz)$ is a polynomial.

If we now refer to the development in section 10, chapter 4, under *case 2,* we see that we may write the generatrix in the form,

$$G(xz) = e^{cxz} \to P[xz/(1+c)] \ ,$$

the inverse function $h(x)$ in the theory referred to being derived from the equation,

$$h(x) + c\, h(x) = x \ .$$

The original equation (1.2) can now be written,

$$\{e^{cxz} \to P[xz/(1+c)]\} \to u(x) = F(x) \ ,$$

from which we obtain by operating on both sides with e^{-kxz}, where $k = c/(1+c)$,* the result,

$$P[xz/(1+c)] \to u(x) = F[(x/(1+c)] \ .$$

The problem of solving equation (1.2) is thus reduced to that of solving an Euler equation of finite order.

Let us consider the following example:

$$u(x) + \tfrac{1}{2}\, x\, u'(x) + (\frac{1}{2^2 2!} - 1)\, x^2 u'' + (\frac{1}{2^3 3!} - 1)\, x^3 u''' + \cdots = \tfrac{1}{4}\, x^2 \ .$$

The generatrix function is seen to be,

$$G(xz) = e^{\frac{1}{2}xz}(1 - x^2 z^2) \ .$$

The equation is then equivalent to,

$$[e^{\frac{1}{2}xz} \to (1 - \frac{4}{9}\, x^2 z^2)] \to u(x) = \tfrac{1}{4}\, x^2 \ ,$$

which reduces to,

$$\frac{4}{9}\, x^2\, u'' - u = -\, x^2/9 \ .$$

The solution of this equation is,

$$u(x) = c_1\, x^{a_1} + c_2\, x^{a_2} + x^2 \ ,$$

where $a_1 = \tfrac{1}{2}(1 + \sqrt{10})$ and $a_2 = \tfrac{1}{2}(1 - \sqrt{10})$. By direct substitution, this solution will be found to reduce the right member of the original equation uniformly to $\tfrac{1}{4}\, x^2$.

PROBLEMS

1. Given equation (1.3), show that

$$f_0(\lambda) = \frac{1}{\lambda+1} - \frac{2}{3}(\tfrac{1}{2})^{\lambda} - \frac{1}{6} - \frac{1}{6}\delta(\lambda) \ ,$$

where $\delta(\lambda) = 0,\ \lambda > 0,\ \delta(0) = 1$.

Hence show that

$$u(x) = \sum_{n=0}^{3} a_n x^n + g(x) \ ,$$

where $g(x) = \sum_m x^{a_m} \quad [\cos(\beta_m \log x) + i \sin(\beta_m \log x)]$,

in which $\lambda_m = \alpha_m + i\beta_m$ are complex roots of $f_0(\lambda) = 0$.

[Equation (1.3), in which the right hand member is regarded as an approximation for the integral on the left, is due to Thomas Simpson (1710-1761) {*Math.*

*The reader can readily show by means of the Bourlet product that this is the inverse of the operator e^{cxz}.

Dissertations, London (1743)}. It is commonly referred to as *Simpson's* or the *parabolic rule*. The reader will find an illuminating discussion of the equation, regarded as a functional equation for the evaluation of $u(x)$, in the references cited in the footnote to section 1].

2. Given equation (1.3), show that if $u(x)$ is continuous for $0 \leqq x \leqq b$ and has six continuous derivatives for $0 \leqq x \leqq H < b$, and if the formula evaluates the integral for each value of x, $0 \leqq x \leqq b$, then $u(x)$ is a polynomial of degree not higher than 3, $0 \leqq x \leqq b$. [Gillespie: *Mathematical Monthly*, vol. 27 (1920), pp. 405-406].

3. Consider Weddle's formula for approximate integration

$$\int_0^x u(x)\,dx = (x/20)\,[u(0) + 5u(x/6) + u(x/3)$$
$$+ 6u(x/2) + u(2x/3) + 5u(5x/6) + u(x)]$$

as a functional equation in $u(x)$.

7. *Analogous Study of Ordinary Linear Differential Equations.* The methods employed in section 2 for the solution of the homogeneous Euler differential equation of infinite order can be applied to the problem of ordinary differential equations. This application was first made by G. W. Hill and was extended by H. Poincaré and H. von Koch. (For historical account, see next section). We follow the development given by the latter.

Let us consider the ordinary differential equation

$$L(u) \equiv P_0(x)u(x) + P_1(x)u'(x) + P_2(x)\,u''(x) + \cdots$$
$$+ P_{n-1}(x)u^{(n-1)}(x) + P_n(x)\,u^{(n)}(x) = 0\ , \qquad (7.1)$$

where the coefficients are developable in Laurent series of the form

$$P_r(x) = \sum_{k=-\infty}^{\infty} p_{rk}\,x^k\ . \qquad (7.2)$$

These expansions are assumed to be valid within an annulus of bounding radii R and R', that is to say, for values of x which satisfy the condition: $R < |x| < R'$.

It is always possible to include the unit circle within the annulus by means of the transformation: $x = t(RR')^{\frac{1}{2}}$. We shall therefore make the assumption that the unit circle lies within the annulus.

Let us now assume that the solutions of (7.1) can be expanded in the following series:

$$u(x) = \sum_{m=-\infty}^{\infty} a_m\,x^{\lambda+m}\ . \qquad (7.3)$$

If we place (7.3) in (7.1) and take account of (7.2), we shall obtain

$$L(u) = \sum_{r=0}^{n} \sum_{m,k=-\infty}^{\infty} a_m\,p_{rk}\,x^{\lambda+m+k-r}(\lambda+m)^{(r)}\ ,$$

where $(\lambda+m)^{(r)}$ has the same significance as in section 2.

Making the transformation: $s = m + k - r$, we obtain

$$L(u) \equiv \sum_{m,k=-\infty}^{\infty} \sum_{r=0}^{n} a_m (\lambda+m)^{(r)} p_{r,s+r-m} \, x^{\lambda+s}$$

$$= \sum_{m,k=-\infty}^{\infty} a_m \varphi_{s-m}(\lambda+m) x^{\lambda+s} \, , \qquad (s = -\infty \text{ to } +\infty)$$

where we abbreviate

$$\varphi_j(\varrho) = p_{0j} + p_{1,j+1}\,\varrho + p_{2,j+2}\,\varrho^{(2)} + \cdots + p_{n-2,j+n-2}\,\varrho^{(n-2)}$$
$$+ p_{n-1,j+n-1}\,\varrho^{(n-1)} + p_{n,j+n}\,\varrho^{(n)} \, .$$

Since $u(x)$ is a solution of (7.1), we may set the coefficients of $x^{\lambda+s}$ equal to zero, and hence obtain

$$\sum_{m,s=-\infty}^{\infty} a_m \varphi_{s-m}(\lambda+m) = 0 \tag{7.4}$$

for the determination of the coefficients a_m .

Now it is evident that we may adopt in equation (7.1) the simplifying conditions:

$$P_n(x) \equiv 1 \, , \qquad P_{n-1}(x) \equiv 0 \, ,$$

without impairing the generality of the problem. The second assumption merely involves making the transformation:

$$u(x) = y(x)\, g(x) \, , \qquad g(x) = \exp[-\frac{1}{n} \int^{x} P_{n-1}(x)\, dx]$$

in (7.1), which will yield an equation in $y(x)$ in which the term in $y^{(n-1)}(x)$ is missing.

Let us now divide (7.4) by $\varphi_0(\lambda+s)$. Hence, employing the abbreviation $\vartheta_{ms}(\lambda) = [\varphi_{s-m}(\lambda+m)]/[\varphi_0(\lambda+s)]$, we can write (7.4) in the form

$$\sum_{m=-\infty}^{\infty} a_m \vartheta_{ms}(\lambda) = 0 \, , \quad (s = -\infty \text{ to } +\infty) \, . \tag{7.5}$$

Since $\vartheta_{mm} = 1$, the main diagonal of the determinant

$$\Delta(\lambda) = |\vartheta_{ms}(\lambda)|$$

will consist of elements equal to 1.

Let us now consider the series

$$\sum_{m,s=-\infty}^{\infty}{}' |\vartheta_{ms}(\lambda)| = \sum_{m,s=-\infty}^{\infty}{}' \left| \frac{\varphi_{s-m}(\lambda+m)}{\varphi_0(\lambda+s)} \right| = \sum_{s,p=-\infty}^{\infty}{}' \left| \frac{\varphi_p(\lambda+s-p)}{\varphi_0(\lambda+s)} \right| \, ,$$

where the primes on the summation signs indicate that the terms $m = s$ and $p = 0$ are to be omitted.

Let us indicate by $\lambda_1, \lambda_2, \cdots, \lambda_n$, the roots of $\varphi_0(\lambda) = 0$, from which it follows that the zeros of $\varphi_0(\lambda+s)$ will be $\lambda_i - s$. In the fol-

lowing analysis it will be assumed that λ lies in a region, bounded by lines parallel to the axis of imaginaries, which contains none of the roots of $\varphi_0(\lambda) = 0$.

Since $\varphi_p(\lambda)$ is a polynomial of degree $n-2$, $p \neq 0$, we can write

$$\varphi_p(\lambda+s-p) = (\lambda+s)^{n-2} h_2(p) + (\lambda+s)^{n-3} h_3(p) + \cdots + h_n(p) \ ,$$

where $h_r(p)$ consists of a finite number of terms of the form $p_{k,p+k}\, p^q$, $q = 0, 1, 2, \cdots$. Hence the series

$$S_r = \sum_{p=-\infty}^{\infty} |h_r(p)| \ ,$$

will converge in view of the assumption that $P_r(x)$ as defined by (7.2) converges upon the unit circle.

Hence since $\varphi_0(\varrho) = \varrho^n[1+O(1/\varrho)]$, we obtain the inequality

$$\sum_{s,p=-\infty}^{\infty}{}' \left|\frac{\varphi_p(\lambda+s-p)}{\varphi_0(\lambda+s)}\right| < S_2 \sum_{s=-\infty}^{\infty} \left|\frac{(\lambda+s)^{n-2}}{\varphi_0(\lambda+s)}\right| + S_3 \sum_{s=-\infty}^{\infty} \left|\frac{(\lambda+s)^{n-3}}{\varphi_0(\lambda+s)}\right| + \cdots$$
$$+ S_n \sum_{s=-\infty}^{\infty} \left|\frac{1}{\varphi_0(\lambda+s)}\right| .$$

Referring to the theorem on the convergence of infinite determinants given in section 3, chapter 3, we see from the above analysis that $\Delta(\lambda)$ is an analytic function with at most polar singularities at the roots of the equation $\varphi_0(\lambda) = 0$.

Moreover, we observe from the explicit form of the elements of $\Delta(\lambda)$ that

$$\vartheta_{ms}(\lambda+1) = \frac{\varphi_{s-m}(\lambda+1+m)}{\varphi_0(\lambda+1+s)} = \frac{\varphi_{(s-1)-(m+1)}(\lambda+1+m)}{\varphi_0(\lambda+1+s)} = \vartheta_{m+1,s+1}(\lambda) \ .$$

From this we infer that $\Delta(\lambda)$ is a periodic function of unit period,

$$\Delta(\lambda+1) = \Delta(\lambda) \ .$$

Moreover, if $\lambda \to \infty$ in such a way that its real part remains finite, then $\vartheta_{ms}(\lambda) \to 0$, $m \neq s$, and we have

$$\Delta(\lambda) \to 1 \ .$$

In view of this we see that we can write

$$\Delta(\lambda) = M + \pi \sum_{k=1}^{n} M_k \cot(\lambda-\lambda_k)\pi \ .$$

Letting the imaginary part of λ first approach $+\infty$ and then $-\infty$, we shall obtain

$$1 = M - \pi i \sum_{k=1}^{n} M_k \ , \qquad 1 = M + \pi i \sum_{k=1}^{n} M_k \ ,$$

from which we derive

$$M = 1 \ , \qquad \sum_{k=1}^{n} M_k = 0 \ .$$

Also setting $\lambda = 0$, we obtain

$$\Delta(0) = 1 - \pi \sum_{k=1}^{n} M_k \cot \lambda_k \pi \ ,$$

which permits an explicit determination of the constants M_k for the case $n = 2$.

In view of the foregoing analysis it is clear that we can also write $\Delta(\lambda)$ in the form

$$\Delta(\lambda) = \prod_{k=1}^{n} \frac{\sin(\lambda - \lambda^{(k)})\pi}{\sin(\lambda - \lambda_k)\pi} \ .$$

In order to achieve an explicit determination of the solutions of (7.1), let us now multiply the equations of system (7.4) successively by the functions

$$h_0(\lambda) = 1 \ , \qquad h_m(\lambda) = \frac{1}{m^n} \prod_{k=1}^{n} \exp[-(\lambda - \lambda_k)/m] \ .$$

We thus obtain the new system

$$\sum_{m=-\infty}^{\infty} a_m \psi_{ms} = 0 \ , \qquad \psi_{ms} = h_m(\lambda)\ \varphi_{s-m}(\lambda + m) \ ,$$

the determinant of which we may write in the form

$$D(\lambda) = \Delta(\lambda)\ \Pi(\lambda) \ ,$$

where we abbreviate

$$\Pi(\lambda) = \lim_{m=\infty} \prod_{k=-\infty}^{m} h_k(\lambda)\ \varphi_0(\lambda + k) \ .$$

Introducing the explicit values into this expression, we obtain

$$\begin{aligned} \Pi(\lambda) &= \lim_{m=\infty} \prod_{k=-m}^{m} \prod_{i=1}^{n} [1 + (\lambda - \lambda_i)/k]\ e^{-(\lambda-\lambda_i)/k} \\ &= \prod_{i=1}^{n} \lim_{m=\infty} \prod_{k=-m}^{m} [1 + (\lambda - \lambda_i)/k]\ e^{-(\lambda-\lambda_i)/k} \\ &= \prod_{i=1}^{n} \frac{\sin(\lambda - \lambda_i)\pi}{\pi} \ .^* \end{aligned}$$

8. *Hill's Problem.* The theory developed in the preceding section was evoked by a remarkable paper published by G. W. Hill (1838-1914) in 1877 on the problem of the motion of the lunar perigee. In this paper Hill made a bold use of determinants of infinite order and

*See Whittaker and Watson: *Modern Analysis.* (loc. cit.), 7.5.

his daring was rewarded by a very accurate determination of the motion which was the object of his study. However, the novel method, for all of its computational power, was open to serious question from the standpoint of rigorous analysis. This defect was remedied in 1886 by Poincaré, who made a searching investigation of the convergence of infinite determinants in general. This research was followed in 1891 and 1892 by two memoirs by H. von Koch, who extended the methods of Hill to linear differential equations of any finite order. These papers have become the classical references for this problem.

Because of the great interest which the researches of Hill aroused among both mathematicians and astronomers, it will not be out of place to append a brief bibliography of some of the more important references to this subject:

G. W. Hill: On the Part of the Motion of the Lunar Perigee which is a function of the Mean Motion of the Sun and Moon. Cambridge, Mass., (1877). Reprinted in *Acta Mathematica,* vol. 8 (1886), pp. 1-36. Also Hill's *Collected Works,* vol. 1 (1905), pp. 243-270.

H. Poincaré: Sur les déterminants d'ordre infini. *Bulletin de la Soc. Math.,* vol. 14 (1886), pp. 77-90.

H. von Koch: Sur une application des determinants infinis à la théorie des équations différentielles linéaires. *Acta Mathematica,* vol. 15 (1891), pp. 53-63; Sur les déterminants infinis et les équations différentielles linéaires. *Ibid.,* vol. 16 (1892), pp. 217-295.

E. W. Brown: *An Introductory Treatise on the Lunar Theory.* Cambridge (1896), pp. 211-225.

F. Tisserand: *Traité de mécanique céleste,* vol. 3, Paris (1894), chap. 15.

T. Cazzaniga: Sui determinanti d'ordine infinito. *Annali di Mat.,* vol. 26 (2nd ser.) (1897), pp. 143-218; Appunti sulla moltiplicazione dei determinanti normaloidi. *Ibid.,* vol. 2 (3rd ser.) (1899), pp. 229-238.

A. R. Forsyth: *Theory of Differential Equations.* Part 3, vol. 4, Cambridge (1902), chap. 8.

G. H. Darwin: Hill's Lunar Theory. Darwin's *Scientific Papers,* vol. 5, Cambridge (1916), pp. 16-58.

A. Wintner: Zur Hillschen Theorie der Variations des Mondes. *Mathematische Zeitschrift,* vol. 24 (1925-1926), pp. 257-265.

F. R. Moulton and collaborators: *Periodic Orbits.* Washington (1920). Publication 161 of the Carnegie Institution. Chapters 1 and 3.

F. R. Moulton: The Problem of the Spherical Pendulum from the Standpoint of Periodic Solutions. *Rendiconti di Palermo,* vol. 32 (1911), pp. 338-364.

H. C. Plummer: *An Introductory Treatise on Dynamical Astronomy*. Cambridge (1918), chap. 20.

E. T. Whittaker and G. N. Watson: *A Course of Modern Analysis*. Cambridge, (3rd ed.) (1920), pp. 36-37; 406-407; 413-417.

F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues*. Paris (1913), pp. 156-162.

The dynamical system investigated by Hill led to the following ratio of the motion of the perigee to the siderial motion of the moon:

$$\frac{1}{n}\frac{d\omega}{dt} = 1 - \frac{\lambda}{1+m}, \tag{8.1}$$

where m and n are astronomical constants (see problem 2 below), and λ is a characteristic number for the differential equation

$$u'' + (\vartheta_0 + 2\sum_{n=1}^{\infty} \vartheta_n \cos 2nx)\, u(x) = 0\ . \tag{8.2}$$

The coefficients ϑ_n are empirical constants (see problem 2 below) which diminish rapidly with n. We shall assume that the series $\sum\limits_{n=0}^{\infty}\vartheta_n$ converges absolutely.

We assume a solution of the form

$$u(x) = z^{\lambda} \sum_{n=-\infty}^{\infty} a_n z^{2n}\ , \qquad z = e^{xi}\ . \tag{8.3}$$

Upon substituting this series in (8.2), we obtain the equation

$$-\sum_{n=-\infty}^{\infty} (2n+\lambda)^2 a_n z^{2n+\lambda} + \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \vartheta_{n-m}\, a_m z^{2n+\lambda} = 0\ ,$$

where for convenience we define $\vartheta_{-n} = \vartheta_n$.

Equating equal powers of z, we get the following system:

$$-(2n+\lambda)^2 a_n + \sum_{m=-\infty}^{\infty} \vartheta_{n-m}\, a_m = 0\ , \qquad (n = -\infty \text{ to } +\infty)\ .$$

We now divide each equation successively by $\vartheta_0 - (2n+\lambda)^2$ in order to secure a convergent determinant for the system. This determinant we may write as follows:

$$\Delta(\lambda) = \left| \frac{\vartheta_{n-m}}{\vartheta_0 - (2n+\lambda)^2} - \delta_{mn}\frac{(2n+\lambda)^2}{\vartheta_0 - (2n+\lambda)^2} \right|,$$

where δ_{mn} is Kronecker's symbol.

We shall now show that

$$\Delta(\lambda) = \frac{2\sin^2 \frac{1}{2}\pi\sqrt{\vartheta_0}}{\cos\pi\lambda - \cos\pi\sqrt{\vartheta_0}}\,[\Delta(0) - \sin^2 \tfrac{1}{2}\pi\lambda \csc^2(\tfrac{1}{2}\pi\sqrt{\vartheta_0})]\ .$$

Hence the roots of $\Delta(\lambda) = 0$ are given by

$$\sin^2 \tfrac{1}{2}\pi\lambda = \Delta(0) \sin^2(\tfrac{1}{2}\pi\sqrt{\vartheta}) \ , \tag{8.5}$$

from which it is possible to determine the values of λ and hence the coefficients a_n of (8.3) in terms of an arbitrary a_0.

In order to establish (8.3), we first observe that

$$\Delta(\lambda+2) = \Delta(\lambda) \ , \text{ and } \Delta(-\lambda) = \Delta(\lambda) \ , \tag{8.6}$$

Moreover we see that $\Delta(\lambda)$ is analytic throughout the λ-plane except for the obvious simple poles at $\lambda = -2n \pm \sqrt{\vartheta_0}$, and also that

$$\lim_{|\lambda|\to\infty} \Delta(\lambda) = 1 \ . \tag{8.7}$$

Hence, by the argument employed in section 7, we shall have

$$\Delta(\lambda) = 1 + \pi M_1 \cot \tfrac{1}{2}\pi(\lambda + \sqrt{\vartheta_0}) + \pi M_2 \cot \tfrac{1}{2}\pi(\lambda - \sqrt{\vartheta_0}) \ , \tag{8.8}$$

where M_1 and M_2 are to be determined from the equations:

$$M_1 + M_2 = 0 \ ,$$

$$M_1 \cot \tfrac{1}{2}\pi\sqrt{\vartheta_0} - M_2 \cot \tfrac{1}{2}\pi\sqrt{\vartheta_0} = [1 - \Delta(0)]/\pi \ .$$

Replacing in equation (8.8) the values thus found, one verifies by a simple manipulation that $\Delta(\lambda)$ takes the form given in (8.4).

PROBLEMS

1. Discuss the equation

$$u''(x) + (a/x^3 + b/x^2 + c/x)\, u(x) = 0 \ .$$

Show that

$$\Delta(\lambda) = 1 + 2M\pi \sin 2\lambda_1\pi/(\cos 2\pi\lambda_1 - \cos 2\pi\lambda) \ ,$$

where $\lambda_1 = \tfrac{1}{2}(1 - \sqrt{1-4b})$. Evaluate M for $a = -b = c = 1$.

If $1 - 4b = p^2$, where p is an integer, show that

$$\Delta(\lambda) = 1 - N\left[\frac{\pi}{\sin(\lambda + \tfrac{1}{2}p)\pi}\right]^2 \ .$$

Compute N for $a = c = 1$, $b = 0$.

2. Hill, in applying his theory to the numerical problem of computing the motion of the lunar perigee, obtained the following value for the function which multiplies $u(x)$ in equation (8.2):

$$\begin{aligned}\theta(x) = 1.15884\,39395\,96583 &- 0.11408\,80374\,93807 \cos 2x \\ + 0.00076\,64759\,95109 \cos 4x &- 0.00001\,83465\,77790 \cos 6x \\ + 0.00000\,01088\,95009 \cos 8x &- 0.00000\,00020\,98671 \cos 10x \\ + 0.00000\,00000\,12103 \cos 12x &- 0.00000\,00000\,00211 \cos 14x \ .\end{aligned}$$

With these values show that

$$\Delta(0) = 1.00180\,47920\,210112$$

and hence that the smallest positive characteristic number is equal to $\lambda = 1.07158\ 32774\ 16012$.

In (8.1) m is the ratio of the synodic month to the siderial year, n is the moon's sidereal mean motion, and n' the mean angular motion of the sun. Given that $m = n'/(n-n')$, $n = 17325594''.06085$, $n' = 1295977''.41516$, show that

$$\frac{1}{n}\frac{d\omega}{dt} = .00857\ 25730\ 04864 \ .$$

3. Discuss the solution of Legendre's equation

$$(1-x^2)\,u''(x) - 2x\,u'(x) + n(n+1)\,u(x) = 0 \ ,$$

by the methods of this chapter. Show that $f_0(\lambda) = n(n+1) - \lambda - \lambda^2$. Hence for positive integral values of n show that a solution exists (the Legendrian polynomials) of the form

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2}x^n[1 - \frac{n(n-1)}{2(2n-1)}\frac{1}{x^2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}\frac{1}{x^4} - \dots] \ .$$

Discuss the second solution of the equation.

9. Analytical Peculiarities of the Solutions of the Homogeneous Euler Equation. In section 7 we have shown how the solution of a linear homogeneous differential equation of finite order with coeffiicients developable in Laurent series may be reduced to the discussion of a linear system of equations the elements of which are functions of the form

$$\vartheta_{ms}(\lambda) = [\varphi_{s-m}(\lambda+m)]/[\varphi_0(\lambda+s)] \ .$$

The functions $\varphi_i(\varrho)$ in this expression are Newton series of *finite* order, that is to say, polynomials. The convergence of the series which represented the solution of the equation, was seen to depend essentially upon the fact that $|\varphi_i(\varrho)$, $i \neq 0$, was dominated by $|\varphi_0(\varrho)|$ as $\varrho \to \infty$. This domination in turn depended upon the fact that the $\varphi_i(\varrho)$ were polynomials and that the coefficient of the term of order $n-1$ in the original equation could be set equal to zero. Neither of these conditions may be assumed in the problem of the generalized Euler equation.

Some of the analytical peculiarities of the situation can be exhibited by means of the following example:

Let us consider the equation

$$\omega u(x) + (\omega a_1 + q\omega x)u'(x) + (\omega a_2 + q\omega a_1 x + q\omega^2 x^2/2!)u''(x)$$
$$+ (\omega a_3 + q\omega a_2 x + q\omega^2 a_1 x^2/2! + q\omega^3 x^3/3!)u^{(3)}(x) + \cdots = 0 , \tag{9.1}$$

where we abbreviate $\omega = q - 1$, $|\omega| < 1$.

Substituting the values of the coefficients in (2.3), we get

$$f_0(\lambda) = \omega + q\omega\lambda + q\omega^2\lambda^{(2)}/2! + \cdots$$
$$= \omega + q[(1+\omega)^\lambda - 1] = q^{\lambda+1} - 1 ,$$
$$f_1(\lambda) = a_1\lambda(q^\lambda - 1) , \quad f_2(\lambda) = a_2\lambda^{(2)}(q^{\lambda-1} - 1) .$$

and in general

$$f_n(\lambda) = a_n\lambda^{(n)}(q^{\lambda-n+1} - 1) = a_n\lambda^{(n)} f_0(\lambda - n) . \tag{9.2}$$

The existence of infinitely many roots of $f_0(\lambda) = 0$, namely,

$$\lambda_n = -1 + 2n\pi i/(\log q) , \qquad n = 0, \pm 1, \pm 2, \cdots$$

implies the existence of infinitely many solutions of the original question. We shall limit our discussion to the single root $\lambda = -1$, in terms of which $(\lambda - r)^{(n)}$ becomes $(-1)^n(r+n)!/r!$.

Noting (9.2), we reduce system (2.9) for the determination of the values of the φ_i to the following

$$a_n(-1)^n\varphi_0 + (-1)^{n-1}a_{n-1}\varphi_1 + (-1)^{n-2}a_{n-2}\varphi_2/2! + \cdots$$
$$- a_1\varphi_{n-1}/(n-1)! + a_0\varphi_n/n! = 0 . \tag{9.3}$$

Let us now specialize the parameters a_1, a_2, a_3, $\cdots$ by assuming that they satisfy the following set of equations:

$$a_1 = 1 ,$$
$$a_2 - a_1 = 0 ,$$
$$a_3 - a_2 = 0 ,$$
$$a_4 - a_3 = -1/4! ,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_n - a_{n-1} + a_{n-4}/4! - a_{n-9}/9! + a_{n-16}/16! - \cdots = (-1)^n\delta_{p^2,n}/n! ,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where we use the symbol, $\delta_{p^2,n} = \begin{cases} 0 , & n \neq p^2 , \\ 1 , & n = p^2 . \end{cases}$

Summing columns of the array we see that $\sum_{i=1}^{\infty} a_i$ converges and is equal to $\varepsilon/(1-\varepsilon)$, where $\varepsilon = 1 - 1/4! + 1/9! - 1/16! + \cdots$.

When the values of a_i are substituted in (9.3) and an explicit determination of the φ_i effected and substituted in (2.2), the following solution of (9.1) is obtained:

$$u(x) = x^{-1}(1 + 1/x + 1/x^4 + 1/x^9 + \cdots + 1/x^{n^2} + \cdots) \ .$$

The series obviously converges for all values of x greater than 1 in absolute value, but it possesses the unit circle as a natural bound. The function which this series represents in the region exterior to the unit circle is not found in the theory of ordinary differential equations.

Let us consider a second case by assuming for a_n the value $(-1)^{n-1}$. The coefficients of the solution, (2.2), are easily computed and we obtain the totally divergent series

$$u(x) = \tfrac{1}{2} x^{-1}\{[1 + 1!(\tfrac{1}{2}x) + 2!(\tfrac{1}{2}x)^2 + 3!(\tfrac{1}{2}x)^3 + \cdots] - 1\} \ .$$

We note, however, that this series is summable by the method of Borel in the negative half-plane and we may thus compute the following integral representation of the formal solution:

$$u(x) = \frac{1}{2x}\int_0^\infty \frac{e^{-t}}{1 - \tfrac{1}{2}xt}\,dt - \frac{1}{2x} \ , \qquad R(x) < 0 \ .$$

A third special case, namely, where we assume that $a_n = 1/n!$, reduces system (9.3) to the following simple form:

$$\varphi_n - {}_nC_1\,\varphi_{n-1} + {}_nC_2\,\varphi_{n-2} - {}_nC_3\,\varphi_{n-3} + \cdots + (-1)^n\varphi_0 = 0 \ ,$$

where ${}_nC_r$ is the rth binomial coefficient. Obviously $\varphi_k = 1$, for all positive integral values of k, furnishes a solution and we thus obtain

$$u(x) = x^{-1}(1 + 1/x + 1/x^2 + 1/x^3 + \cdots) = 1/(x-1) \ .$$

If the values of the coefficients a_i are substituted in the original equation (9.1) and the Taylor transform (see section 6, chapter 2) employed to simplify it, one sees that the special case under discussion yields essentially one solution of the q-difference equation

$$q\,u(qx + 1) - u(x + 1) = 0 \ .$$

In order to examine more generally the convergence of series (2.2), let us construct the function

$$f(s,\lambda) = \sum_{n=0}^{\infty} f_n(\lambda)s^n \ , \tag{9.4}$$

in which the real part of λ is assumed less than or equal to the real part of λ_0, where λ_0 is any root of $f_0(\lambda) = 0$.

Let us assume that this series is uniformly convergent about the origin within a circle of radius R. Then $f(s, \lambda)$ is an analytic function, which, if the right hand member of (9.4) fails to converge on the boundary of the circle, possesses a singularity of modulus R. If $M(\lambda)$ designates the upper bound of $f(s, \lambda)$ in the neighborhood of the circle $|s| = R$, let us say on the circle $|s| = r = R - \varepsilon$, then by Cauchy's inequality [see (2.1) of chapter 5], we shall have

$$|f_n(\lambda)| \leqq M(\lambda)/r^n .$$

Returning now to (2.9) we derive the inequality

$$|\varphi_n| \leqq \{\sum_{m=0}^{n-1} |f_{n-m}(\lambda-m)|\,|\varphi_m|\}/|f_0(\lambda)-n)|$$

$$\leqq \{\sum_{m=0}^{n-1} M(\lambda-m)\,r^{m-n}\,|\varphi_m|/|f_0(\lambda-n)| .$$

If we abbreviate this last expression by Φ_n, we may write

$$\Phi_n = \frac{|\varphi_{n-1}|\,M(\lambda-n+1)}{|f_0(\lambda-n)|} + \frac{|f_0(\lambda-n+1)|\,\Phi_{n-1}}{r\,|f_0(\lambda-n)|} .$$

Employing the abbreviation $\lambda - n = \varrho$, and noting that $|\varphi_{n-1}| < \Phi_{n-1}$, we attain the desired inequality

$$\frac{\Phi_n}{\Phi_{n-1}} < \frac{M(\varrho+1)}{|f_0(\varrho)|} + \frac{|f_0(\varrho+1)|}{r\,|f_0(\varrho)|} .$$

If we now assume (1) that $\lim_{\rho=-\infty} f(\varrho)$ exists; (2) that $\lim_{\rho=-\infty} |f_0(\varrho + 1)|/|f_0(\varrho)| = 1/q$; (3) that $\lim_{\rho=-\infty} M(\varrho+1)/|f_0(\varrho)| = 0$, then series (2.2) converges outside the circle of radius $1/qR$ and converges uniformly outside the circle of radius $1/qr$, since $\sum_{n=0}^{\infty} \Phi_n s^n$ is the majorant of $\sum_{n=0}^{\infty} \varphi_n s^n$.

It is at this point that we differ from the classical case of differential equations of finite order, since these assumptions imply the existence of all the limits $\lim_{\rho=-\infty} f_n(\varrho)$, $n = 0, 1, 2, \cdots$. In the finite case the functions $f_n(\varrho)$ are polynomials of bounded degree and conditions (1), (2), and (3) are immediately satisfied, the value of q being equal to 1. In the present instance, however, these limits depend upon the properties of Newton series, particularly their behaviour at infinity, and we must await the exploration of their asymptotic properties before attaining more general theorems.

PROBLEMS

1. In the third case of the illustrative example of this section, evaluate the function $f(s,\lambda)$ and discuss the character of the solution from the properties of $f(s,\lambda)$.

2. Show that the series representing $f(s, \lambda)$ for the second case of the illustrative example of this section is totally divergent. Can a meaning be given to it by Borel summability?

3. Discuss the solution of the third case of the illustrative example for one of the roots other than $\lambda = -1$.

4. Employing the analysis of this section, discuss the solution of the q-difference equation.

$$u(q\,x) + \lambda\,u(x) = 0\ , \qquad q \neq 1\ ,$$

CHAPTER X

Differential Operators of Infinite Order of Fuchsian Type — Infinite Systems

1. Preliminary Remarks. The object of the present chapter is the inversion of the following equation:

$$\{A_0(x) + A_1(x)xz + A_2(x)x^2z^2/2! + \cdots \\ + A_n(x)x^nz^n/n! + \cdots\} \to u(x) = f(x) \quad , \qquad (1.1)$$

where the symbol z^n as usual denotes the differential operator d^n/dx^n and $f(x)$ is a function with limitations to be specified later. The coefficients,

$$A_i(x) = a_{i0} + a_{i1}x + a_{i2}x^2 + \cdots \quad , \qquad (1.2)$$

are functions analytic at $x = 0$ and subject to the restriction $A_n(0) \neq 0$ for values of n greater than or equal to a given n'. In the case of finite order we have $n' = p$, where p is the order of the equation. It will be observed that this restriction brings the finite case within the theory of linear differential equations with regular singular points initiated by the celebrated papers of L. Fuchs (1833-1902).*

Let us designate by $F(x,z)$ the Fuchsian operator in the braces of (1.1) so that we can write the equation in the abbreviated form,

$$F(x,z) \to u(x) = f(x) \quad .$$

It will be observed that the theory of equation (1.1) formally unifies the theories of the Volterra integral equation on the one hand and q-difference equations on the other. To show this let us write the integral equation,

$$u(x) + \lambda \int_0^x K(x,t)u(t)\,dt = f(x) \quad ,$$

in the form,

$$\{1 + \lambda \int_0^x K(x,t)\,e^{(t-x)z}dt\} \to u(x) = f(x) \quad .$$

The coefficient of z^n, $n > 0$, in the expansion of the operator is

$$\lambda \int_0^x K(x,t)\,(t-x)^n\,dt/n! \quad .$$

**Journal für Mathematik*, vols. 66 (1866), p, 121, and 68 (1868), p, 354; *Ges. Werke*, vol. 1, pp. 159 and 205.

For other reference see L. W. Thomé: *Journal für Mathematik*, vols. 74 (1872), p. 193, 75 (1873), p. 265, and 76 (1873), p. 273; E. Picard: *Traité d'analyse*, vol. 3 (1908), chap. 12; E. L. Ince: *Ordinary Differential Equations* (1927), chaps. 15 and 16.

For a statement of the theorem relating to regular singular points see the footnote to section 7, chapter 8.

Operating on this function with z^m, $m \leqq n$, we get,

$$z^m \to \lambda \int_0^x K(x,t)\,(t-x)^n\,dt/n!$$

$$= \lambda \int_0^x z^m \to K(x,t)\,(t-x)^n\,dt/n!\ ,$$

which obviously equals zero for $x = 0$. Letting $m = n+1$ we obtain

$$z^{n+1} \to \lambda \int_0^x K(x,t)\,(t-x)^n\,dt/n!$$

$$= \lim_{t=x} z^n \to \lambda\,K(x,t)\,(t-x)^n/n! + \int_0^x z^{n+1} \to K(x,t)\,(t-x)^n\,dt/n!$$

$$= (-1)^n \lambda\,K(x,x) + \lambda \int_0^x z^{n+1} \to K(x,t)\,(t-x)^n\,dt/n!\ .$$

For $x = 0$ this function is seen to equal $(-1)^n\,\lambda\,K(0,0)$. We thus prove that the coefficient of z^n vanishes at least to the nth order, and hence contains x^n as a factor.

Similarly the q-difference equation,*

$$\varphi_0(x)u(x) + \varphi_1(x)u(q_1x) + \varphi_2(x)u(q_2x) + \cdots + \varphi_r(x)u(q_rx) = g(x)\ ,$$

can be written in the form,

$$\{\varphi_0(x) + \varphi_1(x)e^{(q_1-1)xz} + \varphi_2(x)e^{(q_2-1)xz} + \cdots + \varphi_r(x)e^{(q_r-1)xz}\} \to u(x) = g(x)\ .$$

Expanding this equation into a power series in z and employing the abbreviation, $q_\iota - 1 = w_\iota$, we obtain,

$$\{\varphi_0(x) + \varphi_1(x) + \varphi_2(x) + \cdots + \varphi_r(x) + (w_1\varphi_1 + w_2\varphi_2 + \cdots + w_r\varphi_r)xz + (w_1^2\varphi_1 + w_2^2\varphi_2 + \cdots + w_r^2\varphi_r)x^2z^2/2! + \cdots\} \to u(x) = g(x)\ ,$$

which yields an equation of general Fuchsian type.

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*For the theory of this equation see: R. D. Carmichael: *American Journal of Math.*, vol. 34 (1912), pp. 147-168; G. D. Birkhoff: *Proc. Amer. Academy of Arts and Sciences*, vol. 49 (1913), pp. 521-568; T. E. Mason: *American Journal of Math.*, vol. 37 (1915), pp. 439-444; C. R. Adams: *Annals of Math.*, vol. 27 (1925-1926), pp. 73-83; *ibid.*, vol. 30 (1928-1929), pp. 195-205.

For an important summary and additional bibliography see C. R. Adams: Linear q-Difference Equations, *Bulletin of the American Math Soc.*, vol. 37 (1931), pp. 361-400.

2. The Homogeneous Equation. It will be convenient for us first to discuss the formal solution of the homogeneous case of equation (1.1):

$$\{A_0(x) + A_1(x)xz + A_2(x)x^2z^2/2! + \cdots$$
$$+ A_n(x)x^nz^n/n! + \cdots\} \to u(x) = 0 \,. \qquad (2.1)$$

Our attack will be to exhibit the essential connection, through a Laplace transformation, of the homogeneous case of the generalized Euler equation that we solved in section 2, chapter 9, with the homogeneous Fuchsian equation (2.1).

To show this relationship let us first seek a solution of the Euler equation,

$$a_0v(t) + (a_1 + b_1t)v'(t) + (a_2 + b_2t + c_2t^2/2!)v''(t)$$
$$+ (a_3 + b_3t + c_3t^2/2! + d_3t^3/3!)v'''(t) + \cdots = 0 \,, \qquad (2.2)$$

in the form of a Laplace transformation,

$$v(t) = \int_L e^{xt}\,u(x)\,dx \,, \qquad (2.3)$$

where L is a path in the complex plane.

If we make the abbreviation,

$$A(x) = a_0 + a_1x + a_2x^2 + \cdots \,,$$
$$B(x) = x(b_1 + b_2x + b_3x^2 + \cdots) \,,$$
$$C(x) = (x^2/2!)(c_2 + c_3x + c_4x^2 + \cdots) \,,$$
$$\cdots\cdots\cdots\cdots$$

then we know from an obvious generalization of the theory of section 3, chapter 8 [in particular, equations (3.6) and (3.7)], that if $u(x)$ is a solution of the equation,

$$A(x)u(x) - [B(x)u(x)]' + [C(x)u]''$$
$$- [D(x)u]''' + \cdots = 0 \,, \qquad (2.4)$$

and if L is a path so chosen that the functions

$$\{B(x)u(x) - [C(x)u(x)]' + [D(x)u]'' - \cdots\}\,e^{xt} \,,$$
$$\{C(x)u(x) - [D(x)u(x)]' + [E(x)u]'' - \cdots\}\,e^{xt} \,,$$
$$\{D(x)u(x) - [E(x)u(x)]' + [F(x)u]'' - \cdots\}\,e^{xt} \,,$$
$$\cdots\cdots\cdots\cdots \qquad (2.5)$$

vanish at the extremities of L, which in particular may be a closed circuit, the function $v(t)$ as given by (2.3) is a formal solution of equation (2.2).

In order to identify equation (2.4) with equation (1.1) we expand (2.4) as follows:

$$\begin{aligned}&\{A(x) - B'(x) + C''(x) - D''' + E^{(4)} - \cdots\}\, u(x)\\ &\quad + \{-B + 2C' - 3D'' + 4E''' - \cdots\}\, u'(x)\\ &\quad + \{C - 3D' + 6E'' - \cdots\}\, u''(x) + \{-D + 4E'\\ &\quad - \cdots\}\, u'''(x) + \{E - \cdots\}\, u^{(4)}(x) + \cdots = 0 .\end{aligned}$$

Equating these coefficients successively to

$$A_0(x)\ , \quad x\, A_1(x)\ , \quad x^2 A_2(x)/2!\, , \cdots ,$$

and differentiating the second equation of the resulting set once, the third equation twice, etc., we obtain the following systems of equations:

$$\begin{aligned}A - B' + C'' - D^{(3)} + E^{(4)} - \cdots &= A_0(x)\ ,\\ -B' + 2C'' - 3D^{(3)} + 4E^{(4)} - \cdots &= [x\, A_1]'\ ,\\ C'' - 3D^{(3)} + 6E^{(4)} - \cdots &= [x^2 A_2]''/2!\ ,\\ - D^{(3)} + 4E^{(4)} - \cdots &= [x^3 A_3]'''/3!\ ,\\ E^{(4)} - \cdots &= [x^4 A_4]^{(4)}/4!\ ,\\ \cdots\cdots\cdots &\end{aligned}$$

This system can be solved explicitly for $A(x)$, $B'(x)$, $C''(x)$, etc. and we obtain without difficulty the following results:

$$\begin{aligned}A(x) &= A_0(x) - [x\, A_1(x)]' + [x^2 A_2(x)/2!]'' - \cdots ,\\ B'(x) &= - [x\, A_1]' + 2[x^2 A_2/2!]'' - 3[x^3 A_3/3!]''' + \cdots ,\\ C''(x) &= [x^2 A_2(x)/2!]'' - 3[x^3 A_3/3!]''' + 6[x^4 A_4/4!]^{(4)} - \cdots ,\\ \cdots\cdots\cdots &\end{aligned}$$

Integrating the second row once, the third row twice, etc., we finally get the desired transformation:

$$\begin{aligned}A(x) &= A_0 - (x\, A_1)' + (x^2 A_2/2!)'' - (x^3 A_3/3!)^{(3)} + \cdots ,\\ B(x) &= -(x\, A_1) + 2(x^2 A_2/2!)' - 3(x^3 A_3/3!)'' + \cdots ,\\ C(x) &= (x^2 A_2/2!) - 3(x^3 A_3/3!)' + 6(x^4 A_4/4!)'' - \cdots ,\\ \cdots\cdots\cdots &\end{aligned} \tag{2.6}$$

Equating coefficients of x^r in each of these series we obtain the following explicit transformation which is very useful in the direct application of the results of this section:

$$\begin{aligned} a_r &= a_{0r} - (r+1)!\, a_{1r}/r!\cdot 1! + (r+2)!\, a_{2r}/r!\cdot 2! \\ &\qquad - \cdots + (-1)^s (r+s)!\, a_{sr}/r!\cdot s! + \cdots , \\ b_r &= -a_{1,r-1} + (r+1)!\, a_{2,r-1}/r!\cdot 1! - (r+2)!\, a_{3,r-1}/r!\cdot 2! \\ &\qquad + \cdots + (-1)^s (r+s-1)!\, a_{s,r-1}/r!\cdot (s-1)! + \cdots , \\ c_r &= a_{2,r-2} - (r+1)!\, a_{3,r-2}/r!\cdot 1! + (r+2)!\, a_{4,r-2}/r!\cdot 2! \\ &\qquad - \cdots + (-1)^s (r+s-2)!\, a_{s,r-2}/r!\cdot (s-2)! + \cdots , \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{2.7}$$

Transformation (2.6) can also be stated somewhat differently as follows:

Let us define the functions,

$$a_n(t) = \sum_{m=0}^{\infty} (-1)^n t^{m+n} a_{mn}/n! , \quad n = 0, 1, 2, \cdots ;$$

in terms of these expansions, evaluated at the point $t = 1$, we have,

$$\begin{aligned} A(x) &= a_0(1) + a_1'(1)x + a_2''(1)x^2 + a_3'''(1)x^3 + \cdots , \\ B(x) &= a_0'(1) + a_1''(1)x/2 + a_2'''(1)x^2/3 + a_3^{(4)}(1)x^3/4 + \cdots , \\ C(x) &= a_0''(1)/2 + a_1'''(1)x/6 + a_2^{(4)}(x)x^2/12 + \cdots , \qquad (2.8) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ N(x) &= a_0^{(n-1)}(1)/(n-1)! + a_1^{(n)}(1)x\, 1!/n! \\ &\qquad + a_2^{(n+1)}(1)x^2\, 2!/(n+1)! + \cdots . \end{aligned}$$

We are now in a position to obtain the solution of equation (2.1). To do this we first find the coefficients of a generalized Euler equation by means of the transformation (2.7) or its equivalent (2.8). The solution of this new equation is then obtained by means of the formulas of section 2, chapter 9, and hence, for the determination of the unknown function, $u(x)$, we reach the following equation of Laplace type:

$$\begin{aligned} v(t) &= t^\lambda(\varphi_0 + \varphi_1/t + \varphi_2/t^2 + \cdots \\ &\qquad + \varphi_n/t^n + \cdots) = \int_L e^{xt} u(x)\, dx , \end{aligned}$$

where L is a path to be properly determined. But since the inversion of this integral equation has been accomplished in a number of ways,

the solution of the original problem has been reduced to a form rich in specific results.*

In particular, if L is the path from 0 to $-\infty$, we are concerned with the solution of the Laplace equation:

$$v(t) = \int_0^\infty e^{-xt}\, u(-x)\, dx \ . \tag{2.9}$$

But since

$$\int_0^\infty e^{-xt}\, x^\mu\, dx = \Gamma(\mu+1)/x^{\mu+1} \ ,$$

we see that a solution of equation (2.1) is given by

$$u(x) = e^{2\pi i(\lambda+1)}\{x^{-\lambda-1}/\Gamma(-\lambda)\}\{\varphi_0 + x\,\varphi_1/\lambda + x^2\,\varphi_2/[\lambda(\lambda-1)] + x^3\,\varphi_3/[\lambda(\lambda-1)(\lambda-2)] + \cdots\} \ , \tag{2.10}$$

provided the real part of λ is negative.

These results are stated in the following theorem:

Theorem 1. If the coefficients of a generalized Euler equation are computed by means of the transformation (2.7) [*or* (2.8)] *from the coefficients of equation* (2.1) *and if a set of n solutions exist,*

$$v_i(t) = t^{\lambda_i}\varphi_i(t) \ , \ i = 1, 2, \cdots, n \ ,$$

where the $\varphi_i(t)$ *are functions of the form,*

$$\varphi_i(t) = \varphi_0 + \varphi_1/t + \varphi_2/t^2 + \cdots \ ,$$

then equation (2.1) *possesses n formal solutions,*

$$u_i(x) = e^{2\pi i(\lambda_i+1)}\ x^{-\lambda_i-1}\ \psi_i(x)/\Gamma(-\lambda_i) \ , \ \text{real part of } \lambda_i < 0 \ , \tag{2.11}$$

where the $\psi_i(x)$ *are functions of the form,*

$$\psi_i(x) = \varphi_0 + x\,\varphi_1/\lambda + x^2\,\varphi_2/\lambda(\lambda-1) + x^3\,\varphi_3/\lambda(\lambda-1)(\lambda-2) + \cdots \ . \tag{2.12}$$

Other solutions, if they exist, are obtained from the inversion of the Laplace integral equation,

$$v(t) = \int_L e^{xt}\, u(x)\, dx \ ,$$

where L is a path chosen so that the functions (2.5) *vanish at its extremities.*

From the foregoing analysis it will be clear that the analytical validity of any formal solution of the homogeneous Fuchsian equa-

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*See section 7, chapter 1.

tion will depend upon the validity of the solution of the auxiliary Euler equation. As we have indicated in the last section of chapter 9, this problem is not completely resolved since it depends in an essential manner upon the asymptotic properties of Newton series and these properties have been only meagerly explored.

It is obvious, however, from the equation

$$v(t) = \int_L e^{xt}\, u(x)\, dx$$

that the solution, $v(t)$, of the auxiliary Euler equation and the solution, $u(x)$, of the Fuchsian equation bear the relationship of a function to its Laplace transform. Hence the validity of the solution $u(x)$ as it depends upon the validity of $v(t)$ can in any special case be referred to the general theorems of the generatrix calculus as set forth in section 7, chapter 1.

As an example illustrating theorem 1 let us consider the hypergeometric equation for which we have already found the solution about ∞ by regarding it as a special case of the generalized Euler equation. (See section 2, chapter 9). We observe, however, that the hypergeometric equation may also be written as a Fuchsian equation if it be multiplied by x:

$$-\alpha\beta x u + [\gamma - (\alpha+\beta+1)x]\, x u' + (2-2x)x^2 u''/2! = 0 \ . \qquad (2.13)$$

Making use of the transformation given by (2.7), we obtain as the auxiliary Euler equation the following:

$$(2-\gamma)v(t) + [-(\alpha-2)(\beta-2) + (4-\gamma)t]\, v'(t) + [(\alpha+\beta-5)t + t^2]\, v''(t) - t^2 v'''(t) = 0 \ .$$

From this we obtain the functions,

$$f_0(\lambda) = (2-\gamma) + (4-\gamma)\lambda + \lambda(\lambda-1) = (\lambda+1)(\lambda+2-\gamma) \ ,$$

$$f_1(\lambda) = \lambda(\alpha-1-\lambda)(1-\beta+\lambda) \ .$$

The characteristic numbers are thus, $\lambda = -1$, $\lambda = \gamma - 2$, and the equation determining successive values of φ_n is

$$f_1(\lambda-n)\varphi_n + f_0(\lambda-n-1)\varphi_{n+1} = 0 \ .$$

Whence we get for $\lambda = -1$,

$$\varphi_{n+1}/\varphi_n = -(\alpha+n)(\beta+n)/(\gamma+n) \ ,$$

and for $\lambda = \gamma - 2$,

$$\varphi_{n+1}/\varphi_n = -(\alpha-\gamma+1+n)(\beta-\gamma+1+n)/(n+1) \ .$$

The solutions of (2.13) corresponding to these two values are readily found from (2.11) to be,

$$u_1(x) = F(\alpha, \beta ; \gamma ; x)$$

$$u_2(x) = (-x)^{1-\gamma} F(1+\alpha-\gamma, 1+\beta-\gamma ; 2-\gamma ; x)/\Gamma(2-\gamma) ,$$

that is to say, the solutions of the hypergeometric equation about $x = 0$.

3. Application to a q-difference Equation. As we have indicated in the first section the theory of q-difference equations is formally embraced by the methods which we have developed above. It is beyond the scope of this book to explore the full consequences of an application to q-difference equations, but the following example will illustrate the salient features of such an application.

The following equation,

$$u(qx) + (-1+x)u(x) = 0 , \quad |q| > 1 , \qquad (3.1)$$

plays a rôle in the theory of q-difference equations analogous to that of the equation,

$$u(x+1) - x\, u(x) = 0 ,$$

in the theory of difference equations.

The solution of this equation about $x = 0$ is obtained without difficulty from the equivalent equation,

$$x\, u(x) + w\, x\, u'(x) + w^2 x^2 u''(x)/2! + \cdots = 0 , \quad w = q-1 .$$

From the fact that $A_0(x) = x$, $A_1 = w$, $A_2 = w^2, \cdots , A_n = w^n$, we derive by means of the transformation (2.7) the coefficients of the auxiliary Euler equation to be,

$$a_0 = -w + w^2 - w^3 + \cdots = -w/(1+w) = 1/q - 1 , \; a_1 = 1 ;$$

$$b_1 = -w/q^2 ; \; c_2 = w^2/q^3 ; \; d_3 = -w^3/q^4 ; \; \cdots .$$

From these values we compute the functions:

$$f_0(\lambda) = q^{-\lambda-1} - 1 , \quad f_1(\lambda) = \lambda ,$$

and hence the infinitely many characteristic numbers,

$$\lambda_m = -1 - 2\pi\, m\, i/\log q , \quad m = 0, 1, 2, \cdots .$$

Confining our attention to $\lambda = -1$, we have,

$$\varphi_{n+1}/\varphi_n = -f_1(\lambda-n)/f_0(\lambda-n-1) = -(\lambda-n)/(q^{n+1}-1) ,$$

and hence we attain the solution,

$$u(x) = [1 - x/(q-1) + x^2/(q-1)(q^2-1) \\ - x^3/(q-1)(q^2-1)(q^3-1) + \cdots] \ ,$$

$$= (1-x/q)(1-x/q^2)(1-x/q^3) \cdots .$$

In order to obtain the solution about infinity we must first compute the solution about the origin of the auxiliary equation,

$$(1/q-1)v(t) + (1-wt/q^2)v'(t) + w^2 t^2 v''(t)/q^3 \cdot 2! \\ - w^3 t^3 v'''(t)/q^4 \cdot 3! + \cdots = 0 \ , \qquad (3.2)$$

provided such a solution exists. In order to do this we multiply (3.2) through by t thus transforming it into an equation of Fuchsian type and for this new equation we now compute a corresponding Euler equation by means of (2.7).

Since we have,

$$a_{00} = 0 \ , \ a_{01} = 1/q - 1 \ , \ a_{10} = 1 \ , \ a_{11} = -w/q \ ,$$

$$a_{20} = a_{30} = \cdots = a_{n0} = 0 \ , \ a_{21} = w^2/q^3 \ , \ a_{31} = -w^3/q^4 \ , \cdots \ ,$$

$$a_{n1} = (-1)^n w^n/q^{n+1} \ ,$$

we find,

$$a_0 = -1 \ , \ a_1 = -1 + 1/q + 2w/q^2 + 3w^2/q^3 \\ + 4w^3/q + \cdots = q - 1 \ ;$$

$$b_1 = -1 \ , \ b_2 = w\,q \ ; \ c_2 = 0 \ , \ c_3 = w^2 q \ ; \ d_4 = w^3 q \ ; \ \cdots \ ,$$

and thus the equation,

$$- V(t) + [(q-1) - t]\, V'(t) + w\,q\, V''(t) \\ + w^2 q\, t^2\, V'''(t)/2! + w^3 q\, t^3\, V^{(4)}(t)/3! + \cdots = 0 \ . \ (3.3)$$

We then obtain,

$$f_0(\lambda) = -(1+\lambda) \ , \qquad f_1(\lambda) = (q-1)\lambda + w\,q\,\lambda(\lambda-1) \\ + w^2 q\,\lambda(\lambda-1)(\lambda-2)/2! + \cdots = \lambda(q^\lambda-1) \ ,$$

and from the equation,

$$f_1(\lambda-n)\varphi_n + f_0(\lambda-n-1)\varphi_{n+1} = 0 \ , \quad \lambda = -1 \ ,$$

the successive values,

$$\varphi_0 = 1 \ , \ \varphi_1 = 1/q - 1 \ , \ \varphi_2 = (q^{-1}-1)(q^{-2}-1) \ , \\ \varphi_3 = (q^{-1}-1)(q^{-2}-1)(q^{-3}-1) \ , \cdots .$$

Hence the solution of (3.3) is the function,

$$V(t) = t^{-1} \sum_{i=0}^{\infty} \varphi_i / t^i \ ,$$

and by means of theorem 1, section 2, the solution of (3.2) is,

$$v(t) = 1 - t\,\varphi_1 + t^2\,\varphi_2/2! - t^3\,\varphi_3/3! + \cdots$$
$$= 1 + (1-q^{-1})\,t + (1-q^{-1})\,(1-q^{-2})\,t^2/2! + \cdots \ .$$

Returning now to the original equation we see that its solution is transformed into the problem of inverting the equation,

$$\int_L e^{xt}\,u(x)\,dx = v(t) \ . \qquad (3.4)$$

An obvious choice of L is a path about the origin and for this we find formally,

$$U(s) = s + (1-q^{-1})\,s^2 + (1-q^{-1})\,(1-q^{-2})\,s^3$$
$$+ (1-q^{-1})\,(1-q^{-2})\,(1-q^{-3})\,s^4 + \cdots \ ,$$

where we abbreviate $s = 1/x$. But this function fails to converge for $s = 1$, and its analytic extension possesses poles at the points, $s = q$, $s = q^2$, $\cdots$.

To show this let us first apply Raabe's test* to the series. Computing the ratio of succesive terms we have, $u_{n+1}/u_n = s\,(1 - q^{-n}) = s \cdot 1/(1 + a_n)$, where $a_n = 1/(q^n - 1)$. Since, then, $\lim_{n=\infty} n\,a_n = 0$, the series diverges for $|\,s\,| > 1$, and in particular for $s = 1$.

In order to discuss the function $U(s)$ outside the unit circle we observe that it satisfies the following functional equation:

$$U(s) = [s/(1-s)]\,[1 - U(s/q)] \ .$$

We thus conclude that as s approaches q, $s \to q$, we have the limit,

$$U(s) \to U(q) \to [q/(1-q)]\,[1 - U(1)] = \infty \ ,$$

and hence by induction,

$$s \to q^n\,,\ U(s) \to U(q^n) \to [q^n/(1-q^n)]\,[1 - U(q^{n-1})] = \infty \ ,$$

for all integral values of n.

From the facts thus deduced we conclude that the residues of $u(x)$ in the left member of (3.4) must be computed not merely at $x = 1$, but also at $1/q$, $1/q^2$, $1/q^3$, $\cdots$. Hence we write,

$$u(x) = \varphi(x)/[(1-1/x)\,(1-1/qx)\,(1-1/q^2x)\cdots(1-1/q^nx)\cdots] \ ,$$

*See E. Goursat: *Cours d'analyse*, vol. 1 (Hedrick translation) (1904), p. 341.

where $\varphi(x)$ is a function regular at $x = 1, 1/q, 1/q^2, \cdots$, which must be so determined that equation (3.4) is satisfied.

Integrating $\int_L e^{xt}\, u(x)\, dx$, where the path L is now a series of loops about $x = 1$, $x = 1/q$, $x = 1/q^2$, $\cdots$, we find

$$\int_L e^{xt}\, u(x)\, dx = \{e^t\, \varphi(1) + e^{t/q}\, \varphi(1/q)/q(1-q)$$
$$+ e^{t/q^2}\, \varphi(1/q^2)/q^2(1-q)(1-q^2) + \cdots\}\, G(q)\ , \qquad (3.5)$$

where we abbreviate, $G(q) = (1-1/q)(1-1/q^2)(1-1/q^3)\cdots$.

If we now write the function,

$$\begin{aligned} v(t) &= 1 + (1-q^{-1})t + (1-q^{-1})(1-q^{-2})t^2/2! + \cdots \\ &= G(q)\{1/G(q) + (1-q^{-1})t/G(q) \\ &\qquad + (1-q^{-1})(1-q^{-2})t^2/2!\cdot G(q) + \cdots\}\ , \end{aligned}$$

and observe the following expansions:*

$$\begin{aligned} 1/G(q) &= 1 + 1/q(1-q^{-1}) + 1/q^2\cdot(1-q^{-1})(1-q^{-2}) + \cdots\ , \\ (1-q^{-1})/G(q) &= 1 + 1/q^2\cdot(1-q^{-1}) \\ &\qquad + 1/q^4\cdot(1-q^{-1})(1-q^{-2}) + \cdots\ , \\ (1-q^{-1})(1-q^{-2})/G(q) &= 1 + 1/q^3\cdot(1-q^{-1}) \\ &\qquad + 1/q^3\cdot(1-q^{-1})(1-q^{-2}) + \cdots\ , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\ , \end{aligned}$$

we see that we can write,

$$v(t) = G(q)\{e^t + e^{t/q}/q(1-q^{-1})$$
$$+ e^{t^2/q^2}/q^2\cdot(1-q^{-1})(1-q^{-2}) + \cdots\}\ . \qquad (3.6)$$

Comparing (3.5) with (3.6) we find, $\varphi(1) = 1$, $\varphi(1/q) = -q$, $\varphi(1/q^2) = q^3$, and in general, $\varphi(1/q^n) = (-1)^n\, q^{\frac{1}{2}n(n+1)}$. These equations are obviously satisfied by the function,

$$\varphi(x) = e^{A\,\log^2 x + B\,\log x + C\pi i\,\log x}\ ,$$

where $A = 1/\log q^2$, $B = -\frac{1}{2}$, $C = -1/\log q$.

Hence the solution of the original equation about $x = \infty$ is found to be,

$$u(x) = e^{A\,\log^2 x}\, x^{B+C\pi i}/[(1-1/x)(1-1/qx)(1-1/q^2x)$$
$$\cdots(1-1/q^nx)\cdots]\ .$$

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*See G. Chrystal: *Algebra*, part 2, (1889), p. 320, or L. Euler: *Int. in Anal. Inf.* (1797), § 313, p. 262.

4. The Resolvent Generatrix of the Fuchsian Equation. We proceed next to the formal determination of the resolvent generatrix of the non-homogeneous equation. If we attempt to solve (1.1) by the method of infinite differentiation, we obtain, after differentiating (1.1) an infinite number of times and collecting the coefficients of $u^{(i)}(x)$, a system of the form,

$$\begin{aligned} b_{11}u_1 + b_{12}u_2 + b_{13}u_3 + \cdots + b_{1n}u_n + \cdots &= f_1 , \\ b_{21}u_1 + b_{22}u_2 + b_{23}u_3 + \cdots + b_{2n}u_n + \cdots &= f_2 , \\ b_{31}u_1 + b_{32}u_2 + b_{33}u_3 + \cdots + b_{3n}u_n + \cdots &= f_3 , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{4.1}$$

where $u_i = u^{(i-1)}(x)$, $f_i = f^{(i-1)}(x)$, and the elements of the matrix $\|b_{ij}\|$ are determined from the algorithm,

$$b_{1j} = A_{j-1}(x)x^{j-1}/(j-1)!, \quad b_{i1} = A_0^{(i-1)}(x) ,$$

$$b_{ij} = b_{i-1,j-1} + \frac{d}{dx} b_{i-1,j} .$$

The practical difficulties of solving (4.1) are, in general, great so we turn to another method which makes use of an interesting connection between the Fuchsian equation and the generalized Euler equation discussed in the last chapter. We formulate the method in the following theorem:

Theorem 2. The resolvent generatrix $X_0(x,z)$ for equation (1.1) *is given formally by the equation,*

$$X_0(x,z) = Y_0(0,x) + Y_0'(0,x)z + Y_0''(0,x)z^2/2! + Y_0'''(0,x)z^3/3 + \cdots ,$$

where we use the abbreviation,

$$Y_0^{(r)}(0,x) = \frac{\partial^r}{\partial t^r} Y_0(t,x)\,|_{t=0} ,$$

in which $Y_0(x,z)$ is the resolvent of the generalized Euler equation, (1.7) *of chapter* 9,

$$G(x,z) \to u(x) = g(x) ,$$

where the coefficients are computed from the transformation,

$$S: \quad a_m = a_{0m} , \quad b_{m+1} = a_{1m} , \quad c_{m+2} = a_{2m} , \cdots . \tag{4.2}$$

The resolvent $Y_0(x,z)$ is explicitly calculated by means of (4.4) *of chapter* 9.

Proof: In order to prove this theorem we employ the Bourlet product, equation (3.1) of chapter 4, instead of the system (4.2) of

chapter 9, for the determination of the resolvent $Y_0(x,z)$, which satisfies the equation,

$$Y_0(x,z) \to G(x,z) = 1 \ .$$

Substituting the explicit form of $G(x,z)$ in the Bourlet product formula, we get,

$$\begin{aligned}
&Y_0(x,z) \to G(x,z) \equiv \\
&[a_0 + (a_1 + b_1 x)z + (a_2 + b_2 x + c_2 x^2/2!)z^2 + \cdots]\, Y_0(x,z) \\
&+ [b_1 + (b_2 + c_2 x)z + (b_3 + c_3 x + d_3 x^2/2!)\, z^2 + \cdots] z\, \partial Y_0/\partial z \\
&+ [c_2 + (c_3 + d_3 x)z + (c_4 + d_4 x + e_4 x^2/2!)z^2 + \cdots]\frac{z^2}{2!}\, \partial^2 Y_0/\partial z^2 \\
&+ \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\
&\qquad = 1 \ . \qquad\qquad (4.3)
\end{aligned}$$

But this equation we notice is of Fuchsian type in the variable z as defined by equation (1.1) with the right hand member set equal to 1. The auxiliary variable x may be regarded as an independent parameter and can be chosen in particular to equal zero.

If we now set $x = 0$ in (4.3) and identify the coefficients of z^r with the coefficients of z^r in (1.1), we obtain the transformation S (4.2).

Referring to (12.5) of chapter 4, we reach the conclusion: *The resolvent generatrix, $Y_0(0,x)$, is the solution of equation* (1.1) *for the case $f(x) = 1$.*

In order to solve (1.1) for the more general case $f(x) = x^m$, we make use of the fact (see section 12, chapter 4) that if $Y_0(x,z)$ is the resolvent of $G(x,z)$ and if $Y_m(x,z)$ is an operator such that $Y_m(x,z) \to G(x,z) = z^m$, then we have $Y_m(x,z) = z^m \to Y_0(x,z)$.

But from the operational product, equation (3.1) of chapter 4, we obtain

$$\begin{aligned}
Y_m(x,z) &= z^m\, Y_0 + m\, z^{m-1}\frac{\partial Y_0}{\partial x} + \frac{m(m-1)}{2!}\, z^{m-2}\frac{\partial^2 Y_0}{\partial x^2} + \cdots \\
&= [z + Y_0(x,z)]^{(m)} \ . \qquad\qquad (4.4)
\end{aligned}$$

Now equation (4.3) is identical with equation (1.1) for the special case under consideration provided x is set equal to zero, the right hand member is replaced by z^m, and the z is finally changed into an x.

Hence if we designate by $u_m(x)$ the solution of equation (1.1) for $f(x) = x^m$, we obtain the result

$$u_m(x) = [x + Y_0(t,x)]^{(m)}|_{t=0} \ , \quad u_0(x) = Y_0(0,x) \ , \qquad (4.5)$$

where the differentiation in the symbolic expansion is with respect to t.

We now finally refer to equation (12.5) of chapter 4 in which we replace the values of $u_m(x)$ by the right hand member of (4.5). From this substitution we immediately derive the identity

$$(\partial^m X_0/\partial z^m)|_{z=0} = [\partial^m Y_0(t,x)/\partial t^m]|_{t=0} ,$$

and hence obtain the desired expansion

$$\begin{aligned} X_0(x,z) &= X_0(x,0) + X_0'(x,0)z + X_0''(x,0)z^2/2! + \cdots \\ &= Y_0(0,x) + Y_0'(0,x)z + Y_0''(0,x)z^2/2! + \cdots \end{aligned} \tag{4.6}$$

where we abbreviate

$$[\partial^r X_0(x,z)/\partial z^r]|_{z=0} = X_0^{(r)}(x,0) \quad , [\partial^r Y_0(t,x)/\partial t^r]|_{t=0} = Y_0^{(r)}(0,x) \quad .$$

In this discussion nothing has yet been said about the nature of the function $f(x)$ and the final definition of the solution

$$u(x) = X_0(x,z) \rightarrow f(x) \quad .$$

If we limit $f(x)$ to the class of functions of finite grade q, then simple conditions for the existence and analyticity of $u(x)$ can be given.

Under the restrictions imposed in theorem 3, chapter 9, we see that $Y_0(x,z)$ exists within a region Y, when x is restricted to the region S and z to the interior of the unit circle. Moreover, when the variables x and z are thus limited, $Y_0(x,z)$ is an analytic function within each of the two specified regions.

If furthermore S includes the origin in its interior, then the limit

$$\lim_{n=\infty} | Y_0(0,x)/n! |^{1/n} = Q$$

exists provided x lies within the unit circle. From this it follows that the expansion comprising the right hand member of (4.6) represents an analytic function within the circle $|z| \leqq Q$.

Hence if $f(x)$ is restricted to the class of functions of bounded grade q, and if $q \leqq Q$, it follows from theorem 6, *chapter* 5, *that the series $X_0(x,z) \rightarrow f(x)$ converges uniformly when x lies within the unit circle, and hence represents an analytic function there.*

As an example let us compute the resolvent for the following differential equation:

$$u(x) + e^{ax}\, x\, u'(x) + e^{2ax}\, x^2\, u''(x)/2! + e^{3ax}\, x^3\, u(x)/3! + \cdots = f(x) \quad ,$$

where we limit ourselves to the case where a is small.

We find from transformation (4.2) the following generalized Euler equation:

$$
\begin{aligned}
v(t) &+ tv'(t) + (ta + t^2/2!)v''(t) + (a^2t/2! + at^2 + t^3/3!)v'''(t) \\
&+ (a^3t/3! + a^2t^2 + at^3/2! + t^4/4!)v^{(4)}(t) + (a^4t/4! \\
&+ 4a^3t^2/3! + 3a^2t^3/4 + at^4/3! + t^5/5!)v^{(5)}(t) + (a^5t/5! \\
&+ a^4t^2/3 + 3a^3t^3/4 + a^2t^4/3 + at^5/4! + t^6/6!)v^{(6)}(t) \\
&+ \cdots = g(t) \ . \qquad (4.7)
\end{aligned}
$$

By means of equation (4.4) of chapter 9 we compute the resolvent generatrix of (4.7) and thus obtain (using $p = d/dt$)

$$
\begin{aligned}
Y_0(t,p) &= 1 - tp/2 + (t^2 - at)p^2/4 \cdot 2! - (t^3 - 3at^2 \\
&- 3a^2t/2)p^3/8 \cdot 3! + (t^4 - 6at^3 - 3a^2t^2 - 2a^3t)p^4/16 \cdot 4! \\
&- (t^5 - 10t^4a + 5t^2a^3 + 5a^4t/2)p^5/32 \cdot 5! + \cdots \ .
\end{aligned}
$$

The resolvent for the original equation is then derived for a few terms by means of (4.6) as follows:

$$
\begin{aligned}
X_0(x,z) &= 1 + (-x/2 - ax^2/4 \cdot 2! + 3a^2x^3/16 \cdot 3! - a^3x^4/8 \cdot 4! \\
&- 5a^4x^5/64 \cdot 5! + \cdots)z + (x^2/4 + ax^3/8 - a^2x^4/64 \\
&- a^3x^5/16 \cdot 4! + \cdots)z^2/2! + (-x^3/8 - 3ax^4/32 + 0 \cdot x^5/32 \\
&+ \cdots)z^3/3! + (x^4/16 + 2ax^5/32 + \cdots)z^4/4! \\
&+ (-x^5/32 + \cdots)z^5/5! + \cdots \ .
\end{aligned}
$$

For small values of a we obtain by induction the following solution:

$$
\begin{aligned}
X_0(x,z) &= 1 - (\tfrac{1}{2}xz) + (\tfrac{1}{2}xz)^2/2! - (\tfrac{1}{2}xz)^3/3! + \cdots \\
&\qquad - \tfrac{1}{2}ax\{\tfrac{1}{2}xz - (\tfrac{1}{2}xz)^2 + (\tfrac{1}{2}xz)^3/2! - \cdots\} \ , \\
&= e^{-\frac{1}{2}xz}(1 - ax^2z/8) \ .
\end{aligned}
$$

PROBLEMS

1. Given the q-difference equation

$$u(x) = x\,u(qx) \ , \quad |q| < 1 \ ,$$

show that the following functions are solutions:

$$u(x) = K \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n(n-1)} x^n \ ,$$

$$v(x) = \prod_{n=1}^{\infty} (1 + q^{n-1}x)(1 + q^n x^{-1}) \ .$$

If $u(x) = v(x)$, show that

$$1/K = \prod_{n=1}^{\infty} (1-q^n) \ .$$

Finally, employing the abbreviation

$$u(x;q) = \prod_{n=1}^{\infty} (1+q^{n-1}x)(1+q^n x^{-1})(1-q^n) \ ,$$

make the following identification with the theta functions:

$$u(q\, e^{2iz};\, q^2) = \theta_3(z) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz} \ ,$$

$$u(-q\, e^{2iz};\, q^2) = \theta_4(z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz} \ ,$$

$$q^{1/4} e^{iz} u(q^2 e^{2iz};\, q^2) = \theta_2(z) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{(2n+1)iz} \ ,$$

$$-i\, q^{1/4} e^{iz} u(-q^2 e^{2iz};\, q^2) = \theta_1(z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)iz} \ .$$

[This problem is due to G. W. Starcher: A Solution of a Simple Functional Equation as a Basis for Readily Obtaining Certain Fundamental Formulas in the Theory of Elliptic Functions. *Bulletin of the American Math. Soc.*, vol. 36 (1930), pp. 577-581].

2. Show that if $G(x,z) \to u(x) = f(x)$ is a generalized Euler equation of infinite order, then $G(z,x) \to u(x) = f(x)$ is formally a general Fuchsian equation of infinite order. Prove also that if $Y(x,z)$ is the resolvent generatrix of the first equation, then $Y(z,x)$ is the resolvent generatrix of the second.

5. Systems of Differential Equations in Infinitely Many Variables. Before passing on from the general differential equation treated in this chapter to the theory of integral operators, it will be useful to consider the problem presented by systems of differential equations in infinitely many variables. We have already given the historical origins and the present status of this problem in section 6, chapter 1; the special case of systems of differential equations with constant coefficients has been treated in section 7, chapter 7.

The problem considered in this section is that of solving a system of differential equations of the form

$$\frac{du_i}{dx} - \sum_{j=1}^{\infty} a_{ij}(x) u_j(x) = f_i(x) \ , \quad (i = 1, 2, \cdots, \infty) \qquad (5.1)$$

where we shall assume that the functions $a_{ij}(x)$ and $f_i(x)$ are analytic within a circular region $|\, x \,| < R$.

It is well known that an ordinary linear differential equation of finite order can be reduced to a finite system of differential equations of finite order and conversely. In fact, general existence theo-

rems for ordinary differential equations are perhaps most effectively achieved through their equivalent systems.*

This important equivalence is not carried over to differential equations of infinite order, however, as one may see from the reduction of the equation

$$a_0(x)u(x) + a_1(x)u'(x) + \cdots + a_n(x)\, u^{(n)}(x) + \cdots = f(x)$$

to the system

$$\frac{du_n}{dx} = u_{n+1} \ , \quad u_0 = u(x) \ ,$$

$$a_0\, u_0 + a_1\, u_1 + a_2\, u_2 + \cdots + a_n\, u_n + \cdots = f(x) \ .$$

T. Lalesco [*Bibliography*: Lalesco (1)] has suggested the following as a proper generalization:

Let us write

$$L_n(u) \equiv a_0(x)\, u^{(n)} + a_1(x)\, u^{(n-1)} + \cdots + a_n(x)u \ .$$

Then the equation

$$\lim_{n\to\infty} L_n(u) = f(x)$$

is equivalent to the system

$$a_0(x)\, \frac{du_1}{dx} + a_1\, u_1 + a_2\, u_2 + \cdots + a_n\, u_n + \cdots = f(x) \ ,$$

$$\frac{du_n}{dx} = u_{n-1} \ , \quad (n = 1, 2, \cdots, \infty) \ .$$

From this we see that infinite systems of differential equations present a problem distinct from that of differential equations of infinite order.

We shall first consider the homogeneous case of equation (5.1) and shall prove the following theorem:†

Theorem 3. Given the system

$$\frac{du_i}{dx} = \sum_{j=1}^{\infty} a_{ij}(x) u_j \ , \ (i = 1, 2, \cdots, \infty) \ , \qquad (5.2)$$

let $a_{ij}(x)$ *be given functions of* x *analytic for* $|x| \leqq R$, *and let* $\{S_i\}$, $\{T_i\}$ *be a set of positive numbers such that* (a) $K = \sum_{i=1}^{\infty} S_i\, T_i$ *exists, and* (b) $|a_{ij}| < S_i\, T_j$, *for* $|x| \leqq R_i < R$.

*See, for example, E. L. Ince: *Ordinary Differential Equations.* London (1927), chapter 3.

†In this development we follow H. von Koch. [See *Bibliography*: von Koch (1)].

If $\{u_i^{(0)}\}$ represents a set of constants such that the series

$$a(u) = \sum_{i=1}^{\infty} u_i^{(0)} T_i \tag{5.3}$$

converges absolutely, then system (5.2) *is satisfied by a unique set of functions $\{u_i(x)\}$, which assume the initial values $\{u_i^{(0)}\}$ for $x = 0$ and which are analytic in the circle $|x| \leqq R$.*

Proof: Let us consider the functions

$$u_i(x) = \sum_{k=0}^{\infty} u_i^{(k)} x^k , \quad (i = 1, 2, \cdots, \infty) . \tag{5.4}$$

It is clear that the boundary conditions are satisfied and it remains to be shown that the values $u_i^{(k)}$, $k \neq 0$, can be chosen so that (5.2) is satisfied and (5.4) converges uniformly in the circle $|x| < R$.

Employing the method of majorants, we consider the system

$$\frac{dv_i}{dx} = \frac{S_i}{1 - \frac{x}{R}} \sum_{j=1}^{\infty} T_j v_j , \quad (i = 1, 2, \cdots, \infty) \tag{5.5}$$

and the boundary conditions $\{v_i(0)\} = \{v_i^{(0)}\}$, where we assume that $|u_i^{(0)}| \leqq v_i^{(0)}$.

We next observe that

$$v_i(x) = v_i^{(0)} + \frac{S_i}{K} \left[\frac{a(v)}{(1 - \frac{x}{R})^{KR}} - a(v) \right]$$

is a solution of (5.5), since we have

$$\frac{dv_i}{dx} = S_i \frac{a(v)}{(1 - \frac{x}{R})^{KR+1}} = \frac{S_i}{(1 - \frac{x}{R})} \frac{a(v)}{(1 - \frac{x}{R})^{KR}}$$

$$= \frac{S_i}{(1 - \frac{x}{R})} \sum_{j=1}^{\infty} T_j v_j .$$

For $|x| < R$ we can develop v_i in a power series

$$v_i(x) = v_i^{(0)} + v_i^{(1)} x + v_i^{(2)} x^2 + \cdots$$

where all the values $\{v_i^{(k)}\}$ are positive.

From the method of formation of the majorant system (5.5), it is clear that the functions $v_i(x)$ will serve as majorants for $u_i(x)$ and that we shall have $|u_i^{(k)}| < v_i^{(k)}$. Hence a solution of the prescribed kind exists for (5.2) and it will converge uniformly in the circle $|x| < R$.

If the functions $a_{ij}(x)$ have the following development within the circle $|x| < R$:

$$a_{ij}(x) = a_{ij}^{(0)} + a_{ij}^{(1)} x + a_{ij}^{(2)} x^2 + \cdots ,$$

then $u_i(x)$ can be explicitly determined. Thus, substituting (5.4) in (5.2) and equating coefficients, we obtain the following linear system:

$$\begin{aligned} u_i^{(1)} &= \sum_{j=1}^{\infty} a_{ij}^{(0)} u_j^{(0)} , \\ u_i^{(2)} &= \frac{1}{2} \sum_{j=1}^{\infty} [a_{ij}^{(1)} u_j^{(0)} + a_{ij}^{(0)} u_j^{(1)}] , \\ u_i^{(3)} &= \frac{1}{3} \sum_{j=1}^{\infty} [a_{ij}^{(2)} u_j^{(0)} + a_{ij}^{(1)} u_j^{(1)} + a_{ij}^{(0)} u_j^{(2)}] , \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{5.6}$$

This explicit evaluation of $u_i^{(j)}$ also establishes the uniqueness of the solution.

Example. As an elementary example consider the system

$$\frac{du_i}{dx} = u_{i+1} , \quad (i = 1, 2, \cdots, \infty) .$$

Since $a_{ij} = \delta_{i+1,j}$ where δ_{ij} is the Kronecker symbol, we get from (5.6)

$$u_i^{(1)} = u_{i+1}^{(0)} , \quad u_i^{(2)} = u_{i+2}^{(0)}/2! , \quad u_i^{(3)} = u_{i+3}^{(0)}/3! , \cdots$$

and hence achieve the solution

$$u_i(x) = \sum_{n=0}^{\infty} u_{i+n}^{(0)} x^n/n! .$$

The solution (5.4) can be given another form which is often convenient, namely,

$$u_i(x) = \sum_{j=1}^{\infty} u_j^{(0)} u_{ij}(x) , \tag{5.7}$$

where the functions $u_{ij}(x)$ are members of a doubly infinite set of normal solutions of the original equations; that is, they satisfy the boundary conditions

$$u_{ij}(0) = \delta_{ij} ,$$

where δ_{ij} is the Kronecker symbol.

That such a system exists is proved from the easily established inequalities

$$
\begin{aligned}
| u_{ij} | &\leqq \frac{S_i T_j}{K} \left[\frac{1}{(1-\frac{x}{R})^{KR}} - 1 \right] , \quad (i \neq j) , \\
| u_{ii} - 1 | &\leqq \frac{S_i T_i}{K} \left[\frac{1}{(1-\frac{x}{R})^{KR}} - 1 \right] .
\end{aligned} \tag{5.8}
$$

From these inequalities we then obtain the majorant

$$| u_i(x) | \leqq u_i^{(0)} + \frac{a(u) S_i}{K} \left[\frac{1}{(1-\frac{x}{R})^{KR}} - 1 \right] ,$$

which establishes the convergence of (5.7).

It is easily apprehended that the set of functions $u_{ij}(x)$ forms a fundamental system of solutions of equations (5.2). One may also establish from the inequalities (5.8) and theorem 1, chapter 3 that the determinant $[u_{ij}]$ converges absolutely together with all of its minors.

Finally, it should be pointed out that the more general system

$$\frac{du_i}{dx} = \sum_{j=-\infty}^{\infty} a_{ij}(x) u_j , \quad (i = -\infty, \cdots, +\infty)$$

may be included under theorem 3. This is accomplished by writing the system in the form

$$\frac{du_i}{dx} = a_{i0} u_0 + \sum_{j=1}^{\infty} (a_{ij} u_j + a_{i,-j} u_{-j}) , \quad (i = 0, 1, -1, 2, -2, \cdots) .$$

The Non-homogeneous Equation

We next consider the non-homogeneous equation (5.1) where we assume that the functions $f_i(x)$, analytic within the circle $| x | < R$, have the formal expansions

$$f_i(x) = f_i^{(0)} + f_i^{(1)} x + f_i^{(2)} x^2 + \cdots .$$

If the power series (5.4) is substituted in (5.1) and corresponding powers of x equated, we obtain the following linear system for the determination of the coefficients $u_i^{(j)}$:

$$
\begin{aligned}
u_i^{(1)} &= \sum_{j=1}^{\infty} a_{ij}^{(0)} u_j^{(0)} + f_i^{(0)} , \\
2 u_i^{(2)} &= \sum_{j=1}^{\infty} [a_{ij}^{(1)} u_j^{(0)} + a_{ij}^{(0)} u_j^{(1)}] + f_i^{(1)} , \\
3 u_i^{(3)} &= \sum_{j=1}^{\infty} [a_{ij}^{(2)} u_j^{(0)} + a_{ij}^{(1)} u_j^{(1)} + a_{ij}^{(0)} u_j^{(2)}] + f_i^{(2)} , \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{5.9}
$$

An existence theorem analogous to the one given for the homogeneous case can be established by a direct proof of the convergence of the solution obtained formally from system (5.9), but the details of the proof are much more troublesome. It will be sufficient for our purposes to consider a somewhat different argument as follows:

Let us assume that the functions $u_{ij}(x)$ are the members of the fundamental set of solutions of the homogeneous equation given in (5.7) and let us consider the following set of functions:

$$u_i(x) = \sum_{j=1}^{\infty} u_{ij}(x)\, v_j(x) \ , \tag{5.10}$$

where the quantities $v_j(x)$ are to be determined.

Assuming the legitimacy of the process, an immediate consequence of the conditions stated in theorem 3, we now form the derivative of (5.10) and thus obtain

$$\frac{du_i(x)}{dx} = \sum_{j=0}^{\infty} \frac{du_{ij}}{dx} v_j(x) + \sum_{j=1}^{\infty} u_{ij}(x) \frac{dv_j}{dx} \ . \tag{5.11}$$

If now (5.10) and (5.11) are substituted in (5.1), one gets for the determination of the functions $v_i(x)$ the following system:

$$\sum_{j=1}^{\infty} \frac{du_{ij}}{dx} v_j + \sum_{j=1}^{\infty} u_{ij} \frac{dv_j}{dx} = \sum_{k,j=1}^{\infty} a_{ik}\, u_{kj}(x)\, v_j(x) + f_i(x) \ . \tag{5.12}$$

Moreover, since we have

$$\frac{du_{ij}}{dx} = \sum_{k=1}^{\infty} a_{ik}\, u_{kj} \ ,$$

system (5.12) reduces to

$$\sum_{j=1}^{\infty} u_{ij} \frac{dv_j}{dx} = f_i(x) \ . \tag{5.13}$$

If we now return to the inequalities given in (5.8), it is clear that the determinant $[u_{ij}]$ converges absolutely by the criteria of theorem 1, section 3, chapter 3. Moreover, from the corollary of that section and the assumption of the boundedness of the functions $f_i(x)$ within the circle $|x| < R$, it follows that system (5.13) can be solved and the derivatives dv_j/dx thus determined. These derivatives will also be analytic functions within the circle of definition and hence, if we designate them by $F_j(x)$, we shall obtain

$$v_j(x) = \int_0^x F_j(t)\, dt \ , \qquad |x| < R \ .$$

From this it follows that the solution of (5.1), which assumes the values $u_i^{(0)}$ at the origin, will be explicitly given by

$$u_i(x) = \sum_{j=1}^{\infty} [u_j^{(0)} + \int_0^x F_j(t)\, dt]\, u_{ij}(x) \quad . \tag{5.14}$$

The uniqueness of this solution is immediately apprehended from the form of system (5.9) in which the values of the coefficients are determined sequentially and hence uniquely.

These conclusions can be summarized in the following theorem:

Theorem 4. Under the conditions stated in theorem 3 *and with the additional assumption that the functions $f_i(x)$ are analytic within the circle $|x| < R$ and remain bounded with i, there exists a unique solution of system* (5.1), *which assumes the values $u_i^{(0)}$ at the origin and which is analytic within the circle of definition. The solution is given by* (5.14).

The theory of linear systems of form (5.1) has been developed by various authors under different types of conditions. The reader is referred to section 6, chapter 1 for an historical account of these investigations.

F. R. Moulton [see *Bibliography*: Moulton (2)] generalized the problem so as to apply to the non-linear system

$$\begin{aligned} \frac{du_i}{dx} &= f_i(x\,; u_1, u_2, \cdots) \ , \\ &= a_i + f_i^{(1)} + f_i^{(2)} + \cdots \ , \quad (i = 1, 2, \cdots) \ . \end{aligned}$$

In this development the quantities a_i are constants and the functions $f_i^{(j)}$ are the terms of the f_i respectively which are homogeneous of degree j in the variables $x, u_1, u_2, \cdots$. The following restrictions are imposed:

(1) The functions u_i are all zero for $x = 0$.

(2) Finite constants $\{c_i\}$, $\{r_i\}$, A and a exist such that the series

$$s = c_0\, x + c_1\, u_1 + c_2\, u_2 + \cdots$$

converges provided $|x| \leqq r_0$, $|u_i| \leqq r_i$, and the quantities $A\, r_i\, s^j$ dominate f_i^j and $|a_i| \leqq A\, r_i\, a$.

Moulton also showed how the analytical extension of the solution could be accomplished so that the functional system would still be satisfied by the solution thus continued.

The Cauchy-Lipschitz method so useful in obtaining existence proofs for ordinary differential equations has been extended by several writers to the infinite case.

W. L. Hart [see *Bibliography*: Hart (4)] in a paper of considerable elegance considered the problem from the point of view of limited bilinear forms and deduced existence criteria for solutions which belong to Hilbert space.

PROBLEMS

1. Solve the system

$$\frac{du_i}{dx} = u_{i+1} + 1/(1-x) \ , \qquad (i = 1, 2, 3, \cdots) \ ,$$

where the solutions are subject to the condition: $u_i(0) = a^i$, $|a| < 1$.

2. Prove that the determinant

$$\Delta = [u_{ij}]$$

of the fundamental set of solutions of (5.1) is given by the formula

$$\Delta = C\, e^{\int_0^x A(t)\, dt} \ ,$$

where we abbreviate

$$A(t) = \sum_{i=1}^{\infty} a_{ii}(t) \ .$$

3. The system

$$\frac{dv_i}{dx} = -\sum_{j=1} a_{ji}\, v_j(x) \ , \qquad (i = 1, 2, 3, \cdots)$$

is called the system adjoint to (5.2). Prove that if $u_i(x)$ is a solution of (5.2) and if $v_i(x)$ is a solution of the adjoint system, then we have formally

$$\frac{d}{dx}\Big[\sum_{i=1} u_i(x)\, v_i(x)\Big] = 0 \ . \qquad \text{(Hart).}$$

CHAPTER XI

INTEGRAL EQUATIONS OF INFINITE ORDER

1. Introduction. In this chapter we shall consider the problem presented by the operational solution of integral equations. We shall begin with the Volterra integral equation

$$a_0(x)\, u(x) + \int_a^x K(x, t)\, u(t)\, dt = f(x) \;, \qquad (1.1)$$

the history of which has been summarized in the first chapter. This will be followed by an exposition of a theory due to T. Lalesco [see *Bibliography*: Lalesco (2)] which generalizes the *Cauchy problem.* The chapter will conclude with a discussion of the Fredholm integral equation,

$$u(x) = f(x) + \lambda \int_a^b K(x, t)\, u(t)\, dt \;, \qquad (1.2)$$

mainly from the point of view of differential equations of infinite order.

2. The Cauchy Problem of Linear Differential Equations. We shall find it useful in the beginning to trace the development of the *Cauchy problem* from its genesis in the theory of ordinary differential equations to its modern setting in the theory of the Volterra integral equation. Operational methods will then be invoked to solve this latter problem.

Let us begin by considering the problem presented by the non-homogeneous differential equation,

$$\begin{aligned} L(u) \equiv p_n(x) u^{(n)}(x) &+ p_{n-1}(x) u^{(n-1)}(x) \\ &+ \cdots + p_0(x) u(x) = r(x) \;, \qquad (2.1) \end{aligned}$$

the solution of which at some point, $x = a$, shall satisfy the conditions:

$$u(a) = g_0 , \quad u'(a) = g_1, \cdots, \quad u^{(n-1)}(a) = g_{n-1} \;. \qquad (2.2)$$

We shall suppose, unless it is otherwise stated, that the functions $p_i(x)$, $r(x)$ are continuous in a closed interval (ab) and that $p_n(x)$ does not vanish in that interval.

Now assume that we can write the solution of (2.1) in the form,

$$u(x) = c_1(x)\, u_1(x) + c_2(x)\, u_2(x) + \cdots + c_n(x)\, u_n(x) \;, \qquad (2.3)$$

where the $c_i(x)$ are as yet undetermined functions and $\{u_i(x)\} =$

$\{u_1(x),\ u_2(x),\cdots,\ u_n(x)\}$ forms a set of linearly independent solutions of the homogeneous equation,

$$L(u)=0 .$$

Hereafter we shall refer to $\{u_i(x)\}$ as a *fundamental set* of solutions.*

Differentiating equation (2.3) we shall have

$$u'(x)=c_1u_1'+c_2u_2'+\cdots+c_nu_n'+c_1'u_1+c_2'u_2+\cdots+c_n'u_n .$$

As a first condition upon the $c_i(x)$ let us assume that,

$$c_1'u_1+c_2'u_2+\cdots+c_n'u_n=0 .$$

We can continue to differentiate $u(x)$ in this way until we have taken $n-1$ derivatives and imposed $n-1$ conditions upon the $c_i(x)$. For the nth conditions we shall choose the following:

$$c_1'(x)u_1^{(n-1)}(x)+c_2'(x)u_2^{(n-1)}(x)+\cdots$$
$$+c_n'(x)u_n^{(n-1)}(x)=r(x)/p_n(x) .$$

The $c_i(x)$ are now explicitly determined as follows:

$$c_i'(x)=W_{1i}(x)\,r(x)/W(x)\,p_n(x) , \tag{2.4}$$

where $W(x)$, called the *Wronskian* of the fundamental system,† is defined to be,

$$W(x)=\begin{vmatrix} u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \\ u_1^{(n-2)} & u_2^{(n-2)} & \cdots & u_n^{(n-2)} \\ \cdot & \cdot & \cdot & \cdot \\ u_1 & u_2 & \cdots & u_n \end{vmatrix} ,$$

and $W_{1i}(x)$ is the cofactor of the first row and the ith column.

Recalling the conditions imposed upon the functions $c_i(x)$ we may write the values of $u(x)$ and its first n derivatives as follows:

$$u(x)=c_1u_1+c_2u_2+\cdots+c_nu_n ,$$
$$u'(x)=c_1u_1'+c_2u_2'+\cdots+c_nu_n' ,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$u^{(n)}(x)=c_1u_1^{(n)}+c_2u_2^{(n)}+\cdots+c_nu_n^{(n)}+r(x)/p_n(x) .$$

We now observe that if each row be multiplied in turn by p_0, p_1, $\cdots$ p_n and the equations added, we shall obtain equation (2.1).

*This name is due to L. Fuchs, *Journal für Mathematik*, vol. 66 (1866), p. 126.

†After H. Wronski (1778-1853).

That is to say, the function (2.3), subject to conditions (2.4), is a solution of the non-homogeneous equation.

If the value obtained from (2.4),

$$c_i(x) = \int^x \{W_{1i}(t)r(t)/W(t)p_n(t)\}dt \ ,$$

is substituted in (2.3) and if account is taken of the fact that the general solution of the homogeneous equation added to a particular solution of (2.1) is the general solution of (2.1), we shall have as the general solution of the non-homogeneous equation the following:

$$\begin{aligned} u(x) &= A_1u_1(x) + A_2u_2(x) + \cdots + A_nu_n(x) \\ &\quad + \int^x \{r(t)\,[W_{11}(t)u_1(x) + W_{12}(t)u_2(x) \qquad (2.5) \\ &\quad + \cdots + W_{1n}(t)u_n(x)]/W(t)p_n(t)\}dt \ , \end{aligned}$$

where the A_i are constants as yet undetermined.

Suppose now that we take $(n-1)$ derivatives of $u(x)$. If we recall the formula for differentiation under the integral sign and notice that,

$$W_{11}(x)u_1^{(k)}(x) + W_{12}(x)u_2^{(k)}(x) + \cdots + W_{1n}(x)u_n^{(k)}(x) = 0 \ , \quad k = 0, 1, 2, \cdots, n-2 \ , \qquad (2.6)$$

we shall have,

$$\begin{aligned} u^{(j)}(x) &= A_1u_1^{(j)}(x) + A_2u^{(j)}(x) + \cdots + A_nu_n^{(j)}(x) \\ &\quad + \int^x \{r(t)\,[W_{11}(t)u_1^{(j)}(x) + W_{12}(t)u_2^{(j)}(x) + \cdots \\ &\quad + W_{1n}(t)u_n^{(j)}(x)]/W(t)p_n(t)\}dt \ , \\ &\qquad j = 0, 1, 2, \cdots, n-1 \ . \end{aligned}$$

For the determination of the constants A_i we set $x = a$ and make use of conditions (2.2). We thus obtain the following systems of equations:

$$g_j - \int_{(j)}^a = A_1u_1^{(j)}(a) + A_2u_2^{(j)}(a) + \cdots + A_nu_n^{(j)}(a) \ , \quad j = 0, 1, 2, \cdots, n-1 \ , \qquad (2.7)$$

where $\int_{(j)}^a$ has been used for brevity to denote the integral:

$$\int_{(j)}^a = \int^a \{r(t)\,[W_{11}(t)u_1^{(j)}(a) + W_{12}(t)u_2^{(j)}(a) + \cdots + W_{1n}(t)u_n^{(j)}(a)]/W(t)p_n(t)\}\,dt \ .$$

When system (2.7) is solved for A_i we get,

$$A_i = \{g_{n-1}W_{i1}(a) + g_{n-2}W_{i2}(a) + \cdots + g_0W_{in}(a) - W_{11}(a)\int_{(0)}^{a} - W_{12}(a)\int_{(1)}^{a} - \cdots - W_{1n}(a)\int_{(n-1)}^{a}\}/W(a) ,$$

in which we use the abbreviation $W_{ij}(x)$ for the cofactor of the element in the ith row and the jth column of the Wronskian.

When these values of A are substituted in equation (2.5) we get for the solution of (2.1), subject to the conditions (2.2),

$$\begin{aligned} u(x) = (-1)^{n-1}\{ &\begin{vmatrix} g_0 & g_1 & \cdots & g_{n-1} \\ u_2(a) & u_2'(a) & \cdots & u_2^{(n-1)}(a) \\ \cdot & \cdot & \cdot & \cdot \\ u_n(a) & u_n'(a) & \cdots & u_n^{(n-1)}(a) \end{vmatrix} u_1(x)/W(a) \\ &+ \begin{vmatrix} u_1(a) & u_1'(a) & \cdots & u_1^{(n-1)}(a) \\ g_0 & g_1 & & g_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ u_n(a) & u_n'(a) & \cdots & u_n^{(n-1)}(a) \end{vmatrix} u_2(x)/W(a) \\ &+ \cdots + \begin{vmatrix} u_1(a) & u_1'(a) & \cdots & u_1^{(n-1)}(a) \\ u_2(a) & u_2'(a) & \cdots & u_2^{(n-1)}(a) \\ \cdot & \cdot & \cdot & \cdot \\ g_0 & g_1 & \cdots & g_{n-1} \end{vmatrix} u_n(x)/W(a)\} \qquad (2.8) \\ &+ \int_a^x \{r(t)[W_{11}(t)u_1(x) + W_{12}(t)u_2(x) + \cdots \\ &\qquad + W_{1n}(t)u_n(x)]/W(t)p_n(t)\}\,dt . \end{aligned}$$

It is sometimes useful to express this equation as a linear function of the g_i. We thus get,

$$\begin{aligned} u(x) = (-1)^{n-1}\{ &\begin{vmatrix} u_1(x) & u_2(x) & \cdots & u_n(x) \\ u_1'(a) & u_2'(a) & \cdots & u_n'(a) \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-1)}(a) & u_2^{(n-1)}(a) & \cdots & u_n^{(n-1)}(a) \end{vmatrix} g_0/W(a) \\ &+ \begin{vmatrix} u_1(a) & u_2(a) & \cdots & u_n(a) \\ u_1(x) & u_2(x) & \cdots & u_n(x) \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-1)}(a) & u_2^{(n-1)}(a) & \cdots & u_n^{(n-1)}(a) \end{vmatrix} g_1/W(a) + \end{aligned}$$

$$\cdots + \begin{vmatrix} u_1(a) & u_2(a) & \cdots & u_n(a) \\ u_1'(a) & u_2'(a) & \cdots & u_n'(a) \\ \cdot & \cdot & \cdot & \cdot \\ u_1(x) & u_2(x) & \cdots & u_n(x) \end{vmatrix} g_n/W(a) \tag{2.9}$$

$$+ \int_a^x \{r(t)\,[W_{11}(t)u_1(x) + W_{12}(t)u_2(x) + \cdots + W_{1n}(t)u_n(x)]/W(t)p_n(t)\}\,dt \;.$$

The function,

$$W(x,t) = [W_{11}(t)u_1(x) + W_{12}(t)u_2(x) + \cdots + W_{1n}(t)u_n(x)]/W(t)p_n(t)$$

we shall designate as the *Cauchy function* of the differential equation. It is obviously independent of the fundamental set of solutions used in its construction and can be expressed uniquely in terms of the coefficients of the differential equation.

Its most significant property, easily established from (2.6), is the following:

$$\partial^m W(x,t)/\partial x^m|_{t=x} = 0\;, \quad m = 0, 1, 2, \cdots, n-2; \quad \partial^{n-1} W(x,t)/\partial x^{n-1}|_{t=x} = 1/p_n(x)$$

$$\partial^m W(x,t)/\partial t^m|_{t=x} = 0\;, \quad m = 0, 1, 2, \cdots, n-2; \quad \partial^{n-1} W(x,t)/\partial t^{n-1}|_{t=x} = (-1)^{n-1}/p_n(x)\;.$$

A special example, often useful in applications of the theory, is found in the case where the coefficients of the differential equation are constants. If the $p_i(x) = p_i$ are constants, then a set of fundamental solutions is given by $e^{Ax}, e^{Bx}, \cdots, e^{Nx}$ where $A, B, \cdots, N$ are roots of the equation:

$$F(r) \equiv p_n r^n + p_{n-1} r^{n-1} + \cdots + p_1 r + p_0 = 0\;.$$

Substituting this set in the Wronskian we get,

$$W(x) = \begin{vmatrix} A^{n-1} & B^{n-1} & \cdots & N^{n-1} \\ A^{n-2} & B^{n-2} & \cdots & N^{n-2} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdots & 1 \end{vmatrix} e^{(A+B+\cdots+N)x}\,,$$

$$= (A-B)(A-C)\cdots(A-N)(B-C)\cdots(B-N)\cdots(M-N)\,e^{(A+B+\cdots+N)x}$$

Similarly the cofactor of the first element of $W(x)$ becomes,

$$W_{11}(x) = \begin{vmatrix} B^{n-2} & C^{n-2} & \dots & N^{n-2} \\ B^{n-3} & C^{n-3} & \dots & N^{n-3} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdots & 1 \end{vmatrix} e^{(B+C+\dots+N)x}$$

$$= (B-C)(B-C)\cdots(B-N)(C-D)\cdots (C-N)\cdots(M-N)e^{(B+C+\dots+N)x} .$$

Hence we find that

$$W_{11}(x)/W(x) = e^{-Ax}/[(A-B)(A-C)\cdots(A-N)] = p_n e^{-Ax}/F'(A) ,$$

$$W_{12}(x)/W(x) = p_n e^{-Bx}/F'(B) ,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The Cauchy function for this case thus becomes:

$$W(x,t) = e^{A(x-t)}/F'(A) + e^{B(x-t)}/F'(B) + \cdots + e^{N(x-t)}/F'(N) . \qquad (2.10)$$

We shall find it useful in the sequal to examine formulas (2.8) and (2.9) by means of their connection with the *identity of Lagrange.*

Suppose that $L(u)$ is a linear combination of $u(x)$ and its first n derivatives as given in (2.1).

Consider the following identities:

$$\int v(p_0 u)\,dx \equiv \int u(p_0 v)\,dx ,$$

$$\int v(p_1 u')\,dx \equiv (p_1 v)u - \int u(p_1 v)'dx ,$$

$$\int v(p_2 u'')\,dx \equiv (p_2 v)u' - (p_2 v)'u + \int u(p_2 v)''dx ,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\int v(p_n u^{(n)})\,dx \equiv (p_n v)u^{(n-1)} - (p_n v)'u^{(n-2)} + \cdots + (-1)^n \int u(p_n v)^{(n)}dx .$$

If the integrals on the right are transposed and the identites added we shall have,

$$\int \{vL(u) - uM(v)\}\,dx = P(u,v) , \qquad (2.11)$$

where we write

$$M(v) = (-1)^n d^n(p_n v)/dx^n + (-1)^{n-1} d^{n-1}(p_{n-1}v)/dx^{n-1} + \cdots + p_0 v , \qquad (2.12)$$

$$\begin{aligned} P(u,v) &= u\{p_1 v - d(p_2 v)/dx + \cdots + (-1)^{n-1} d^{n-1}(p_n v)/dx^{n-1}\} \\ &+ u'\{p_2 v - d(p_3 v)/dx + \cdots + (-1)^{n-2} d^{n-2}(p_n v)/dx^{n-2}\} \\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &+ u^{(n-1)} p_n v \ . \end{aligned} \tag{2.13}$$

We see that $P(u,v)$ is a bilinear form in the variables $u, u' \cdots u^{(n-1)}$ and $v, v', \cdots, v^{(n-1)}$, whose determinant is

$$\Delta(P) = \begin{vmatrix} \cdots\cdots\ \cdots\cdots & & (-1)^{n-1}p_n \\ \cdots\cdots\ (-1)^{n-2}p_n & & 0 \\ \cdots\cdots \quad 0 & & 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ p_n\ 0\ \cdots\cdots\ 0 & & 0 \end{vmatrix} = (-1)^{n-1}(p_n)^n \ . \tag{2.14}$$

Identity (2.11) was first studied by J. Lagrange* and is referred to as the *identity of Lagrange.* The linear expression (2.12) is called the *Lagrange adjoint* of $L(u)$.

It is clear how these ideas may be extended to differential expressions of infinite order, an extension which has been achieved by V. V. Latshaw to whom the author is indebted for these remarks.

Since the expression

$$M_n(v) = \sum_{m=0}^{n} (-z)^m \to [p_m(x)\ v(x)]$$

is the adjoint of the expression

$$L_n(u) = \sum_{m=0}^{n} p_m(x)\,[z^m \to u(x)] \ ,$$

one may assume naturally that

$$M(v) = \lim_{n=\infty} M_n(v)$$

is the generalized Lagrange adjoint of

$$L(u) = \lim_{n=\infty} L_n(u) \ .$$

From this, more generally, one may define as the adjoint of the expression

$$L(u) = \sum_{m=0}^{\infty} p_m(x)\,[\phi_m(z) \to u(x)] \ ,$$

the corresponding functional expansion

$$M(u) = \sum_{m=0}^{\infty} \phi_m(-z) \to [p_m(x)\ v(x)] \ ,$$

where $\phi_m(z)$ is a differential operator of infinite order with constant coefficients.

*Solution de differents problèmes de calcul intégral. *Miscellanea Faurinensia*, t. III (1762-1765); *Oeuvres complètes*, t. I, p. 471.

If we make proper convergence assumptions to assure the existence of the functions

$$P_k(x) = \sum_{m=0}^{\infty} p_m(x)\ \phi_m^{(k)}(0)/k!\ ,$$

then the Lagrange identity may be written formally

$$\int [v\, L(u) - u\, M(v)]\, dx = \sum_{m=0}^{\infty} u^{(m)}(x) \sum_{k=0}^{\infty} (-1)^k\ (P_{k+m+1}\, v)^{(k)}\ .$$

In this manner all the formal machinery of the finite case is carried over to the infinite case, provided proper attention is paid to the convergence of the operators and series which are involved.

It is also convenient to generalize the Lagrange identity in another way, that is to say, with respect to the operator which multiplies the difference $v\ L(u)$ — $u\ M(v)$. Thus, in the theory of difference equations the symbol of summation, that is to say, the operator $1/(e^z - 1)$, replaces the symbol of integration. Hence we require a generalization of the Lagrange identity such that the bilinear form $P(u,v)$, when operated upon by $e^z - 1$, will reduce to $v\ L(u) - u\ M(v)$.

This generalization is easily achieved as follows:

Let us consider the difference sum

$$L(u) \equiv p_0(x)\ u(x) + p_1(x)\ u(x+1) + \cdots + p_n(x)\ u(x+n)\ ,$$

which is equivalent to

$$L(u) \equiv [p_0(x) + p_1(x)\ e^z + \cdots + p_n(x)\ e^{nz}] \rightarrow u(x)\ .$$

The desired Lagrange adjoint is found to be

$$\begin{aligned} M(v) &\equiv p_0\, v + e^{-z} \rightarrow (p_1\, v) + \cdots + e^{-nz} \rightarrow (p_n\, v) \\ &= p_0(x)\ v(x) + p_1(x-1)\ v(x-1) + \cdots + p_n(x-n)\ v(x-n)\ , \end{aligned}$$

and the corresponding identity of Lagrange may be written

$$\sum [v\, L(u) - u\, M(v)] = P(u,v)\ ,$$

where we abbreviate

$$\begin{aligned} P(u,v) = {} & u[e^{-z} \rightarrow (p_1\, v) + e^{-2z} \rightarrow (p_2\, v) + \cdots + e^{-nz} \rightarrow (p_n\, v)] \\ & + \ \cdots \\ & + [e^{(n-3)z} \rightarrow u][e^{-z} \rightarrow (p_{n-2}\, v) + e^{-2z} \rightarrow (p_{n-1}\, v) + e^{-3z} \rightarrow (p_n\, v)] \\ & + [e^{(n-2)z} \rightarrow u][e^{-z} \rightarrow (p_{n-1}\, v) + e^{-2z} \rightarrow (p_n\, v)] \\ & + [e^{(n-1)z} \rightarrow u][e^{-z} \rightarrow (p_n\, v)]\ . \end{aligned}$$

Noting the identity

$$e^{-nz} \rightarrow (U\, V) \equiv U(x-n)\ V(x-n) \equiv [e^{-nz} \rightarrow U][e^{-nz} \rightarrow V]\ ,$$

the reader may easily establish the identity of Lagrange by evaluating the expression

$$(e^z - 1) \rightarrow P(u,v)\ .$$

Other generalizations and a more extended account of the developments associated with them will be found in the following sources:

S. Pincherle and U. Amaldi: *Bibliography*: Pincherle (4), pp. 237-246.
E. Bortolotti: La forma aggiunta di una data forma lineare alle differenze. *Rendiconti dei Lincei*, vol. 5 (5th series) (1896), pp. 349-356; Le forme lineari alla differenze equivalenti alle loro aggiunte. *Ibid.*, vol. 7 (5th series) (1898), pp. 257-265.
N. E. Nörlund: *Bibliography* (3), pp. 14-21.
G. Wallenberg and A. Guldberg: *Bibliography*, pp. 78-86.

We next establish a fundamental relation which exists between the solutions of $L(u) = 0$ and $M(u) = 0$.

Suppose that $\{u_i\}$ is a fundamental set of solutions of $L(u) = 0$. It is then apparent that we can write $L(u)$ in terms of these solutions as follows:

$$L(u) = \varrho(x) \begin{vmatrix} u^{(n)}(x) & u^{(n-1)}(x) & \cdots\cdot u(x) \\ u_1{}^{(n)}(x) & u_1{}^{(n-1)}(x) & \cdots\cdot u_1(x) \\ \cdot & \cdot & \cdot \\ u_n{}^{(n)}(x) & u_n{}^{(n-1)}(x) & \cdots\cdot u_n(x) \end{vmatrix},$$

where $\varrho(x)$ is a non-vanishing multiplier, since any member of the fundamental set $\{u_i\}$ when substituted for $u(x)$ in the determinant reduces it to zero.

Hence comparing the coefficients of $L(u)$ with the coefficients in the expansion of the determinant we get $W(x) = p_n(x)/\varrho(x)$ and $W'(x) = -p_{n-1}(x)/\varrho(x)$. From these it follows that $W'(x)/W(x) = -p_{n-1}(x)/p_n(x)$; this equation leads us to *Abel's identity*:

$$W(x) = k\, e^{-\int_c^x [p_{n-1}(x)/p_n(x)]dx},$$

where k is an arbitrary constant.

Let us now consider the expression,

$$\vartheta_i(u) = \{ \frac{\partial W}{\partial u_i} u + \frac{\partial W}{\partial u_i'} u' + \cdots + \frac{\partial W}{\partial u_i{}^{(n-1)}} u^{(n-1)} \}/W .$$

It is at once evident that

$$\vartheta_i(u_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol which is 1 for $i = j$ and 0 for $i \neq j$.

Consequently if we substitute in $\vartheta_i(u)$ the function $u = c_1u_1 + c_2u_2 + \cdots + c_nu_n$, where the c_i are constants, we have,

$$\vartheta_i(u) = c_i,$$

from which it follows that,

$$d\vartheta_i(u)/dx = 0 .$$

Since the last equation has for its solution the general solution of $L(u) = 0$, we must have,

$$d\vartheta_i(u)/dx = v_i(x)L(u) \ .$$

Comparing coefficients it is clear that we get,

$$v_i(x) = \{\partial \log W/\partial u_i^{n-1}\}/p_n(x) \ . \tag{2.15}$$

Moreover since $v_i(x)L(u)$ is an exact derivative, $v_i(x)$ must be a particular solution of the adjoint equation. By giving i all values from 1 to n we obtain the solutions, $v_1, v_2, \cdots, v_n$ of the adjoint equation $M(v) = 0$. These solutions are easily seen to be linearly independent and are called the adjoints of the solutions $u_i(x)$.

It follows further from (2.11) that $\vartheta_i(u) = P(u,v)$ and we have as a consequence that,

$$P(u_j,v_i) = \delta_{ij} \ .$$

Making use of this fact we obtain readily from (2.15) by the elementary properties of determinants the following relations:*

$$\begin{aligned} &v_1u_1^{(j)} + v_2u_2^{(j)} + \cdots + v_nu_n^{(j)} = 0 \ , \qquad j < n-1 \ , \\ &v_1u_1^{(n-1)} + v_2u_2^{(n-1)} + \cdots + v_nu_n^{(n-1)} = 1/p_n(x) \ . \end{aligned} \tag{2.16}$$

We consider now the problem of finding a solution of the differential equation,

$$L(u) + \lambda u = r(x) \ ,$$

which at the point $x = a$ satisfies the conditions,

$$u(a) = g_0 \, , u'(a) = g_1 \, , u''(a) = g_2 \, , \cdots , u^{(n-1)}(a) = g_{n-1} \ .$$

Making use of the preceding results it is clear that $u(x)$ will satisfy the following integral equation:

$$u(x) = f(x) - \lambda \int_a^x G(x,t)u(t)\,dt \ , \tag{2.17}$$

where

$$f(x) = U(x) + \int_a^x G(x,t)r(t)\,dt \ , \tag{2.18}$$

in which

$$G(x,t) = \sum_{i=1}^{n} U_i(x)V_i(t)/[W(t)p_n(t)] \ , \tag{2.19}$$

where the $\{U_i(x)\}$ form any fundamental set of solutions of $L(u) = 0$, the $\{V_i(x)\}$ are the corresponding adjoints, and $U(x)$ is the solution of $L(u) = 0$, which satisfies the conditions,

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*For an extensive treatment of this subject see G. Darboux: *La théorie des surfaces*, Paris (1889), part 2, book 4, chapter 5.

$$U(a) = g_0 , U'(a) = g_1 , U''(a) = g_2 , \cdots , U^{(n-1)}(a) = g_{n-1} .$$

In order to prove this directly let us differentiate (2.17) n times. Recalling the fundamental property of $W(x,t)$, we shall then have the following array of equations:

$$u(x) = U(x) + \int_a^x \{r(t) - \lambda u(t)\} G(x,t) dt ,$$

$$u'(x) = U'(x) + \int_a^x \{r(t) - \lambda u(t)\} \frac{\partial}{\partial x} G(x,t) dt ,$$

$$u''(x) = U''(x) + \int_a^x \{r(t) - \lambda u(t)\} \frac{\partial^2}{\partial x^2} G(x,t) dt .$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$u^{(n-1)}(x) = U^{(n-1)}(x) + \int_a^x \{r(t) - \lambda u(t)\} \frac{\partial^{n-1}}{\partial x^{n-1}} G(x,t) dt ,$$

$$u^{(n)}(x) = U^{(n)}(x) + \int_a^x \{r(t) - \lambda u(t)\} \frac{\partial^n}{\partial x^n} G(x,t) dt + \{r(x) - \lambda u(x)\}/p_n(x) .$$

If these equations are multiplied successively by $p_0(x)$, $p_1(x)$, $\cdots$, $p_n(x)$ and the members of the array added together we shall get,

$$L(u) = L(U) + \int_a^x \{r(t) - \lambda u(t)\} L\{G(x,t)\} dt + r(x) - \lambda u(x) .$$

But since $L(U) = 0$ and $L\{G(x,t)\} = 0$, we obtain,

$$L(u) + \lambda u = r(x) .$$

That the boundary conditions are also satisfied by $u(x)$ is seen if we set $x = a$ and notice that $U(x)$ was chosen so as to satisfy these conditions also.

Example: Find the integral equation equivalent to the following differential system:

$$u^{(n+1)}(x) + \lambda u(x) = g(x) ,$$

$$u(0) = u_0 , u'(0) = u_1 , u''(0) = u_2 , \cdots , u^{(n)}(0) = u_n .$$

Obviously a fundamental set of functions for the equation

$$u^{(n+1)}(x) = 0$$

would be

$$\{u_i\} \equiv 1, \ x, \ x^2/2!, \ x^3/3!, \ \cdots , \ x^n/n! .$$

The corresponding adjoint system is,

$$\{v_i\} \equiv (-1)^n x^n/n!\,,\ (-1)^{n-1}x^{n-1}/(n-1)!\,,\cdots,\ -x,\ 1\ ,$$

as may be proved by substituting these values in equations (2.16).

The kernel of the integral equation will then be,

$$G(x,t) = x^n/n! - x^{n-1}t/(n-1)! + x^{n-2}t^2/2!(n-2)! - \cdots - (-1)^n t^n/n!\ ,$$

$$= (x-t)^n/n!\ ,$$

and the function $U(x)$ becomes,

$$U(x) = u_0 + u_1 x + u_2 x^2/2! + u_3 x^3/3! + \cdots + u_n x^n/n!\ .$$

Hence the integral equation equivalent to the differential system under consideration is,

$$u(x) = U(x) + \int_0^x \{(x-t)^n g(t)/n!\}\,dt - \lambda \int_0^x \{(x-t)^n u(t)/n!\}\,dt\ .$$

We shall now consider a problem which is the converse of the one just discussed. *We shall show that if $\varphi(x)$ is a function which, for $x = a$, satisfies the conditions:*

$$\varphi(a) = g_0,\quad \varphi'(a) = g_1,\quad \varphi''(a) = g_2,\cdots,\quad \varphi^{(n-1)}(a) = g_{n-1}\ ,$$

the solution of the integral equation,

$$u(x) = \varphi(x) - \lambda \int_a^x G(x,t)\,u(t)\,dt\ , \qquad (2.20)$$

will be a solution of the differential system,

$$L(u) + \lambda u = L(\varphi)\ , \qquad (2.21)$$

$$u(a) = g_0,\quad u'(a) = g_1,\quad u''(a) = g_2,\cdots,\quad u^{(n-1)}(a) = g_{n-1}\ .$$

From equations (2.17) and (2.18) we have the relation,

$$\varphi(x) = U(x) + \int_a^x G(x,t)\,r(t)\,dt\ .$$

Differentiating this n times, multiplying by the coefficients of $L(u)$ and adding, we are led immediately to the equation,

$$L\{\varphi(x)\} = r(x)\ .$$

Then by setting $x = a$ in each successive equation obtained by differentiating $\varphi(z)$, we get,

$$\varphi(a) = U(a),\quad \varphi'(a) = U'(a),\quad \cdots,\quad \varphi^{(n-1)}(a) = U^{(n-1)}(a)\ ,$$

which uniquely determines $U(x)$.

Because of the equivalence of (2.20) and (2.21) a solution of the integral equation is easily obtained. Making use of the results attained at the beginning of this section, we see that the solution of the differential system (2.21) and hence the solution of the integral equation (2.20) may be written in the form,

$$u(x) = V(x) + \int_a^x \Gamma(x, t)\, L\{\varphi(t)\}\, dt\ , \qquad (2.22)$$

where $V(x)$ is a solution of $L(u) + \lambda u = 0$ and the boundary conditions and where $\Gamma(x, t)$ is the corresponding kernel associated with $L(u) + \lambda u$.

An interesting corollary follows. Suppose that φ is a solution of $L(u) = 0$. We then have $u(x) = V(x)$.

Another result is obtained if we assume that,

$$L(\varphi) = -\lambda\varphi\ .$$

From (2.22) we see that

$$u(x) = \varphi(x) - \lambda \int_a^x \Gamma(x, t)\, \varphi(t)\, dt\ ,$$

will be a solution of the equation,

$$u(x) = \varphi(x) - \lambda \int_a^x G(x, t)\, u(t)\, dt\ .$$

PROBLEMS

1. Prove that (2.10) is identical with the function,

$$W(x,t) = A_0(x-t)^{n-1}/(n-1)! + A_1(x-t)^n/n! + A_2(x-t)^{n+1}/(n+1)! + \dots ,$$

where the constants $A_0, A_1, A_2, \cdots$ are the coefficients of r in the power series expansion of $1/(p_n + p_{n-1}r + \dots + p_0 r^n)$.

2. Solve the equation

$$[(z-1)(z-2)(z+3)(z+1)] \to u(x) = e^{mx}\ .$$

3. Employ the methods of this section to represent the solution of the equation

$$(1-x^2)\, u''(x) - 2x\, u'(x) + n(n+1)\, u(x) = f(x)\ ,$$

in integrable form. Note that the left hand member set equal to zero is Legendre's equation for which a fundamental set of solutions is given by the Legendrian functions $L_n(x)$, $Q_n(x)$.

4. Solve the equation

$$\cos z \to u(x) = f(x)$$

by the methods of this section.

5. Solve the equation

$$\cos \sqrt{z} \to u(x) = f(x) \ .$$

6. Prove that if $U(x)$ is a function which vanishes at a set of discrete points in the interval (a, b) and if the Wronskian formed by this function with a second function $V(x)$ does not vanish in (a, b), then $V(x)$ also vanishes in the interval and its zeros are separated by the zeros of $U(x)$. Hint: Consider the derivative of the function $U(x)/V(x)$.

3. Operational Solution of the Volterra Equation. In the last section we have traced the development of the Volterra integral equation from the Cauchy problem associated with ordinary differential equations. Let us now see how this equation can be solved by means of operators.

We shall assume that the kernel, $K(x, t)$, of the equation

$$a_0(x)\, u(x) + \int_0^x K(x, t)\, u(t)\, dt = f(x) \ , \tag{3.1}$$

can be expanded in a Taylor's series as follows:

$$K(x, t) = a_1(x) + a_2(x)\,(x - t) + a_3(x)\,(x - t)^2/2! + a_4(x)\,(x - t)^3/3! + \cdots \ .$$

Employing identity (6.4) of chapter 2,

$$z^{-n} \to u(x) = \int_0^x [(x - t)^{n-1}\, u(t)/(n - 1)\,!]\, dt \ ,$$

we see that we can write (3.1) in the form

$$\{a_0(x) + a_1(x) z^{-1} + a_2(x) z^{-2} + a_3(x) z^{-3} + \cdots\} \to u(x) = f(x) \ . \tag{3.2}$$

Operating upon both sides of this equation successively by z^{-1}, z^{-2}, etc., and employing the rule of Leibnitz [formula (2.3) of chapter 4] we obtain the following infinite set of linear equations:

$$\{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots\} \to u(x) = f(x) \ ,$$

$$\{a_0 z^{-1} + (a_1 - a_0') z^{-2} + (a_2 - a_1' + a_0'') z^{-3} + \cdots\} \to u(x) = z^{-1} \to f(x) \ ,$$

$$\{a_0 z^{-2} + (a_1 - 2a_0') z^{-3} + (a_2 - 2a_1' + 3a_0'') z^{-4} + \cdots\} \to u(x) = z^{-2} \to f(x) \ ,$$

$$\{a_0 z^{-3} + (a_1 - 3a_0') z^{-4} + (a_2 - 3a_1' + 6a_0'') z^{-5} + \cdots\} \to u(x) = z^{-3} \to f(x) \ ,$$

. . . .

$$\{a_0z^{-n} + (a_1 - na_0')\, z^{-n-1} + [a_2 - na_1' + n(n+1)a_0''/2!]\, z^{-n-2}$$

$$+ [a_3 - na_2' + n(n+1)a_1''/2! - n(n+1)(n+2)a_0'''/3!]\, z^{-n-3}$$

$$+ \cdots\} \to u(x) = z^{-n} \to f(x) \;,$$

. . . .

Solving this system of equations for $u(x)$ we obtain the following expansion:*

$$u(x) = \{b_0 + b_1z^{-1} + b_2z^{-2} + b_3z^{-3} + \cdots + b_mz^{-m} + \cdots\} \to f(x) \tag{3.3}$$

where we use the abbreviations:

$$b_0 = 1/a_0, \quad b_1 = -a_1/a_0^2, \quad b_2 = \begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 - a_0' \end{vmatrix} /a_0^3 \;,$$

$$b_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 - a_0' & a_2 - a_1' + a_0'' \\ 0 & a_0 & a_1 - 2a_0' \end{vmatrix} /a_0^4 \;, \cdots ,$$

$$= \begin{vmatrix} a_1\,, & a_2\,, & a_3 & ,\cdots, & a_m \\ a_1\,, & a_1 - a_0', & a_2 - a_1' + a_0'' & ,\cdots, & a_{m-1} - a_{m-2} + \cdots \pm a_0 \\ 0 & a_0 & a_1 - 2a_0' & ,\cdots, & a_{m-2} - 2a'_{m-3} + \cdots \mp (m-1)a_0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & ,\cdots, & a_1 - (m-1)a_0' \end{vmatrix} /a_0^{m+1}$$

We shall next verify that the function,

$$X(x,z) = b_0 + b_1z^{-1} + b_2z^{-2} + b_3z^{-3} + \cdots \;, \tag{3.4}$$

is in fact a formal solution of the generatrix equation.

Forming the product $[X \cdot F]$ we get

$$[X \cdot F] = \{b_0 + b_1z^{-1} + b_2z^{-2} + \cdots\}\{a_0 + a_1z^{-1} + a_2z^{-2} + \cdots\}$$

$$- z^{-2}\{b_1 + 2b_2z^{-1} + 3b_3z^{-2} + \cdots\}\{a_0' + a_1'z^{-1} + a_2'z^{-2} + \cdots\}$$

$$+ z^{-3}\{2b_1 + 2\cdot 3b_2z^{-1} + 3\cdot 4b_3z^{-2} + \cdots\}$$

$$\times \{a_0'' + a_1''z^{-1} + a_2''z^{-2} + \cdots\}/2! - \cdots$$

$$\equiv a_0b_0 + [a_1b_0 + a_0b_1]z^{-1} + [a_2b_0 + (a_1 - a_0')b_1 + a_0b_2]z^{-2}$$

$$+ [a_3b_0 + (a_2 - a_1' + a_0'')b_1 + (a_1 - 2a_0')b_2 + a_0b_3]z^{-3} + \cdots \;.$$

Equating this expansion to unity we obtain the following set of equations:

*It will be noted that this expansion is derived immediately from formula (2.7) of chapter 3.

$$a_0 b_0 = 1 \;,$$
$$a_1 b_0 + a_0 b_1 = 0 \;,$$
$$a_2 b_0 + (a_1 - a_0') b_1 + a_0 b_2 = 0 \;,$$
$$a_3 b_0 + (a_2 - a_1' + a_0'') b_1 + (a_1 - 2a_0') b_2 + a_0 b_3 = 0 \;,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

From this system we can determine the values of b_0, b_1, b_2, $\cdots$ and the expansion thus obtained is found to coincide with (3.3).

Moreover the solution is unique provided all the $a_i(x)$ exist at the point $x = 0$ and $a_0(0) \neq 0$. To prove this let us assume that X is of the form

$$X(x,z) = A(x) e^{-xz} + X_0(x,z) \;,$$

where $X_0(x,z)$ is the function already obtained. Substituting this in the generatrix equation and recalling that

$$a_i(0) = a_i(x) - x a_i'(x) + x^2 a_i''(x)/2! - \cdots \;,$$

we get,

$$[X \cdot F] = 1 + A(x)^{-xz}\{a_0(0) + a_1(0) z^{-1} + a_2(0) z^{-2} + \cdots\} \;, \tag{3.5}$$

where $A(x)$ is an arbitrary function.

Since we have

$$e^{-xz} \to \{\varphi(z) \to u(x)\} = \{e^{-xz}\varphi(z)\} \to u(x) \;,$$

which follows at once from the generatrix product (3.1) of chapter 4, it is clear that (3.5) will furnish a resolvent generatrix for equation (3.1) provided

$$e^{-xz} \to \{[a_0(0) + a_1(0) z^{-1} + a_2(0) z^{-2} + \cdots] \to f(x)\}$$

is identically zero for all functions $f(x)$ which are suitably defined. It is obvious from the operational identity

$$e^{-xz} \to g(x) = g(0) \;,$$

that the expression in the brackets must vanish at $x = 0$ if (3.5) is to equal 1.

But from the equation

$$\{a_0(0) + a_1(0) z^{-1} + a_2(0) z^{-2} + \cdots\} \to f(x)$$
$$= a_0(0) f(x) + \int_0^x K(x-t) f(t)\, dt \;,$$

it follows that we shall have

$$e^{-xz} \to \{a_0(0) f(x) + \int_0^x K(x-t) f(t)\, dt = a_0(0) f(0) \;,$$

which is zero for all functions $f(x)$ only when $a_0(0) = 0$. Since this

case is explicitly excluded by the conditions imposed, the uniqueness of the operator is thus established.

We thus prove the theorem:

Theorem 1. If we define $F(x,z) = a_0(x) + a_1(x)z^{-1} + a_2(x)z^{-2} + \cdots$, *where the* $a_i(x)$ *are functions analytic in a common region about the origin and where* $a_0(0) \neq 0$, *then the resolvent generatrix* $X(x,z)$ *satisfying the equation* $[X \cdot F] = 1$, *is also an integral operator of infinite order and has the expansion given by equation* (3.2). *This operator is also unique.*

The application of this theorem to the case of the equation belonging to the group of the closed cycle (see section 9, chapter 1) is particularly important in application. In this case the functions $a_i(x)$ are constants and we thus consider the equation

$$u(x) + \int_0^x K(x-t)u(t)\,dt = f(x) \ ,$$

where we write,

$$K(x-t) = K(0) + (x-t)K'(0) + (x-t)^2K''(0)/2! + \cdots \ ;$$

$$a_0 = 1 \ , \quad a_i = K^{(i-1)}(0) \ , \quad i \neq 0 \ .$$

It is immediately seen that the resolvent generatrix will be,

$$X(x,z) = 1/F(x,z) = \{1 + K(0)/z + K'(0)/z^2 + K''(0)/z^2 + \cdots\}^{-1},$$

and we have the following theorem:

Theorem 2. The resolvent kernel for the integral equation of the closed cycle,

$$u(x) + \int_0^x K(x-t)u(t)\,dt = f(x) \ ,$$

is formally equivalent to the expansion,

$$k(x-t) = k(0) + k'(0)(x-t) + k''(0)(x-t)^2/2! + \cdots \ ,$$

where the $k^{(i)}(0)$ *are determined from the equation,*

$$\{1 + K(0)r + K'(0)r^2 + K''(0)r^3 + \cdots\}^{-1}$$
$$= 1 + k(0)r + k'(0)r^2 + k''(0)r^3 \cdots \ .$$

The solution thus becomes,

$$u(x) = f(x) + \int_0^x k(x-t)f(t)\,dt \ .$$

We illustrate this theorem by means of three simple examples.

Example 1. Solve the equation

$$u(x)+\int_0^x e^{(x-t)}u(t)\,dt=f(x)\ .$$

From the resolvent generatrix

$$X=1/(1+1/z^2+\cdots)=1-1/z\ ,$$

the solution is immediately obtained:

$$u(x)=f(x)-\int_0^x f(t)\,dt\ .$$

Example 2. Consider the equation

$$u(x)+\lambda\int_0^x (x-t)u(t)\,dt=f(x)\ .$$

Since the resolvent generatrix is

$$X=1/(1+\lambda/z^2)=1-\lambda/z^2+\lambda^2/z^4-\lambda^3/z^6+\cdots\ ,$$

the solution will be,

$$\begin{aligned}u(x)&=f(x)-\lambda\int_0^x\{(x-t)-\lambda(x-t)^3/3!+\lambda^2(x-t)^5/5!\\&\qquad+\cdots\}f(t)\mathrm{d}t\ ,\\&=f(x)+\sqrt{\lambda}\int_0^x\sin\sqrt{\lambda}(t-x)f(t)\,dt\ .\end{aligned}$$

Example 3. Solve the equation

$$u(x)+\int_0^x \psi(x-t)u(t)\,dt=f(x)\ ,$$

where the kernel is of the form,

$$\begin{aligned}\psi(z)&=1+z/1!2!+z^2/2!3!+z^3/3!4!+\cdots\ ,\\&=-iJ_1(2i\sqrt{z})/\sqrt{z}\ ,\end{aligned}$$

in which $J_1(x)$ is the Bessel function of first order.*

Computing the resolvent generatrix we get,

$$X=\{1+\sum_{n=0}^{\infty}\psi^{(n)}(0)z^{n+1}\}^{-1}=e^{-z}=1-z+z^2/2!-\cdots\ .$$

Hence the resolvent kernel will be,

$$k(z)=-1+z/1!2!-z^2/2!3!+\cdots=-\psi(-z)\ ,$$

*This problem is due to T. Kameda: A Method for Solving Some Integral Equations. *Tohoku Math. Journal,* vol. 23 (1924), pp. 197-209, in particular, p. 202.

and the solution of the integral equation is given by

$$u(x) + \int_0^x \psi(t-x) f(t)\,dt = f(x) \ .$$

PROBLEMS

1. Show that the solution of the integral equation

$$f(x) = \int_0^x K(x-t) u(t)\,dt \,, \ f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$$

where the kernel is of the form

$$K(u) = u^{-a}(a_0 + a_1 u + a_2 u^2 + \cdots + a_n u^n) \ , \ 0 < a < 1 \ ,$$

may be written

$$u(t) = (\sin a\pi/\pi) \int_0^t f'(s) L(t-s)\,ds \ ,$$

in which we abbreviate

$$L(w) \equiv \frac{w^{a-1}}{a_0} + \frac{A^{n-a}}{F'(A)} \gamma_a(Aw) + \frac{B^{n-a}}{F'(B)} \gamma_a(Bw) + \cdots + \frac{N^{n-a}}{F'(N)} \gamma_a(Nw) \ .$$

In this function $A, B, \cdots, N$ are the roots (assumed distinct) of the equation

$$F(z) \equiv a_0 z^n + (1-a) a_1 z^{n-1} + (1-a)(2-a) a_2 z^{n-2} + \cdots + (1-a)(2-a) \cdots (n-a) a_n = 0$$

and $\gamma_a(z)$ denotes the incomplete gamma function

$$\gamma_a(z) = \int_0^z s^{a-1} e^{(z-s)}\,ds = \begin{cases} z^a/a + z^{a+1}/[a(a+1)] + z^{a+2}/[a(a+1)(a+2)] + \cdots \ , \\ \Gamma(a) e^z - z^{a-1}[1 + (a-1)/z + (a-1)(a-2)/z^2 + \cdots . \end{cases}$$

[E. T. Whittaker: On the Numerical Solution of Integral Equations. *Proc. Royal Soc. of London,* vol. 94 (A) (1918), pp. 367-383].

2. Given the integral equation

$$u(x) = f(x) + \lambda \int_0^x K(x-t) u(t)\,dt$$

and its solution

$$u(x) = f(x) + \lambda \int_0^x k(x-t) f(t)\,dt \ ,$$

establish the following relations between the kernels at $t = 0$:

$$\begin{aligned}
k(0) &= K(0) \ , \\
k'(0) &= K'(0) + \lambda K(0)\, k(0) \ , \\
k''(0) &= K''(0) + \lambda[K(0)\, k'(0) + K'(0)\, k(0)] \ , \\
k^{(3)}(0) &= K^{(3)}(0) + \lambda[K(0)\, k''(0) + K'(0)\, k'(0) + K''(0)\, k(0)] \ , \\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

3. Assuming the notation of problem 2, and employing the abbreviation

$$D_i = a_i + a_{i+1}\, q + a_{i+2}\, q^2 + \cdots + a_n\, q^{n-i} \ , \quad q = d/dz \ ,$$

show that, if $K(z)$ satisfies the equation

$$D_0 \rightarrow K(z) = 0 \ ,$$

then the resolvent kernel satisfies the equation

$$D_0 \rightarrow k(z) - \lambda[K(0)\, D_1 \rightarrow k(z) + K'(0)\, D_2 \rightarrow k(z)$$
$$+ K''(0)\, D_3 \rightarrow k(z) + \cdots + K^{(n-1)}(0)\, D_n \rightarrow k(z) = 0 \ ,$$

where the arbitrary constants of the solution are determined from the equations given in problem 2. (G. C. Evans).

4. Show that in terms of the coefficients of the operator of problem 3, the differential equation satisfied by $k(z)$ may be written

$$\sum_{i=0}^{n} (a_i - b_i)\, d^i\, k(z)/dz^i = 0 \ ,$$

where we abbreviate

$$b_i = \sum_{h=0}^{n-1-i} a_{i+1+h}\, K^{(h)}(0) \ , \quad i = 0, 1, 2, \cdots, n-1 \ ; \quad b_n = 0 \ .$$

5. Given the kernel $K(z) = A \cos z + B \sin z$, show that

$$k(z) = [(B + \lambda A^2 - C_2 A)/(C_1 - C_2)]\, e^{C_1 z}$$
$$+ [(B + \lambda A^2 - C_1 A)/(C_2 - C_1)]\, e^{C_2 z} \ ,$$

where C_1 and C_2 are the roots of the equation

$$C^2 + \lambda AC + (1 - \lambda B) = 0 \ .$$

6. If $K(z) = A^2 z$, show that $k(z) = A \sinh Az$.

7. Prove that if $K(z) = A^n z^{(n-1)}/(n-1)!$, then the resolvent kernel is of the form

$$k(z) = A\,(e^{Az} + \omega\, e^{\omega Az} + \omega^2 e^{\omega^2 Az} + \cdots + \omega^{n-1} e^{\omega^{n-1} Az})/n$$

where ω is an nth root of unity. (G. C. Evans).

8. Prove the identity

$$z^{-1} + z^{-2} + z^{-3} + \cdots = (e^x\, e^{-xz} - 1)/(1-z) \ , \quad z = d/dx \ .$$

9. Prove that

$$-\{z^{-1}/x + 2!\, z^{-2}/x^2 + 3!\, z^{-3}/x^3 + \cdots\} = 1 - xz\, e^{-xz} \int_{-\infty}^{z} (e^t/t)\, dt \ .$$

10. Prove that the solution of

$$u(x) = f(x) + \int_0^x K(x-t)\, u(t)\, dt$$

where we write

$$K(z) = K_0 + K_1 z + K_2 z^2/2! + K_3 z^3/3! + \cdots$$

$$f(x) = f_0 + f_1 x + f_2 x^2/2! + f_3 x^3/3! + \cdots ,$$

is given by

$$u(x) = u_0 + u_1 x + u_2 x^2/2! + u_3 x^3/3! + \cdots ,$$

in which the coefficients are determined as follows:

$$u_0 = f_0 , \quad u_1 = f_1 + K_0 u_0 , \quad u_2 = f_2 + K_0 u_1 + K_1 u_0 , \cdots ,$$
$$u_r = f_r + K_0 u_{r-1} + K_1 u_{r-2} + \cdots + K_{r-1} u_0 .$$

(Gorakh Prasad).

11. If in problem 10 the kernel has the more general form

$$K(x,t) = K_{00} + K_{10} x + K_{01} t + (K_{20} x^2 + 2K_{11} xt + K_{02} t^2)/2!$$
$$+ (K_{30} x^3 + 3K_{21} x^2 t + 3K_{12} xt^2 + K_{03} t^3)/3! + \cdots ,$$

show that the solution is given by

$$u(x) = u_0 + u_1 x + u_2 x^2/2! + u_3 x^3/3! + \cdots ,$$

in which the coefficients are determined by

$$u_r = f_r + \Sigma K_{r-p-q-1,p} u_q r!/[(r-p-q-1)!p!q!(p+q+1)!] ,$$

the summation extending over all positive integral values of p and q (including zero), which satisfy the inequality $r-p-q-1 \geqq 0$. (Gorakh Prasad).

12. Prove that the solution of the equation

$$u(x) = f(x) + \int_0^x K(x-t)u(t)\,dt$$

is given by

$$u(x) = (1/2\pi i) \int_{a-i\infty}^{a+i\infty} \{e^{xt} F(t)/[1-g(t)]\}\,dt$$

where we abbreviate,

$$F(t) = \int_0^\infty e^{-xt} f(x)\,dx , \quad g(t) = \int_0^\infty e^{-xt} K(x)\,dx . \quad \text{(V. A. Fock).}$$

The following problems, taken from the domain of econometrics, illustrate applications of the theory of integral equations.

13. Assuming that the demand $y(t)$ for a product depends on the present price $p(t)$ and all previous prices in the range, $t_0 \leqq s \leqq t$, we can write as a first approximation.

$$y(t) = ap(t) + b - \int_{t_0}^t e^{\alpha(t-s)} p(s)\,ds \qquad (1)$$

where a, b, and α are constants. For a monopolist to maximize his net profit over the interval of time $t_1 \leqq s \leqq t_2$,

$$\Pi = \int_{t_1}^{t_2} [py(s) - Ay^2(s) - By(s) - C]\,ds ,$$

where A, B and C are constants, he must choose his price and production satis-

fying (1), the end conditions $p(t_1) = p_1$ and $p(t_2) = p_2$, and the equation

$$ap + y - 2Aay - aB + \int_t^{t_1} e^{-a(t-s)} y(s)\, ds = 0 \ . \tag{2}$$

Find the price $p(t)$ and the production $y(t)$.

The more general problem of competition among n producers, which reduces to the above for $n = 1$, has been treated by C. F. Roos: A Mathematical Theory of Competition, *American Journal of Mathematics*, vol. 47 (1925), pp. 163-175.

14. In general, current demand, $y(t)$, is supplied out of inventories and out of current production. Let T be the period of production, frequently defined as the average time elapsing between the manufacture of a lot of goods and their sale to the consumer. For simplicity assume T constant. Then the inventory v at the time t may be taken as

$$v(t) = \int_{t-T}^{t} [e^{\eta(s-t+T)} - e^{\mu(s-t+T)}]\, u(s)\, ds$$

where $u(s)$ is production at the time s and η and μ are constants. Then, if γ represents the percentage of current production passing to the consumer and λ represents the percentage of inventory going to the consumer, we have

$$y(t) = \gamma\, u(t) + \lambda \int_{t-T}^{t} [e^{\eta(s-t+T)} - e^{\mu(s-t+T)}]\, u(s)\, ds \ .$$

Show that if current production $u(t)$ is equal to current consumption, $y(t)$, then

$$\frac{d^2u}{dt^2} - [\eta + \mu + \lambda_1 (e^{\eta T} - e^{\mu T})] \frac{du}{dt} + \eta \mu u - \lambda_1 (\eta - \mu)\, u(t-T) = 0$$

and find the rate of production for which this balance attains.

[C. F. Roos: *Dynamic Economics*, Bloomington (1934), pp. 224-226.]

4. Lalesco's Theory of Integral Equations of Infinite Order. T. Lalesco [*Bibliography*: Lalesco (2)] in 1910 considered a more general situation than that discussed in the preceding section. His investigation centered around the following integral equation of infinite order:

$$u(x) + a_1(x) \int_1 u(t) + a_2(x) \int_2 u(t)\, dt^2 + \cdots$$
$$+ a_n(x) \int_n u(t)\, dt^n + \cdots = f(x) \ , \tag{4.1}$$

where the functions $a_n(x)$ possess a common domain of existence R, which includes the origin. It is further assumed that the functions $a_n(x)$ are bounded in R and that their maximum value, A_n, does not increase indefinitely with n, that is to say, that there exists a finite quantity A such that $A_n < A$ for all values of n.

If all the integrals in (4.1) vanish at the origin (or at any other single point in R), then the inversion of the equation is obtained by the method described in section 3. Let us, however, assume a more

general situation and assign an arbitrary value, C_n, to each integral at the origin. We shall then have

$$\begin{aligned}
\int_1 u(t)\,dt &= \int_0^x u(t)\,dt + C_1 , \\
\int_2 u(t)\,dt^2 &= \int_0^x u(t)\,dt^2 + C_1 x + C_2 , \\
\int_3 u(t)\,dt^3 &= \int_0^x u(t)\,dt^3 + C_1 x^2/2! + C_2 x + C_3 , \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\int_n u(t)\,dt^n &= \int_0^x u(t)\,dt^n + C_1 x^{n-1}/(n-1)! \\
&\qquad + \cdots + C_{n-1} x + C_n .
\end{aligned} \tag{4.2}$$

Replacing these values in (4.1), we obtain the equation

$$\begin{aligned}
\{[1 + a_1(x)]z^{-1} + a_2(x)z^{-2} + \cdots + a_n(x)z^{-n} + \cdots\} &\to u(x) \\
= f(x) - \sum_{n=1}^{\infty} C_n \varphi_n(x) &= f(x) - \varphi(x) ,
\end{aligned} \tag{4.3}$$

where we abbreviate,

$$\varphi_n(x) = \sum_{p=0}^{\infty} x^p \quad a_{n+p}(x)/p! , \tag{4.4}$$

and z^{-n} has the same significance as in the preceding section.

Because of the restriction imposed on the functions $a_n(x)$, it is at once clear that the sums (4.4) are uniformily convergent in the domain R. Hence the sum

$$\varphi(x) = \sum_{n=1}^{\infty} C_n \varphi_n(x) , \tag{4.5}$$

will be uniformly convergent in R provided the C_n are properly chosen. Since Ae^X, where X is the maximum modulus of any point in R, is a majorant for $\varphi_n(x)$, a sufficient condition for the uniform convergence of (4.5) is that the series $\sum_{n=1}^{\infty} |C_n|$ converge.

We are now in the circumstances of theorem 1 and hence can find a unique formal solution for (4.1). That the solution actually exists is easily established by means of successive approximations.

Let us write (4.3) in the form

$$u(x) - S(u) = f(x) - \varphi(x) .$$

We then construct the sequence,

$$
\begin{aligned}
u_1(x) &= f(x) - \varphi(x) \ , \\
u_2(x) &= S(u_1) \ , \\
u_3(x) &= S(u_2) \ , \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
u_n(x) &= S(u_{n-1}) \ .
\end{aligned}
$$

The function

$$
u(x) = u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots \tag{4.6}
$$

is obviously a formal solution of (4.3). Letting M be the maximum modulus of $f(x) - \varphi(x)$ in R we readily establish the following inequalities:

$$
\begin{aligned}
&|u_1(x)| < M \ , \\
&|u_2(x)| < AM|e^x - 1| < AMkX \ , \\
&|u_3(x)| < A^2Mk|e^x - 1 - x/1!| < A^2Mk^2X^2/2! \ , \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&|u_{n+1}(x)| < A^nMk^{n-1}|e^x - 1 - x/1! - \cdots \\
&\qquad\qquad - x^{n-1}/(n-1)!| < A^nMk^nX^n/n! \ ,
\end{aligned}
$$

where k has been so chosen that

$$
|e^x - 1 - x/1! - x^2/2! - \cdots - x^{n-1}/(n-1)!| < kX^n/n! \ .
$$

A majorant for (4.6) is thus seen to be Me^{AkX} and the series is thus uniformly convergent in R.

The following theorem is thus established:

Theorem 3. If in equation (4.1) *the functions* $a_n(x)$ *share a common domain R about the origin and are subject to the inequality,* $|a_n(x)| < A$, *where A is finite, then equation* (4.1) *has a unique solution such that its integrals of different orders assume preassigned values* C_n *provided the series* $\sum_{n=1}^{\infty} |C_n|$ *converges.*

5. The Differential Equation Background of the Fredholm Integral Equation. In section 2 we showed how the Cauchy problem of ordinary linear differential equations led naturally to the theory of integral equations of Volterra type. It is our purpose here to trace the equivalent development of the problem of Fredholm integral equations from the problem of differential equation with boundary conditions specified at two points. To distinguish it from the Cauchy

problem, this will be called a *Sturmian problem* in recognition of the fact that J. C. F. Sturm (1803-1855) was the first to make a systematic study of this question in his classical memoir of 1836.*

For simplicity we shall consider first the case of homogeneous boundary conditions. Thus we shall attempt to express in terms of an integral equation the following system:

$$L(u) = r(x) \ , \qquad\qquad (5.1)$$
$$U_i(u) = 0 \quad , \qquad i = 1, 2, \cdots, n \ ,$$

where we employ the abbreviations:

$$L(u) = p_n(x)u^{(n)}(x) + p_{n+1}(x)u^{(n-1)}(x) + \cdots + p_0(x)u(x) \ ,$$

$$U_i(u) = A_i(u) - B_i(u) \ ,$$

$$A_i(u) = \alpha_{i1}u(a) + \alpha_{i2}u'(a) + \cdots + \alpha_{in}u^{(n-1)}(a) \ ,$$

$$B_i(u) = \beta_{i1}u(b) + \beta_{i2}u'(b) + \cdots + \beta_{in}u^{(n-1)}(b) \ .$$

Employing an idea introduced by D. Hilbert, we shall define the *Green's function* belonging to the differential system (5.1). By the Green's function, $\Gamma(x,t)$, belonging to (5.1) we mean a function with the following properties:

(a) $\Gamma(x,t)$, with respect to x in the interval (ab) and except at $x = t$, is continuous together with its first n derivatives and satisfies the system

$$L(u) = 0 \ ,$$
$$U_i(u) = 0 \ . \qquad\qquad (5.2)$$

(b) For $x = t$, $\Gamma(x,t)$ is continuous together with its first $n—2$ derivatives.

(c) For $x = t$, the $n—1$st derivative satisfies the condition:

$$\lim_{\varepsilon=0}\left[\frac{\partial^{n-1}\Gamma(x,t)}{\partial x^{n-1}}\right]_{x=t+\varepsilon} - \lim_{\varepsilon=0}\left[\frac{\partial^{n-1}\Gamma(x,t)}{\partial x^{n-1}}\right]_{x=t-\varepsilon} = -1 \ . \qquad (5.3)$$

The actual construction of a Green's function out of a set of fundamental solutions $(u_1, u_2, \cdots, u_n)$ of the differential equation $L(u) = 0$, may be described as follows:

*Sur les équations différentielles linéaires du second ordre. *Journal de Mathématiques*, vol. 1 (1836), pp. 106-186. See also: M. Bôcher: *Leçons sur les méthodes de Sturm.* Paris (1917), 118 p.

Let us first write the function,

$$g(x,t) = \pm[1/2W(t)] \begin{vmatrix} u_1(x) & u_1^{(n-2)}(t) & u_1^{(n-3)}(t) & \cdots & u_1(t) \\ u_2(x) & u_2^{(n-2)}(t) & u_2^{(n-3)}(t) & \cdots & u_2(t) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_n(x) & u_n^{(n-2)}(t) & u_n^{(n-3)}(t) & \cdots & u_n(t) \end{vmatrix} .$$

where the plus sign is to be used when $x \leqq t$ and the negative sign when $x > t$, and where $W(t)$ is the Wronskian defined in section 2.

Introducing a set of fundamental solutions $(v_1, v_2, \cdots, v_n)$ of the adjoint differential equation $M(v) = 0$ [see (2.12)], we can write $g(x, t)$ in the more compact form,

$$g(x,t) = \pm\tfrac{1}{2}[u_1(x)\, v_1(t) + u_2(x)\, v_2(t) + \cdots + u_n(x)\, v_n(t)] .$$

It is now clear that we may represent the Green's function by the expression,

$$\Gamma(x,t) = c_1 u_1(x) + c_2 u_2(x) + \cdots + c_n u_n(x) + g(x,t) , \qquad (5.4)$$

provided the arbitrary constants c_i can be so chosen that $\Gamma(x, t)$ satisfies the boundary conditions $U_i(u) = 0$ and the discontinuity condition (5.3).

When $\Gamma(x, t)$ is substituted in the boundary conditions we are led to the following system of equations:

$$\begin{aligned} c_1 U_{11} + c_2 U_{12} \cdots + c_n U_{1n} = & \\ -\tfrac{1}{2}[(A_{11} + B_{11})\, v_1(t) + (A_{12} + B_{12})\, v_2(t) + \cdots & \\ + (A_{1n} + B_{1n}) v_n(t)] , & \\ c_1 U_{21} + c_2 U_{22} + \cdots + c_n U_{2n} = & \\ -\tfrac{1}{2}[(A_{21} + B_{21})\, v_1(t) + (A_{22} + B_{22})\, v_2(t) + \cdots & \\ + (A_{2n} + B_{2n})\, v_n(t)] , & \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ c_1 U_{n1} + c_2 U_{n2} + \cdots + c_n U_{nn} = & \\ -\tfrac{1}{2}[(A_{n1} + B_{n1})\, v_1(t) + (A_{n2} + B_{n2})\, v_2(t) + \cdots & \\ + (A_{nn} + B_{nn})\, v_n(t)] . & \end{aligned} \qquad (5.5)$$

where we employ the convenient notation, $U_{ij} = U_i(u_j)$, $A_{ij} = A_i(u_j)$, $B_{ij} = B_i(u_j)$.

In order that the constants c_i be determined from the system (5.5) it is necessary and sufficient that the determinant of the coefficients does not vanish, that is to say, that we have

$$| U_{ij} | \neq 0 .$$

But this condition is in turn the necessary and sufficient condition that the homogeneous system (5.2) shall have no solution. Hence we arrive at the important proposition:

The non-homogeneous system (5.1) *has a Green's function when and only when the homogeneous system* (5.2) *is inconsistent.**

It remains for us to show that the Green's function is unique, which is accomplished by means of the discontinuity condition (5.3).

Suppose that $G_1(x,t) = c_1(t)u_1(x) + \cdots + c_n(t)\,u_n(x)$ represents $\Gamma(x,t)$ in the interval (a,t) and that $G_2(x,t) = d_1(t)\,u(x) + \cdots + d_n(t)\,u_n(x)$ represents $\Gamma(x,t)$ in the interval (t,b).

In order that $\Gamma(x,t)$ have the continuity properties specified in (*b*) and (*c*) above it will be sufficient to write

$$\begin{aligned}
&c_1u_1(t) + \cdots + c_nu_n(t) - d_1u_1(t) - \cdots - d_nu_n(t) = 0\ ,\\
&c_1u_1'(t) + \cdots + c_nu_n'(t) - d_1u_1'(t) - \cdots - d_nu_n'(t) = 0\ ,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&c_1u_1^{(n-1)}(t) + \cdots + c_nu_n^{(n-1)}(t) - d_1u_1^{(n-1)}(t) - \cdots\\
&\qquad\qquad\qquad\qquad\qquad - d_nu_n^{(n-1)}(t) = 1\ .
\end{aligned}$$

Employing the abbreviation $f_i = c_i - d_i$, we have

$$\begin{aligned}
&f_1u_1(t) + \cdots + f_nu_n(t) = 0\ ,\\
&f_1u_1'(t) + \cdots + f_nu_n'(t) = 0\ ,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&f_1u_1^{(n-1)}(t) + \cdots + f_nu_n^{(n-1)}(t) = 1\ .
\end{aligned}$$

Since the determinant of this system is the Wronskian, the f_i are uniquely determined.

The Green's function is now substituted in the boundary conditions. Using the notation adopted above, we obtain the following equations:

$$\begin{aligned}
&c_1A_{11} + \cdots + c_nA_{1n} - d_1B_{11} - \cdots - d_nB_{1n} = 0\ ,\\
&c_1A_{21} + \cdots + c_nA_{2n} - d_1B_{21} - \cdots - d_nB_{2n} = 0\ ,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&c_1A_{n1} + \cdots + c_nA_{nn} - d_1B_{n1} - \cdots - d_nB_{nn} = 0\ .
\end{aligned}$$

*Even when the homogeneous system is not consistent a *generalized Green's function* may be constructed. See the discussion given below.

But these equations may be written in the form

$$
\begin{aligned}
c_1U_{11} + \cdots + c_nU_{1n} &= -f_1B_{11} - \cdots - f_nB_{1n} \ , \\
c_1U_{21} + \cdots + c_nU_{2n} &= -f_1B_{21} - \cdots - f_nB_{2n} \ , \\
&\cdots\cdots\cdots \\
c_1U_{n1} + \cdots + c_nU_{nn} &= -f_1B_{n1} - \cdots - f_nB_{nn} \ .
\end{aligned}
$$

Since the determinant $|U_{ij}|$ is different from zero by hypothesis, the functions $c_i(t)$ are now uniquely determined in terms of the $f_i(t)$ and consequently the $d_i(t)$ are also uniquely determined. This establishes the uniqueness of the Green's function.

If for a system $L(u) = 0$, $U_1(u) = 0$, the condition $|U_{ij}| \neq 0$ is not fulfilled, then the concept of a Green's function must be modified. This modification may be accomplished in the following manner: Since we have $|U_{ij}| = 0$, there will exist a function, $u_0(x)$, which satisfies the system. Let this solution be normalized, that is, assume that $u_0(x)$ fulfills the condition,

$$\int_a^b u_0^2(x)\,dx = 1 \ .$$

Then define a function $\Gamma(x,t)$ which satisfies condition (5.3) and the boundary conditions, and which further satisfies the equations

$$L[\Gamma(x,t)] = u_0(x)u_0(t) \ ,$$

$$\int_a^b \Gamma(x,t)u_0(x)\,dx = 0 \ .$$

The function thus constructed is the *generalized Green's function* of the system.

As an example of the construction of a Green's function, let us consider the following simple case:

$$
\begin{aligned}
d^2u/dx^2 &= 0 \ , \\
u(0) = u(1) &= 0 \ .
\end{aligned}
\tag{5.6}
$$

Employing the elementary system, $u_1(x) = x$, $u_2(x) = 1$, we write

$$
\begin{aligned}
\Gamma(x,t) &= c_1u_1(x) + c_2u_2(x) \pm \tfrac{1}{2}[u_1(x)u_2(t) - u_1(t)u_2(x)] \\
&= c_1x + c_2 \pm \tfrac{1}{2}(x - t) \ .
\end{aligned}
$$

Substituting this function in the boundary conditions we obtain,

$$
\begin{aligned}
\Gamma(0,t) &= c_2 + \tfrac{1}{2}\,(-t) = 0 \ , \\
\Gamma(1,t) &= c_1 + c_2 - \tfrac{1}{2}(1-t) = 0 \ .
\end{aligned}
$$

These equations yield the values, $c_1 = \frac{1}{2} - t$, and $c_2 = \frac{1}{2}t$. From them we obtain the Green's function:

$$\begin{aligned} \Gamma(x,t) &= x(1-t) \ , \qquad & x \leqq t \ , \\ &= t(1-x) \ , \qquad & x \geqq t \ . \end{aligned} \tag{5.7}$$

Preliminary to a discussion of adjoint systems, let us now return to the Lagrange identity (2.11) which we shall integrate between the limits a and b:

$$\int_a^b [vL(u) - uM(v)]dx = P(u,v)\Big|_a^b \tag{5.8}$$

The quantity $P(u,v)\big|_a^b$ is at once recognized as a bilinear form in the two sets of $2n$ quantities

$$u(a), u'(a), \cdots, u^{(n-1)}(a), u(b), u'(b), \cdots, u^{(n-1)}(b) \ ,$$
$$v(a), v'(a), \cdots, v^{(n-1)}(a), v(b), v'(b), \cdots, v^{(n-1)}(b) \ .$$

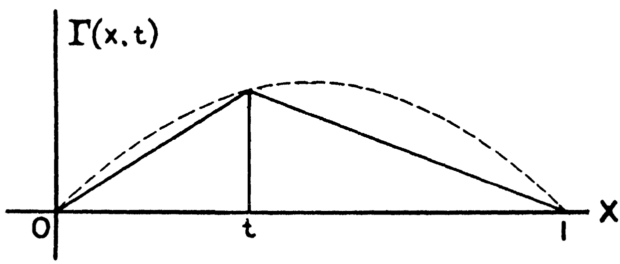

FIGURE 1.

The form is non-singular since its determinant is equal to $[p_n(a)p_n(b)]^n$. It may then be written uniquely as follows:

$$P(u,v)\big|_a^b = U_1V_{2n} + U_2V_{2n-1} + \cdots + U_{2n}V_1 \ ,$$

where $U_i = A_i(u) - B_i(u)$, provided the determinant of the $4n^2$ coefficients is not zero. The quantities V_i are linear forms in $v(a)$, $v'(a), \cdots, v^{(n-1)}(a)$, $v(b)$, $v'(b), \cdots, v^{(n-1)}(b)$ and are uniquely determined from any specification of the linear forms U_i, which may be arbitrarily given subject only to the condition of their linear independence.

We define the system,

$$\begin{aligned} M(v) &= 0 \ , \\ V_i(v) &= 0 \ , \qquad i = 1, 2, \cdots, 2n-m \ , \end{aligned} \tag{5.9}$$

as the *adjoint* of

$$\begin{aligned} L(u) &= 0 \ , \\ U_i(u) &= 0 \ , \qquad i = 1, 2, \cdots, m \ . \end{aligned} \tag{5.10}$$

In most applications $m = n$, and we shall confine our attention to this case. Under special conditions we may also have system (5.9) identical with system (5.10). The system is then said to be *self-adjoint.* If $M(u) = -L(u)$, then the system is said to be *anti-self-adjoint.*

In a paper of much elegance D. Jackson: *Transactions of the Amer. Math. Soc.*, vol. 17 (1916), pp. 418-424 has given general conditions for self-adjointness. Thus suppose that

$$\Pi = \begin{pmatrix} \pi_1 & \pi_3 \\ \pi_4 & \pi_2 \end{pmatrix}$$

is the matrix of the bilinear form $P(u,v)\Big|_a^b$ in (5.8). Let A_1 be the square matrix of the coefficients of n variables in $U_i(u)$, these coefficients being so chosen that A_1 is non-singular. Then A_2 will be the square matrix of the remaining n variables. If δ denotes the product $A_1^{-1} A_2$ and δ' the conjugate of δ then Jackson's theorem asserts:

"If the differential expression $L(u)$ is self-adjoint, the condition that the boundary conditions be self-adjoint is that the matrix

$$\delta'\pi_1\delta - 2\pi_4\delta + \pi_2$$

by symmetric; if $L(u)$ is an anti-self-adjoint expression or differs from such an expression only in the term of order zero, the condition is that the matrix just written down be skew-symmetric."

It is obvious that this theorem is very difficult to apply in the direct calculation of conditions for self-adjointness. V. V. Latshaw: *Bulletin of the Amer. Math. Soc.*, vol. 39 (1933), pp. 969-978, has materially simplified the explicit calculation of these conditions.

Assuming the self-adjoint expression

$$L(u) = [p_m u^{(m)}]^{(m)} + [p_{m-1} u^{(m-1)}]^{(m-1)} + \cdots + p_0 u \ ,$$

Latshaw, removing the factor A_1 from Jackson's matrix, obtained the following $m(2m-1)$ conditions:

$$T_{ij} = \sum_{r=1}^{2m-1} \sum_{s=r+1}^{2m} \pi_{rs}^{(1)} D_{rs}(i,j) - \sum_{r=1}^{2m} \pi_{ir}^{(4)} D_r(j)$$

$$+ \sum_{r=1}^{2m} \pi_{jr}^{(4)} D_r(i) + \bar{A}_1 \pi_{ij}^{(2)} = 0 \ , \qquad i < j \leqq 2m \ ,$$

where $\pi_{ij}^{(k)}$ denotes the element in the rth row and sth column of the matrix π_k, $\bar{A}_1$ is the determinant of the matrix A_1, the symbol $D_{rs}(i,j)$ denotes the determinant of A_1 with the rth and sth columns replaced by the ith and jth columns respectively of A_2, and $D_r(i)$ indicates one replacement only.

Thus for the second order case,

$$(p_1 u')' + p_0 u = 0 \, , \quad \sum_{i=1}^{2} [a_{ki} u^{(i-1)}(a) + b_{ki} u^{(i-1)}(b)] = 0 \, , \quad k = 1, 2 \ ,$$

Latshaw assumes the non-singular matrix to be

$$A_1 = \begin{pmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{pmatrix} \text{, and hence has } A_2 = \begin{pmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{pmatrix}.$$

According to this specification of the constants in the boundary conditions he obtains the matrices of the bilinear form as follows:

$$\pi_1 = 0 \ , \qquad \pi_2 = 0 \ ,$$

$$\pi_3 = \begin{pmatrix} -p_1(a) & 0 \\ 0 & p_1(b) \end{pmatrix} , \qquad \pi_4 = \begin{pmatrix} p_1(a) & 0 \\ 0 & -p_1(b) \end{pmatrix} .$$

The theorem at once leads to the single condition:

$$T_{12} = \pi_{12}{}^{(1)} D_{12}(1,2) - \pi_{11}{}^{(4)} D(2) - \pi_{12}{}^{(4)} D(2) + \pi_{21}{}^{(4)} D(1) + \pi_{22}{}^{(4)} D_2(1) + \bar{A}_1 \pi_{12}{}^{(2)} = 0 \ ,$$

which immediately reduces to the familiar equation,

$$p_1(a)\,(b_{11}b_{22} - b_{21}b_{12}) = p_1(b)\,(a_{11}a_{22} - a_{21}a_{12}) \ .$$

We shall now prove the following theorem:

Theorem 4. If $G(x,t)$ is the Green's function of a differential system,

$$L(u) = 0 \ , \qquad U_i(u) = 0 \ , \qquad i = 1, 2, \cdots, n \ , \tag{5.11}$$

and $H(x,t)$ the Green's function of the adjoint system,

$$M(u) = 0 \ , \qquad V_i(u) = 0 \ , \qquad i = 1, 2, \cdots n \ , \tag{5.12}$$

the Green's functions of the two systems are connected by the relation

$$p_n(x) H(x,t) \equiv (-1)^n p_n(t) G(t,x) \tag{5.13}$$

Proof: In order to establish this theorem we define t_1 and t_2 as any two points in (a,b), where for convenience we assume $t_1 < t_2$. In the Lagrange identity (5.8), we now set $u = G(x,t_1)$ and $v = H(x,t_2)$. Breaking up the interval (a,b) into the parts $(a, t_1 - \varepsilon)$, $(t_1 + \varepsilon, t_2 - \varepsilon)$, $(t_2 + \varepsilon, b)$ and noting that $L(G) = 0$, $M(H) = 0$, we get the relation

$$P(G,H) \Big|_a^{t_1-\varepsilon} + P(G,H) \Big|_{t_1+\varepsilon}^{t_2-\varepsilon} + P(G,H) \Big|_{t_2+\varepsilon}^{b} = 0 \ . \tag{5.14}$$

Because of the conditions satisfied by G and H at the two end point a and b, the first and last terms of this equation are zero. Also because of the continuity of G and H and their derivatives to, but not including, the n—1st order it is clear that when $\varepsilon = 0$ most of the terms of (5.14) cancel one another. But from the explicit form of $P(u,v)$ given in (2.13) we see that we are left with the terms

$$u(x)\,(-1)^{n-1} p_n(x)\, v^{(n-1)}(x) + u^{(n-1)}(x)\, p_n(x)\, v(x) \ ,$$

which involve the n—1st derivatives for which a discontinuity in the Green's functions has been postulated.

Thus we shall have

$$\lim_{\varepsilon=0} p_n(x)\,[H(x,t_2)G^{(n-1)}(x,t_1)+(-1)^{n-1}G(x,t_1)H^{(n-1)}(x,t_2)\Big|^{t_1-\varepsilon}$$

$$+p_n(x)\,[H(x,t_2)G^{(n-1)}(x,t_1)+(-1)^{n-1}G(x,t_1)H^{(n-1)}(x,t_2)\Big|^{t_2-\varepsilon}_{t_1+\varepsilon}$$

$$+p_n(x)\,[H(x,t_2)G^{(n-1)}(x,t_1)+(-1)^{n-1}G(x,t_1)H^{(n-1)}(x,t_2)\Big|_{t_2+\varepsilon}=0\,.$$

Noting the fact that $G(t_1,t_1)=H(t_2,t_2)=0$ and that

$$\lim_{\varepsilon-0}[G^{(n-1)}(t_1-\varepsilon,\,t_1)-G^{(n-1)}(t_1+\varepsilon,\,t_1)]$$
$$=\lim_{\varepsilon=0}[H^{(n-1)}(t_2-\varepsilon,\,t_2)-H^{(n-1)}(t_2+\varepsilon,\,t_2)]=1\,,$$

we immediately establish the relation,

$$p_n(t_1)H(t_1,\,t_2)+p_n(t_2)\,(-1)^{n-1}\,G(t_2,\,t_1)=0\,.$$

The same reasoning may be applied for $t_1>t_2$ and obviously holds for $t_1=t_2$ so that identity (5.13) is completely established.

Since the differential equations $L(u)=0$ and $M(u)=0$ are non-singular in (a,b) it is possible to make $p_n(x)=1$ through division by the coefficients of $u^{(n)}(x)$. If, moreover, the two systems are self-adjoint, we derive an interesting corollary of theorem 3 for then

$$G(x,t)=(-1)^nG(t,x)\,,$$

and the Green's functions are either *symmetric* or *skew-symmetric* as the order is even or odd.

We are now in a position to solve the general system

$$\begin{aligned} &L(u)=r(x)\,, \\ &U_i(u)=k_i\,,\quad i=1,2,\cdots,n\,, \end{aligned}\tag{5.15}$$

where the k_i are given constants. We shall assume that the corresponding homogeneous system (5.11) is incompatible, that is to say that

$$U_i(u_j)\neq 0\,.$$

Let us represent by $\Gamma(x,t)$ the Green's function of (5.11) and then define

$$G(x,t)=-\Gamma(x,t)/p_n(t)\,.$$

Under these conditions there will exist a unique solution of the system (5.15) and this solution is given by

$$u(x)=k_1\,G_1(x)+\cdots+k_n\,G_n(x)+\int_a^b G(x,t)r(t)\,dt\,,\tag{5.16}$$

where $G_j(x)$ is the solution of the particular system

$$L(u) = 0 \ ,$$
$$U_i(u) = 0 \ , \ i \neq j \ , \ U_j(u) = k_j \ .$$

In order to prove this we compute $L(u)$, noting the continuity of the first n—2 derivatives of $G(x,t)$ at $t = x$, and the discontinuity, equal to $1/p_n(x)$, at $t = x$ of the n—1st derivative. We thus obtain

$$\begin{aligned} L(u) &= \sum_{i=1}^{n} k_i L(G_i) + L \to \int_a^b G(x,t) r(t) dt \\ &= \int_0^b r(t) \left[\sum_{r=0}^{n} p_r(t) \frac{\partial^r G(x,t)}{\partial x^r} \right] dt \\ &\qquad + \lim_{\varepsilon=0} p_n(x) r(x) \left[\frac{\partial^{n-1} G(x,x+\varepsilon)}{\partial x^{n-1}} - \frac{\partial^{n-1} G(x,x-\varepsilon)}{\partial x^{n-1}} \right] \\ &= r(x) \ . \end{aligned}$$

Since $U_i(u)$ involves no derivatives of u higher than n—1, we get

$$\begin{aligned} U_i(u) &= k_i U_i(G_i) + \int_a^b U_i(G) r(t) dt \\ &= k_i \ . \end{aligned}$$

Many of the applications of the boundary value problem of the type considered in this section are associated with a differential system with a parameter, that is to say, with a system of the form

$$\begin{aligned} L(u) + \lambda u(x) &= r(x) \\ U_i(u) &= k_i \, , i = 1, 2, \cdots, n \ , \end{aligned} \tag{5.17}$$

If, as above, $\Gamma(x,t)$ is defined as the Green's function for the system (5.11), then from (15.16) it is evident that we can express (5.17) in the form

$$u(x) = f(x) + \lambda \int_a^b K(x,t) u(t) dt \ , \tag{5.18}$$

where we use the abbreviations,

$$K(x,t) = \Gamma(x,t)/p_n(x) \ ,$$
$$f(x) = \sum_{i=1}^{n} k_i G_i(x) - \int_a^b K(x,t) r(t) dt \ .$$

Equation (5.18) is a Fredholm integral equation of second kind as we see from the first section of this chapter. The theory of this equation will be discussed in the next sections and in chapter 12.

PROBLEMS

1. Prove that the Green's function associated with $L(u) \equiv u''(x)$ and the boundary conditions

$$u(0)=0\ ,\quad u'(1)=0\ ,$$

is given by

$$\begin{aligned}\Gamma(x,t) &= x\ ,\quad x \leqq t\ ,\\ &= t\ ,\quad x \geqq t\ .\end{aligned}$$

2. Show that for $L(u) \equiv u''(x)$ and the boundary conditions

$$u(0)=0\ ,\quad u'(1)=h\,u(1)\ ,$$

the Green's function is

$$\begin{aligned}\Gamma(x,t) &= x + \frac{h\,x\,t}{1-h}\ ,\quad x \leqq t\ ,\\ &= t + \frac{h\,x\,t}{1-h}\ ,\quad x \geqq t\ ,\end{aligned}$$

3. Prove that the Green's function for

$$L(u) \equiv u''(x) + \lambda^2 u(x)\ ,\quad u(0)=u(1)=0\ ,$$

is given by

$$\begin{aligned}\Gamma(x,t) &= \frac{\sin\{\lambda(1-t)\}\cdot\sin\lambda x}{\lambda\sin\lambda}\ ,\quad x \leqq t\ ,\\ &= \frac{\sin\{\lambda(1-x)\}\cdot\sin\lambda t}{\lambda\sin\lambda}\ ,\quad x \geqq t\ .\end{aligned}$$

4. Show that for

$$L(u) \equiv u''(x) - \lambda^2 u(x)\ ,\quad u(0)=u(1)=0\ ,$$

the Green's function is

$$\begin{aligned}\Gamma(x,t) &= \frac{\sinh\{\lambda(1-t)\}\cdot\sinh\lambda x}{\lambda\sinh\lambda}\ ,\quad x \leqq t\ ,\\ &= \frac{\sinh\{\lambda(1-x)\}\cdot\sinh\lambda t}{\lambda\sinh\lambda}\qquad x \geqq t\ .\end{aligned}$$

5. Show that the generalized Green's function belonging to $L(u) \equiv u''(x)$ and the boundary conditions $u(1) = u(-1)$, $u'(1) = u'(-1)$, is the following function:

$$\Gamma(x,t) = -\tfrac{1}{2}\,|\,x-t\,| + \tfrac{1}{4}\,(x-t)^2 + 1/6\ .$$

6. Construct the generalized Green's function belonging to the differential expression: $L(u) \equiv (1-x^2)\,u''(x) - 2x\,u'(x)$, which, together with its first two derivatives, remains finite at the points $x=1$, $x=-1$.

6. The Fredholm Integral Equation as a Differential Equation of Infinite Order. In the preceding section we have shown how the Sturmian problem of linear differential equations leads to an equation of the form

$$u(x) = f(x) + \lambda\int_a^b K(x,t)u(t)\,dt\ . \tag{6.1}$$

In the first chapter, section 10, there was given a short survey of the status of this *Fredholm integral equation* in the general theory of linear operators. It is our present purpose to show how the solution of the Fredholm equation may be attained by the methods developed in the preceding pages of this book.

In section 11 of chapter 4 we considered the formal inversion of the equation,

$$S(u) + \varphi(x)u(x) = f(x) \ ,$$

and obtained, as one inversion, expansion (11.3) of that chapter, namely,

$$u(x) = f(x)/\varphi(x) - (1/\varphi)S \rightarrow (f/\varphi) + (1/\varphi)[S \rightarrow \{(1/\varphi)S \rightarrow (f/\varphi)\}] - \cdots .$$

If we now specialize this equation by setting $\varphi(x) = 1$, $S(u) = -\lambda\int_a^b K(x,t)u(t)\,dt$, we shall get the expansion,

$$u(x) = f(x) + \lambda \int_a^b K(x,t)f(t)\,dt + \lambda^2 \int_a^b K_2(x,t)f(t)\,dt + \lambda^3 \int_a^b K_3(x,t)f(t)\,dt + \cdots \qquad (6.2)$$

where $K_2(x,t)$, $K_3(x,t)$ etc. are *successive iterated kernels* defined as follows:

$$K_2(x,t) = \int_a^b K(x,s)K(s,t)\,ds \ ,$$

$$K_3(x,t) = \int_a^b K_2(x,s)K(s,t)\,ds \ ,$$

$$\cdots\cdots\cdots\cdots$$

$$K_n(x,t) = \int_a^b K_{n-1}(x,s)K(s,t)\,ds \ .$$

If F is the maximum value of $f(t)$ in the interval $a \leqq x \leqq b$ and K the maximum value of $K(x,t)$ in the rectangle $a \leqq x \leqq b$, $a \leqq t \leqq b$, then the series

$$F + |\lambda|KF + |\lambda|^2K^2F + |\lambda|^3K^3F + \cdots = F/(1 - |\lambda|K)$$

forms a majorant for series (6.2). *Hence for values of the parameter less in absolute value than* $1/K$, ($|\lambda| < 1/K$), *a solution exists for the integral equation.*

It is our purpose now to investigate this solution more closely and by applying the methods of section 13, chapter 4, to obtain the classical solution of Fredholm. For this purpose we first reduce (6.1)

formally to a differential equation of infinite order by expanding $u(t)$ into the Taylor's series:

$$u(t) = u(x) + (t-x)u'(x) + (t-x)^2 u''(x)/2! + \cdots .$$

When this is substituted in (6.1) we obtain an equation of the form

$$[1 + a_0(x)]\, u(x) + a_1(x)u'(x) + \cdots + a_n(x)u^{(n)}(x) + \cdots = f(x) \ , \qquad (6.3)$$

where we write

$$a_i(x) = -\lambda \int_a^b K(x,t)\,(t-x)^i dt/i! \ ,$$

After these functions have been substituted in system (13.4) of chapter 4 the coefficients of the unknowns reduce in a remarkable way and we obtain for $D(n,x)$, i. e. the determinant of the reduced system obtained by suppressing terms of order greater than $n-1$ and all rows beyond the *nth*, the following:

$$D(n,x) = \qquad (6.4)$$

$$\left|\begin{matrix} 1-\lambda \int_a^b K(x,t)\,dt & , -\lambda \int_a^b K(x,t)\,(t-x)\,dt & ,\cdots, & -\lambda \int_a^b K(x,t)\, \frac{(t-x)^{(n-1)}}{(n-1)!}\,dt \\ -\lambda \int_a^b K'(x,t)\,dt & ,1-\lambda \int_a^b K'(x,t)\,(t-x)\,dt & ,\cdots, & -\lambda \int_a^b K'(x,t)\, \frac{(t-x)^{(n-1)}}{(n-1)!}\,dt \\ \cdot & \cdot & \cdot & \cdot \\ -\lambda \int_a^b K^{(n-1)}(x,t)\,dt & , -\lambda \int_a^b K^{(n-1)}(x,t)\,(t-x)\,dt & ,\cdots & 1-\lambda \int_a^b K^{(n-1)}(x,t)\, \frac{(t-x)^{(n-1)}}{n-1)!}\,dt \end{matrix}\right|$$

where the differentiation is with respect to x.

If we now refer to the criterion of theorem 8, chapter 3, pertaining to the inversion of a system of equations of determinant

$$|\delta_{ik} - a_{ik}|, \quad \delta_{ik} = 0 \ , \quad i \neq k \ , \quad \delta_{ii} = 1 \ ,$$

namely, that

$$S_i = \sum_{k=1}^{\infty} |a_{ik}| < 1 \ , \qquad i = 1, 2, \cdots, \infty$$

we see that the existence theorem for the integral equation will reduce to a consideration of the inequalities

$$\sigma_m = |\lambda| \int_b^a |K_x^{(m)}(x,s)|\, e^{s-a}\, ds < 1 \ , \qquad m = 0, 1, 2, \cdots .$$

Assuming that $K(x,s)$ is a function of x of grade equal to $q < 1$, then we know from theorem 5, chapter 5 that $|K_x^{(m)}| < (q+\varepsilon)^m K$

for $m > M$, where K and M are finite positive numbers independent of m, and ε is a small positive quantity so chosen that $q + \varepsilon \leqq 1$.

Hence we have

$$\sigma_m < |\lambda| K(e^{b-a}-1) < 1 \ ,$$

and from this we obtain the result that a solution of (6.1) exists and can be found by the method of segments for values of λ, such that

$$|\lambda| < 1/[K(e^{b-a}-1)] \ .$$

This inequality illustrates an interesting feature of the method of segments. Although the limitation on λ resembles, although it fails to be as sharp as, the one previously obtained by means of iterated kernels, the actual solution which we shall find by the method of segments is the meromorphic function of Fredholm.

We shall first identify the Fredholm determinant, $D(\lambda)$, with the limit $\lim\limits_{n=\infty} D(n,x)$. For this purpose we use the development for the secular determinant $\Delta(n)$ given by (2.2) in chapter 3.

Furthermore it is easily shown* that

$$\begin{vmatrix} \int_a^b \phi_{11}(t)\,dt, & \int_a^b \phi_{12}(t)\,dt, \cdots, & \int_a^b \phi_{1n}(t)\,dt \\ \int_a^b \phi_{21}(t)\,dt, & \int_a^b \phi_{22}(t)\,dt, \cdots, & \int_a^b \phi_{2n}(t)\,dt \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \int_a^b \phi_{n1}(t)\,dt, & \int_a^b \phi_{n2}(t)\,dt, \cdots, & \int_a^b \phi_{nn}(t)\,dt \end{vmatrix}$$

$$= \underbrace{\int_a^b \cdot\ \cdot \int_a^b}_{n} \begin{vmatrix} \phi_{11}(t_1), \phi_{12}(t_2), \cdots, \phi_{1n}(t_n) \\ \phi_{21}(t_1), \phi_{22}(t_2), \cdots, \phi_{2n}(t_n) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \phi_{n1}(t_1), \phi_{n2}(t_2), \cdots, \phi_{nn}(t_n) \end{vmatrix} dt_1 dt_2 \cdots dt_n \qquad (6.5)$$

Introducing the notation

$$K_{ij} = \mathrm{K}^{(j-1)}(x,t_i) \ \text{ and } \ S_{ij} = (t_i - x)^{j-1}/(j-1)\,!$$

and taking note both of the expansion (2.2) in chapter 3 and (6.5) above, we can write $D(n,x)$ in the following form:

$$D(n,x) = 1 - \frac{\lambda}{1!}\int_a^b \Sigma\Delta_1 dt_1 + \frac{\lambda^2}{2!}\int_a^b \int_a^b \Sigma\Delta_2 dt_1 dt_2 + \cdots \qquad (6.6)$$
$$\cdots + (-1)^n \frac{\lambda^n}{n!}\int_b^a \cdots \int_a^b \Sigma\Delta_n dt_1 dt_2 \cdots dt_n \ ,$$

*See E. Goursat: Sur un cas élémentaire de l'équation de Fredholm. *Bulletin de la Société Mathématique,* vol. 35 (1907) pp. 165-166.

where we have used the abbreviation,

$$\Delta_n = \begin{vmatrix} K_{1r_1} & S_{1r_1}, & K_{2r_1} & S_{2r_2}, & \cdots, & K_{nr_1} S_{nr_n} \\ K_{1r_2} & S_{1r_1}, & K_{2r_2} & S_{2r_2}, & \cdots, & K_{nr_2} S_{nr_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{1r_n} & S_{1r_1}, & K_{2r_n} & S_{2r_2}, & \cdots, & K_{nr_n} S_{nr_n} \end{vmatrix}$$

and the values of $r_1, r_2, \cdots, r_n$ run independently over all integers from 1 to n.

But the mth sum, if we interchange the K_{ij} by row and column as we can do since the S_{ik} are factors of each column, is precisely the definition of the product of the two matrices*

$$P_{mn} = \begin{Vmatrix} K_{11}, & K_{12}, & \cdots, K_{1n} \\ K_{21}, & K_{22}, & \cdots, K_{2n} \\ \cdot & \cdot & \cdot \quad \cdot \\ K_{m1}, & K_{m2}, & \cdots, K_{mn} \end{Vmatrix} \cdot \begin{Vmatrix} S_{11}, & S_{12}, & \cdots, S_{1n} \\ S_{21}, & S_{22}, & \cdots, S_{2n} \\ \cdot & \cdot & \cdot \\ S_{m1}, & S_{m2}, & \cdots, S_{mn} \end{Vmatrix}$$

Making use of the identity $\sum K_{ij} S_{kj} = K(t_i, t_k)$, we have the following remarkable reduction:†

$$\lim_{n=\infty} P_{mn} = |K(t_i, t_j)| \ ,$$

which, using the notation of Fredholm, we may designate by the symbol

$$K\begin{pmatrix} t_1, t_2, \cdots, t_m \\ t_1, t_2, \cdots, t_m \end{pmatrix} .$$

In the limiting case of (6.6) we then get the value of Fredholm's determinant,

$$\begin{aligned} \lim_{n=\infty} D(n,x) &= D(\lambda) = 1 - \lambda \int_a^b K(t_1,t_1)\, dt_1 \\ &+ \frac{\lambda^2}{2!} \int_b^a \int_b^a K\begin{pmatrix} t_1, t_2 \\ t_1, t_2 \end{pmatrix} dt_1 dt_2 + \cdots \qquad (6.7) \\ \cdots &+ \frac{(-1)^n}{n!} \lambda^n \int_a^b \cdots \int_a^b K\begin{pmatrix} t_1, t_2, \cdots, t_n \\ t_1, t_2, \cdots, t_n \end{pmatrix} dt_1 dt_2 \cdots dt_n + \cdots . \end{aligned}$$

We turn next to the computation of the cofactors of the elements of the first column of (6.4). These cofactors for brevity we may

---

*G. Kowalewski: *Einführung in die Determinantentheorie.* Leipzig, (1909), 6. 72.

†We are here using the convention that the product of two matrices is a determinant. See Kowalewski, *loc. cit.*, p. 68.

designate by D_i and the solution of the original integral equation as given by the method of segments will be, in terms of them, the limit as $n \to \infty$ of the following sum:

$$u_n(x) = [D_1 f(x) + D_2 f'(x) + D_3 f''(x) + \cdots + D_n f^{(n-1)}(x)]/D(n,x) \ . \qquad (6.8)$$

For our purpose we consider the determinant (2.3) of chapter 3 and the expansion of the cofactor of a_{sr}, $r \neq s$, which is given by (2.4) of that chapter.

Applying this formula to the elements of the first column of (6.4) above we have, for $i \neq 1$,

$$D_i = \lambda \int_a^b K_{11} S_{1i} dt_1 - \frac{\lambda^2}{1!} \int_a^b \int_a^b \sum_{r_1} \delta_{i1} dt_1 dt_2 + \cdots$$
$$\cdots + (-1)^{n-2} \frac{\lambda^{n-1}}{(n-2)!} \int_a^b \cdots \int_a^b \sum_{r_1 r_2 \ldots r_{n-2}} \delta_{i,n-1} dt_1 dt_2 \cdots dt_{n-1} \ ,$$

where we use the abbreviation

$$\delta_{i,\,m-1} = \begin{vmatrix} K_{11} & S_{1i} \ , K_{21} & S_{2r_1}, \cdots, K_{m1} & S_{mr_{m-1}} \\ K_{1r_1} & S_{1i} \ , K_{2r_1} & S_{2r_1}, \cdots, K_{mr_1} & S_{mr_{m-1}} \\ \cdot & \cdot & \cdot & \cdot \\ K_{1r_{m-1}} K_{1i} & , K_{2r_{m-1}} S_{2r_1}, & \cdots, K_{mr_{m-1}} & S_{mr_{m-1}} \end{vmatrix} .$$

D_1, the cofactor of the first element, is computed by observing that we have [referring to (2.3) of chapter 3],

$$\Delta_{11} = \Delta - a_{11}\Delta_{11} - a_{21}\Delta_{21} - \cdots - a_{n1}\Delta_{n1}$$
$$= \Delta - a_{11} - \frac{1}{1!}\sum \begin{vmatrix} a_{11}, & a_{1r_1} \\ a_{r_1 1}, & a_{r_1 r_1} \end{vmatrix} - \frac{1}{2!}\sum \begin{vmatrix} a_{11}, & a_{1r_1}, & a_{1r_2} \\ a_{r_1 1}, & a_{r_1 r_1}, & a_{r_1 r_2} \\ a_{r_2 1}, & a_{r_2 r_1}, & a_{r_2 r_2} \end{vmatrix} - \cdots$$
$$\cdots - \frac{1}{(n-1)!}\sum \begin{vmatrix} a_{11} & , a_{1r_1} & , \cdots, a_{1r_{n-1}} \\ a_{r_1 1} & , a_{r_1 r_1} & , \cdots, a_{r_1 r_{n-1}} \\ \cdot & \cdot & \cdot \\ a_{r_{n-1} 1}, & a_{r_{n-1} r_1}, & \cdots, a_{r_{n-1} r_{n-1}} \end{vmatrix} .$$

where $r_1, r_2, \cdots, r_{n-1}$ range over the values $2, 3, \cdots, n$.

Replacing the a_{ij} by the elements of (6.4) we get,

$$D_1 = D(n,x) + \lambda \int_a^b K(x,t_1)\, dt_1 - \frac{\lambda^2}{2!} \int_a^b \int_a^b \sum_{r_1} \delta_{11} dt_1 dt_2 + \cdots$$
$$\cdots + (-1)^{n-1} \frac{\lambda^n}{(n-1)!} \int_a^b \cdots \int_a^b \sum_{r_1 r_2 \ldots r_{n-1}} \delta_{1,n-1} dt_1 dt_2 \cdots dt_n$$

Substituting the values of D_i in the numerator of (6.8) we see that it can be written as follows

$$
\begin{aligned}
& f(x)D(n,x) + \lambda \int_a^b \sum_i K_{1i}S_{1i}f^{(i-1)}(x)\,dt_1 \\
& \quad -\frac{\lambda^2}{1!}\int_a^b \int_a^b \sum_{i r_1} \delta_{i1} f^{(i-1)}(x)\,dt_1 dt_2 + \cdots \qquad (6.9)\\
& \cdots + (-1)^{n-1}\frac{\lambda^{n-1}}{(n-2)!}\int_a^b \cdots \int_a^b \sum_{i r_1 r_2 \ldots r_{n-2}} \delta_{i,n-2} f^{(i-1)}(x)\,dt_1 dt_2 \cdots dt_{n-1} \\
& \quad + (-1)^{n-1}\frac{\lambda^n}{(n-1)!}\int_a^b \cdots \int_a^b \sum_{r_1 r_2 \ldots r_{n-1}} \delta_{i,n-1} f(x)\,dt_1 dt_2 \cdots dt_n ,
\end{aligned}
$$

where $r_1\ r_2, \cdots, r_{n-1}$ run over all values from 2 to n and i runs from 1 to n.

In simplifying this expression we note the two following identities:

$$\sum_{i=1}^{\infty} S_{1i} f^{(i-1)}(x) = f(t_1) ,$$

and

$$
\begin{Vmatrix} K(x,t_1), & K'(x,t_1), & \cdots \\ K(x,t_2), & K'(x,t_2), & \cdots \\ \cdot & \cdot & \cdot \\ K(x,t_n), & K'(x,t_n), & \cdots \end{Vmatrix}
\begin{Vmatrix} 1, & 0, & 0, & \cdots \\ 1, & t_2 - x, & \dfrac{(t_2-x)^2}{2!}, & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ 1, & t_n - x, & \dfrac{(t_n-x)^2}{2!}, & \cdots \end{Vmatrix}
$$

$$
= \begin{vmatrix} K(x,t_1), & K(x,t_2), & \cdots, & K(x,t_n) \\ K(t_2,t_1), & K(t_2,t_2), & \cdots, & K(t_2,t_n) \\ \cdot & \cdot & \cdot & \cdot \\ K(t_n,t_1), & K(t_n,t_2), & \cdots, & K(t_n,t_n) \end{vmatrix} .
$$

It is convenient to use Fredholm's abbreviation

$$K\begin{pmatrix} x, t_2, \cdots, t_n \\ t_1, t_2, \cdots, t_n \end{pmatrix}$$

for this last determinant.

If now in (6.9) we let n approach infinity and make use of the two identities just written down, we shall have, since S_{1i} is a factor of each determinant, the following limit:

$$f(x)D(\lambda)+\lambda\int_a^b K(x,t)f(t)\,dt-\frac{\lambda^2}{1!}\int_a^b\int_a^b K\begin{pmatrix}x, t_2\\ t_1, t_2\end{pmatrix}f(t_1)\,dt_1dt_2+\cdots$$

$$+(-1)^{n-1}\frac{\lambda^n}{(n-1)!}\int_a^b\cdots\int_b^a K\begin{pmatrix}x, t_2, \cdots, t_n\\ t_1, t_2, \cdots, t_n\end{pmatrix}f(t_1)\,dt_1dt_2\cdots dt_n+\cdots .$$

The limiting form of (6.8) is thus seen to be

$$\lim_{n=\infty} u_n(x)=u(x)=f(x)+\lambda\int_a^b\frac{D(x,t;\lambda)}{D(\lambda)}f(t)\,dt\,, \tag{6.10}$$

where we employ the usual abbreviation

$$D(x,y;\lambda)=K(x,y)-\frac{\lambda}{1!}\int_a^b K\begin{pmatrix}x, t_2\\ y, t_2\end{pmatrix}dt_2$$

$$+\frac{\lambda^2}{2!}\int_a^b\int_a^b K\begin{pmatrix}x, t_2, t_3\\ y, t_2, t_3\end{pmatrix}dt_2dt_3-\cdots . \tag{6.11}$$

The formal verification that (6.10) is actually a solution of equation (6.1) when $D(\lambda)\neq 0$ can be made by direct substitution.

7. *The Resolvent Kernel and the Fredholm Minors.* The function

$$\begin{aligned} k(x,t;\lambda) &= D(x,t;\lambda)/D(\lambda) \qquad (7.1)\\ &= K(x,t)+\lambda K_2(x,t)+\lambda^2K_3(x,t)+\cdots \end{aligned}$$

is called the *resolvent kernel* of equation (6.1), since it yields the solution [see equation (6.10)] of the non-homogeneous equation.

In order to show that $k(x,t;\lambda)$ exists for all values of λ for which $D(\lambda)$ does not vanish, it will be necessary to prove the following theorem:

Theorem 5. If $K(x,t)$ is of limited variation and integrable in the square $a\leqq x,\ t\leqq b$, then both $D(x,t;\lambda)$ and $D(\lambda)$ are entire functions of λ of genus not greater than 2.

Proof: Let us designate by K the maximum value of the kernel in the fundamental square. Then by Hadamard's theorem (see section 4, chapter 3), we have

$$\left| K\begin{pmatrix}t_1\, t_2 \cdots t_n\\ t_1\, t_2 \cdots t_n\end{pmatrix}\right| \leqq K^n n^{\frac{1}{2}n} .$$

Hence the nth term of (6.7) is inferior to $a_n=\lambda^nK^n n^{\frac{1}{2}n}|b-a|^n/n!$. Since we have by Stirling's approximation

$$(a_n)^{1/n}\backsim \lambda K|b-a|/n^{\frac{1}{2}} ,$$

it is clear that $D(\lambda)$ is an entire function of genus at most equal to 2. The argument applies without change to $D(x, t; \lambda)$.

We next observe the important, but obviously derived, relation

$$\frac{d}{d\lambda} D(\lambda) = - \int_a^b D(t, t; \lambda)\, dt \; . \tag{7.2}$$

Making use of (6.7) and (7.1), we obtain the expansions

$$\frac{D'(\lambda)}{D(\lambda)} = -(S_1 + \lambda S_2 + \lambda^2 S_3 + \lambda^3 S_4 + \cdots) \; , \tag{7.3}$$

$$\log D(\lambda) = -(\lambda S_1 + \frac{\lambda}{2} S_2 + \frac{\lambda^2}{3} S_3 + \frac{\lambda^3}{4} S_4 + \cdots) \; . \tag{7.4}$$

The quantities

$$S_p = \int_a^b K_p(t,t)\, dt$$

are called the *traces* of the kernel.

We shall find it useful later to have certain identities between the kernel and its resolvent. Let us first observe that

$$\begin{aligned} k(x, t; \lambda) &= K(x, t) + \lambda \int_a^b K(x, s)\,[K(s, t) + \lambda K_2(s, t) + \cdots]\, ds \; , \\ &= K(x, t) + \lambda \int_a^b K(x, s)\, k(s, t; \lambda)\, ds \; , \end{aligned} \tag{7.5}$$

and similarly,

$$\begin{aligned} k(x, t; \lambda) &= K(x, t) + \lambda \int_a^b [K(x,s) + \lambda K_2(x,s) + \cdots]\, K(s,t)\, ds \; , \\ &= K(x, t) + \lambda \int_a^b k(x, s; \lambda)\, K(s, t)\, ds \; . \end{aligned} \tag{7.6}$$

From these equations we immediately derive the important relations

$$\begin{aligned} k(x, t; \lambda) - K(x,t) &= \lambda \int_a^b K(x, s)\, k(s, t; \lambda)\, ds \\ &= \lambda \int_a^b k(t, s; \lambda)\, K(s, t)\, ds \; . \end{aligned} \tag{7.7}$$

Employing the equations just established, we can prove the following elegant identity:

$$k(x, t; \mu) - k(x, t; \lambda) = (\mu - \lambda) \int_a^b k(x, s; \lambda)\, k(s, t; \mu)\, ds \; . \tag{7.8}$$

Using for convenience the notation of Volterra's theory of functions of composition (see section 4, chapter 4), we can write (7.7) in the symbolic form

$$\begin{aligned} k_\lambda - K &= \lambda\, K * k_\lambda = \lambda\, k_\lambda * K \ , \\ k_\mu - K &= \mu\, K * k_\mu = \mu\, k_\mu * K \ . \end{aligned} \tag{7.9}$$

Subtracting the first of these from the second, we obtain

$$k_\mu - k_\lambda = \mu\, K * k_\mu - \lambda\, K * k_\lambda \ . \tag{7.10}$$

Now multiply the first equation in (7.9) on the right by $\mu\, k_\mu$ and the second equation on the left by $\lambda\, k_\lambda$. We thus get

$$\mu\, k_\lambda * k_\mu - \mu\, K * k_\mu = \lambda\, \mu\, K * k_\lambda * k_\mu \ ,$$

$$\lambda\, k_\lambda * k_\mu - \lambda\, k_\lambda * K = \lambda\, \mu\, k_\lambda * K * k_\mu \ .$$

From the permutability of k with K, this second equation may be written

$$\lambda\, k_\lambda * k_\mu - \lambda\, K * k_\lambda = \lambda\, \mu\, K * k_\lambda * k_\mu \ .$$

Subtracting this equation from the first and noting (7.10), we finally obtain the symbolic equivalent of (7.8)

$$(\mu - \lambda)\ k_\lambda * k_\mu = \mu\, K * k_\mu - \lambda\, K * k_\lambda = k_\mu - k_\lambda \ .$$

From the identities (7.7) we reach the conclusion that the solution of the equation *adjoint* to (6.1), namely,

$$u(x) = f(x) + \lambda \int_a^b K(t, x)\ u(t)\ dt \ , \tag{7.11}$$

is given by

$$u(x) = f(x) + \lambda \int_a^b k(t, x; \lambda)\ f(t)\ dt \ . \tag{7.12}$$

Moreover, if we replace $k(x, t; \lambda)$ by $D(x, t; \lambda)/D(\lambda)$, we shall obtain

$$\begin{aligned} D(x, t; \lambda) - D(\lambda)\ K(x, t) &= \lambda \int_a^b K(x, s)\ D(s, t; \lambda)\ ds \\ &= \lambda \int_a^b K(s, x) D(t, s; \lambda)\ ds \ . \end{aligned} \tag{7.13}$$

The theory which we have sketched above can be extended in an analogous manner to the higher minors of $D(\lambda)$. These functions, called the *minors of Fredholm*, are defined by the series

$$D\begin{pmatrix} x_1\, x_2 \cdots x_n \\ t_1\, t_2 \cdots t_n \end{pmatrix} = K\begin{pmatrix} x_1\, x_2 \cdots x_n \\ t_1\, t_2 \cdots t_n \end{pmatrix}$$

$$+\sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int_a^b \cdots \int_a^b K\begin{pmatrix} x_1\, x_2 \cdots x_n\, s_1\, s_2 \cdots s_p \\ t_1\, t_2 \cdots t_n\, s_1\, s_2 \cdots s_p \end{pmatrix} ds_1\, ds_2 \cdots ds_p\ .$$

These minors may be shown to have the following expansions:

$$D\begin{pmatrix} x_1 x_2 \cdots x_n \\ t_1\, t_2 \cdots t_n \end{pmatrix} = K(x_1,t_1)\, D\begin{pmatrix} x_2 \cdots x_n \\ t_2 \cdots t_n \end{pmatrix} - K(x_1, t_2)\, D\begin{pmatrix} x_2 x_3 \cdots x_n \\ t_1\, t_3 \cdots t_n \end{pmatrix}$$

$$+ \cdots + (-1)^{n-1} K(x_1, t_n)\, D\begin{pmatrix} x_2 \cdots x_n \\ t_1 \cdots t_{n-1} \end{pmatrix}$$

$$+ \lambda \int_a^b K(x_1, s)\, D\begin{pmatrix} s\, x_2 \cdots x_n \\ t_1 t_2 \cdots t_n \end{pmatrix} ds\ ;$$

and also

$$D\begin{pmatrix} x_1 x_2 \cdots x_n \\ t_1\, t_2 \cdots t_n \end{pmatrix} = K(x_1, t_1)\, D\begin{pmatrix} x_2 \cdots x_n \\ t_2 \cdots t_n \end{pmatrix} - K(x_2, t_1)\, D\begin{pmatrix} x_1 x_3 \cdots x_n \\ t_2 t_3 \cdots t_n \end{pmatrix}$$

$$+ \cdots + (-1)^{n-1} K(x_n, t_1)\, D\begin{pmatrix} x_1 \cdots x_{n-1} \\ t_1 \cdots t_n \end{pmatrix}$$

$$+ \lambda \int_a^b K(s, t_1)\, D\begin{pmatrix} x_1 x_2 \cdots x_n \\ s\, t_2 \cdots t_n \end{pmatrix} ds\ .$$

The first of these expressions is established from the development of the determinant

$$K\begin{pmatrix} x_1 x_2 \cdots x_n s_1 s_2 \cdots s_p \\ t_1 t_2 \cdots t_n s_1 s_2 \cdots s_p \end{pmatrix} \tag{7.14}$$

by the elements of the first row. This expansion is then multiplied by $[(-1)^p/p!]\ ds_1\, ds_2 \cdots ds_p$ and integrated with respect to each variable from a to b. The sum of these integrals from 1 to ∞ yields the desired expression. The second expansion is obtained similarly from the development of (7.14) by the elements of the first column.

The nth derivative of $D(\lambda)$ can be obtained from the nth Fredholm minor. We thus have

$$\frac{d^n}{d\lambda^n} D(\lambda) = (-1)^n \int_a^b \cdots \int_a^b D\begin{pmatrix} t_1 t_2 \cdots t_n \\ t_1 t_2 \cdots t_n \end{pmatrix} dt_1\, dt_2 \cdots dt_n\ . \tag{7.15}$$

PROBLEMS

1. Compute $D(\lambda)$ for the following kernels with respect to the indicated intervals:

(a) $K(x,t) = x - t,\ (0, 1)$; (b) $K(x,t) = x + t,\ (0, 1)$;

(c) $K(x,t) = xt + x^2t^2 + x^3t^3,\ (0, 1)$; (d) $K(x,t) = Ax^2 + 2Bxt + Ct^2 + Dx + Et + F,\ (a, b)$.

2. Prove that

$$K_{m+n}(x,y) = \int_a^b K_m(x,t)\, K_n(t,y)\, dt .$$

3. Prove identity (7.8) by noting

$$\begin{aligned} k(x,s;\mu)\, k(s,t;\lambda) = K(x,s)\, K(s,t) &+ \mu K_2(x,s)\, K(s,t) + \mu^2 K_3(x,s)\, K(s,t) \\ &+ \lambda K(x,s) K_2(s,t) + \mu\lambda K_2(x,s)\, K_2(s,t) \\ &+ \lambda^2 K(x,s)\, K_3(s,t) \\ &+ \cdots\cdots \end{aligned}$$

Now integrate with respect to s from a to b and observe that

$$\frac{\mu^2 - \lambda^2}{\mu - \lambda} = \mu + \lambda , \qquad \frac{\mu^3 - \lambda^3}{\mu - \lambda} = \mu^2 + \mu\lambda + \lambda^2 , \qquad \text{etc.}$$

(Kowalewski).

4. Show that the nth iterated kernel formed from

$$K(x,y) = (x - y)^{-a}\, \phi(x,y) , \qquad a < 1 ,$$

where $\phi(x,y)$ is bounded in the fundamental square, is bounded when $n > 1/(1 - a)$. (Fredholm).

5. Prove that

$$\frac{\partial^n k(x,t;\lambda)}{\partial\lambda^n} = n!\, k_{n+1}(x,t;\lambda) ,$$

where we define

$$k_n(x,t;\lambda) = \int_a^b k(x,s;\lambda)\, k_{n-1}(s,t;\lambda)\, ds .$$

6. From the results of problem 5 prove that

$$\begin{aligned} k(x,t;\lambda + m) &= k(x,t;\mu) + \lambda k_2(x,t;\mu) + \lambda^2 k_3(x,t;\mu) + \cdots \\ &= k(x,t;\mu) + \lambda \int_a^b k(x,s;\mu)\, k(s,t;\lambda + \mu)\, ds . \end{aligned}$$

7. If $D(\lambda)$ has no zero, prove that the traces after the third are zero.

Hint: Since $D(\lambda)$ is at most of genus 2, it must have the form

$$D(\lambda) = \exp(a\lambda + b\lambda^2) .$$

8. The functions $K(x,y)$, $L(x,y)$ are called *orthogonal* provided

$$\int_a^b K(x,s)\, L(s,y)\, ds = 0 , \quad \int_a^b L(x,s)\, K(s,y)\, ds = 0 .$$

If only one of these relations holds the functions are said to be *semi-orthogonal.*

Prove that if

$$M(x,y) = K(x,y) + L(x,y) \ ,$$

where K and L are orthogonal, then

$$D_M(\lambda) = D_K(\lambda) \, D_L(\lambda) \ .$$

Show also that

$$m(x,y;\lambda) = k(x,y;\lambda) + l(x,y;\lambda) \ ,$$

where m, k, and l are respectively the resolvents of M, K, and L.

9. If $D_1(\lambda)$ is the Fredholm determinant corresponding to a kernel $K_1(x,t)$ and if $D_2(\lambda)$ is the Fredholm determinant corresponding to a second kernel $K_2(x,t)$, show that

$$\frac{d}{d\lambda}\left(\frac{D_1}{D_2}\right) = -\frac{D_1}{D_2}\int_a^b [K_1(t,t) - K_2(t,t)] \, dt \ .$$

Hence prove that if a sequence of zeros exists for $D_1(\lambda)$, then a second sequence of zeros, either coincident with or separated by those of $D_1(\lambda)$, will exist for $D_2(\lambda)$, provided

$$\int_a^b [K_1(t,t) - K_2(t,t)] \, dt \neq 0 \ .$$

10. Given the two systems

$$u'' + \lambda u = 0 \ ,$$

$$\text{I.} \quad \begin{cases} u(0) = 0 \ , \\ u(1) = 0 \ , \end{cases} \qquad\qquad \text{II.} \quad \begin{cases} u(0) = 0 \ , \\ u'(1) = 0 \ , \end{cases}$$

prove that

$$\int_0^1 [\Gamma_1(t,t) - \Gamma_2(t,t)] \, dt = -1/3 \ ,$$

where $\Gamma_1(x,t)$ and $\Gamma_2(x,t)$ are the Green's functions respectively of systems I and II. Show by explicit calculation that the values of λ for which the two systems are respectively consistent separate one another.

11. If $\Gamma_1(x,t)$ and $\Gamma_2(x,t)$ are the Green's functions for the following two systems:

$$(pu')' + qu + \lambda u = 0$$

$$\text{I.} \quad \begin{cases} u(a) = 0 \ , \\ u(b) = 0 \ , \end{cases} \qquad\qquad \text{II.} \quad \begin{cases} u(a) - u(b) = 0 \ , \\ u'(a) - u'(b) = 0 \ , \end{cases}$$

and if $u_1(x)$, $u_2(x)$ form a fundamental set of solutions of the equation $(pu')' + q\,u = 0$, such that $u_1(a) = 0$, $u_1'(a) = 1$, $u_2(a) = 1$, $u_2'(a) = 0$, show that

$$\int_a^b [\Gamma_1(t,t) - \Gamma_2(t,t)] \, dt = \frac{1}{D_1 D_2} \int_a^b \{ [u_2^2(b) - (1+p_1)\,u_2(b) + p_1] \, u_1^2(t)$$

$$- [2u_1(b)\,u_2(b) - (p_1+1)\,u_1(b)] \, u_1(t)\,u_2(t) + u_1^2(b)\,u_2^2(t) \} \, dt \ ,$$

where D_1 and D_2 are the functions $|U_{ij}|$ (see section 5) corresponding respectively to each of the two systems, and where $p_1 = p(a)/p(b)$.

Prove that the quadratic form under the integral sign is definite provided $p(a) = p(b)$, that is to say, provided system II is self-adjoint. Hence show that the zeros of D_1 and D_2 alternate provided system II is self-adjoint.

12. If $D_n(\lambda)$ is the Fredholm determinant which corresponds to the nth iterated kernel, $K_n(x,t)$, prove the identity

$$D_n(\lambda) = e^{-(S_n\lambda + S_{2n}\lambda^2/2 + \cdots + S_{pn}\lambda^p/p + \cdots)} .$$

13. From the results of problem 12 prove the identity

$$D_n(\lambda^n) = D(\lambda)\, D(\lambda\omega) \cdots D(\omega^{n-1}\lambda) ,$$

where ω is a primitive nth root of unity.

14. If $K(x,y)$ is a kernel of the form

$$K(x,y) = \sum_{i=1}^{n} X_i(x)\, Y_i(y) ,$$

where the functions $X_i(x)$ and $Y_i(y)$ are integrable over the fundamental square, show that the Fredholm minor is a polynomial of degree n. Prove also that $D(x,y)$ is a bilinear form in $X_i(x)$ and $Y_i(y)$.

CHAPTER XII

THE THEORY OF SPECTRA

1. Introduction—Examples of Spectra. In the preceding chapters of this book the principal concern was with the problem of the inversion of operators, that is to say, the problem of solving for X in the equation

$$X(x,z) \to F(x,z) = 1 \ . \tag{1.1}$$

In the course of this investigation it was found that $X(x,z)$ was not always unique, and we were led to consider the problem of the homogeneous equation

$$F(x,z) \to u(x) = 0 \ . \tag{1.2}$$

The solution of this problem usually introduced certain *characteristic values* with which were associated the *characteristic functions,* or solutions, of (1.2).

It will be illuminating before we enter into more general aspects of the problem to consider several examples, which have been selected as typical of the difficulties in the extended theory.

Example 1. Let us begin with the theory of the Fredholm integral equation given in the preceding chapter,

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\,dt \ , \tag{1.3}$$

the solution of which appears in the form

$$u(x) = f(x) + \lambda \int_a^b \frac{D(x,s;\lambda)}{D(\lambda)} f(s)\,ds \ . \tag{1.4}$$

In this expression the functions $D(x,s;\lambda)$ and $D(\lambda)$, given explicitly by (6.11) and (6.7) of chapter 11, are entire functions of the parameter λ provided $K(x,t)$ is bounded and integrable in the rectangle $a \leqq x \leqq b$, $a \leqq t \leqq b$. It will be clear, however, that in general a solution of (1.3) will not exist for values of λ for which $D(\lambda) = 0$. These roots, $\lambda_1, \lambda_2, \cdots, \lambda_n$, are called the *principal or proper values* (*Eigenwerte*) of the kernel, $K(x,t)$ and together form what is known as the *spectrum* of the integral equation. Although the principal values are frequently referred to as *characteristic numbers,* D. Hilbert has suggested that their reciprocals, namely, $\mu_i = 1/\lambda_i$, should be called by this name (*charakteristischen Zahlen*). Because of obvious advantage in being able to refer separately to λ and its reciprocal, we shall adopt this nomenclature here.

Associated with the principal values there exists a set of *principal or proper functions* (*Eigenfunktionen*) $u_1(x), u_2(x), \cdots, u_n(x)$, which are solutions of the homogeneous equation

$$u(x) = \lambda \int_a^b K(x,t)\,u(t)\,dt \ , \tag{1.5}$$

when λ is replaced successively by the principal values.

In the subsequent development of this example it will be convenient to consider the principal values in relation to kernels of special types. These may be defined as follows:

(1) A *symmetric* kernel is one for which

$$K(x,y) \equiv K(y,x) \ .$$

(2) A *skew-symmetric* kernel is defined by the identity

$$K(x,y) \equiv -K(y,x) \ .$$

(3) A symmetric kernel is said to be *positive definite* provided

$$I \equiv \int_a^b \int_a^b K(x,y)\,h(x)\,h(y)\,dx\,dy > 0 \ ,$$

for all functions $h(x)$, $h(y)$ of integrable square. The kernel is called *negative definite* provided $I < 0$. If $I \geqq 0$, or $I \leqq 0$, then the kernel is called *positive* or *negative* respectively.

(4) Symmetric kernels belong to a more general class called *Hermitian kernels,* that is to say, kernels for which

$$K(x,y) \equiv \overline{K}(y,x) \ ,$$

where $\overline{K}$ denotes the complex conjugate of K.

A kernel is said to *belong to class* p provided the integral

$$\int_a^b \int_a^b | K(x,y) |^p \, dx\,dy$$

exists. If $p = 2$, then $K(x,y)$ belongs to Hilbert space.

The principal values may be characterized by means of the kernels to which they belong.

Thus it may be proved that at least one principal value exists provided a kernel is symmetric or hermitian. Moreover, all the principal values of such kernels are real. If a symmetric kernel is also positive (or negative) definite, then all the principal values are positive (or negative). If a kernel is skew-symmetric, then at least two principal values exist and these are pure imaginaries.

The sum

$$S_p(\lambda) = \frac{1}{\lambda_1^p} + \frac{1}{\lambda_2^p} + \frac{1}{\lambda_3^p} + \frac{1}{\lambda_4^p} + \cdots$$

is of major importance in the classification of kernels. Thus, if the kernel is of class 2, then $S_2(\lambda)$ converges; if $K(x,y)$ is a kernel formed by the iteration of two kernels of class 2, then $S_1(\lambda)$ converges; if $K(x,y)$ is positive (or negative) definite, then $S_1(\lambda)$ converges.

Example 2. As we have seen in the preceding chapter (see section 5), the solution of the differential system

$$\begin{aligned} L(u) &= \lambda\, u(x) \\ U_i(u) &= 0 \ , \qquad i = 1, 2, \cdots, n \ , \end{aligned} \tag{1.6}$$

where $L(u)$ and $U_i(u)$ are defined in (5.1) of chapter 11, does not exist for all values of λ. However, if $u_1(x,\lambda)$, $u_2(x,\lambda)$, $\cdots$, $u_n(x,\lambda)$ form a set of linearly independent solutions of (1.6), then the zeros of the function $D(\lambda)$, where we define

$$D(\lambda) = | U_i[u_j(\lambda)] | \ ,$$

form a spectrum for which solutions of (1.6) in general exist. These solutions are also solutions of the integral equation

$$u(x) = \lambda \int_a^b G(x,t) u(t)\, dt \ ,$$

where $G(x,t)$ is the Green's function associated with the differential form $L(u)$ and λ belongs to the spectrum.

For example, the system

$$\begin{aligned} u''(x) &= -\lambda\, u(x) \\ u(a) &= u(b) = 0 \end{aligned}$$

has the spectral function

$$D(\lambda) = [\sin \sqrt{\lambda}\,(b-a)]/\sqrt{\lambda} \ ,$$

from which one obtains the infinite set of principal values

$$\lambda_n = n^2 \pi^2/(b-a)^2 \ , \quad n = 1, 2, 3, \cdots \ .$$

If we make the simplifying assumption that $a = 0$, $b = 1$, then the kernel of the integral equation becomes [see (5.7), chapter 11]

$$G(x,t) = \begin{cases} x(1-t) \ , & x \leqq t \ , \\ t(1-x) \ , & x \geqq t \ . \end{cases}$$

We now note that $S_1(\lambda)$ as defined in the first example is equal to the integral of $G(t,t)$,

$$S_1(\lambda) = \frac{\pi^2}{1} + \frac{\pi^2}{4} + \frac{\pi^2}{9} + \cdots = \int_0^1 G(t,t)\,dt = 1/6 \ .$$

The principal functions, *normalized* for the interval (0,1), that is to say, so that

$$\int_0^1 u_n^2(x)\,dx = 1 \ ,$$

are given by

$$u_n(x) = \sqrt{2}\sin n\pi x \ , \quad n = 1, 2, 3, \cdots \ .$$

Let us now consider the sum

$$K(x,t) = \sum_{n=1}^{\infty} \frac{u_n(x)\,u_n(t)}{\lambda_n} = \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi t}{n^2} \ .$$

It is well known* that

$$\sum_{n=1}^{\infty} \frac{\cos n\pi X}{\pi^2\,n^2} = \tfrac{1}{4}(X^2 - 2X + 2/3) \ , \quad 0 \leqq X \leqq 2 \ .$$

If in this identity we now set $X = t - x$, and $X = t + x$, where $x < t$, $0 \leqq x$, $t \leqq 1$, we shall obtain

$$\sum_{n=1}^{\infty} \frac{\cos n\pi x \cos n\pi t}{\pi^2\,n^2} + \tfrac{1}{2}K(x,t)$$
$$= \tfrac{1}{4}(x^2 - 2xt + t^2 - 2t + 2x + 2/3) \ ,$$

$$\sum_{n=1}^{\infty} \frac{\cos n\pi x \cos n\pi t}{\pi^2\,n^2} - \tfrac{1}{2}K(x,t)$$
$$= \tfrac{1}{4}(x^2 + 2xt + t^2 - 2t - 2x + 2/3) \ .$$

Subtracting the second of these equations from the first, we obviously obtain

$$K(x,t) = x(1 - t) \ , \quad x < t \ .$$

Interchanging the rôle of x and t, we similarly find

$$K(x,t) = t(1 - x) \ , \quad x > t \ .$$

In this manner we show that $K(x,t)$ is identical with the Green's function $G(x,t)$ as previously defined.

*See H. T. Davis: *Tables of the Higher Mathematical Functions*, vol. 2, p. 18.

Example 3. Another example of a somewhat different kind is furnished by a theory due to E. Schrödinger* in which proper values are introduced in order to account for the discrete nature of radiant energy, which from the phenomena of diffraction and polarization must also be described by means of the continuous solutions of Maxwell's well known equations.

Basic to the Schrödinger tradition of the quantum theory we find the so-called *wave equation*

$$\nabla^2 \psi + \frac{8\pi^2 m}{h^2}(W - V)\psi = 0 \; . \tag{1.8}$$

In this equation m is the mass of a moving particle, $h = 6.547 \times 10^{-27}$ erg secs, is Planck's constant, W is the total energy of the particle, $V(x,y,z)$ is the potential energy of the particle, and ψ represents the space part of the function

$$\Psi = \psi(x,y,z)\, e^{2\pi i \nu t} \; ,$$

which describes the undulatory principle associated with the particle.

Our purpose here will be served by the simplest specialization, namely, that of the *linear oscillator*. The potential energy of such an oscillator is equal to $\frac{1}{2}\, a\, x^2$ and equation (1.8), being now restricted to a single dimension, reduces to

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(W - \tfrac{1}{2}ax^2)\psi = 0 \; . \tag{1.9}$$

In this equation W is to be regarded as a parameter, which is to be determined in such a manner that $\psi(x)$ exists as a single valued, continuous function throughout the length of the real axis and is further subjected to the boundary values

$$\lim_{x=-\infty} \psi(x) = 0 \; , \quad \lim_{x=+\infty} \psi(x) = 0 \; . \tag{1.10}$$

If we make the transformation: $x = kt$, $\psi(kt) = u(t)$, $k^2 = h/[4\pi(am)^{\frac{1}{2}}]$, and employ the abbreviation

$$n + \tfrac{1}{2} = \frac{2\pi\sqrt{m}\, W}{h\sqrt{a}} \; , \tag{1.11}$$

then equation (1.9) becomes

$$\frac{d^2u(t)}{dt^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}t^2)u(t) = 0 \; . \tag{1.12}$$

*Quantisierung als Eigenwertproblem. *Annalen der Physik* (4), vol. 79 (1926), pp. 361-376; 489-527; vol. 80, pp. 437-490; vol. 81, pp. 109-139. English translation by J. F. Shearer and W. M. Deans, (1928).

This is the standard form of what it called *Weber's equation.** Two linearly independent solutions of this equation exist in the form

$$u_1(t) = e^{-\frac{1}{4}t^2}[1 - \frac{n}{2!}t^2 + \frac{n(n-2)}{4!}t^4 - \frac{n(n-2)(n-4)}{6!}t^6 + \cdots]$$

$$u_2(t) = e^{-\frac{1}{4}t^2}[t - \frac{(n-1)}{3!}t^3 + \frac{(n-1)(n-3)}{5!}t^5 - \frac{(n-1)(n-3)(n-5)}{7!}t^7 + \cdots] .$$

Moreover, since the equation (1.12) remains unchanged if t is replaced by it and n by $-n-1$, a second set of linearly independent solutions exists of the form

$$v_1(t) = e^{\frac{1}{4}t^2}[1 - \frac{(n+1)}{2!}t^2 + \frac{(n+1)(n+3)}{4!}t^4 - \frac{(n+1)(n+3)(n+5)}{6!}t^6 + \cdots]$$

$$v_2(t) = e^{\frac{1}{4}t^2}[t - \frac{(n+2)}{3!}t^3 + \frac{(n+2)(n+4)}{5!}t^5 - \frac{(n+2)(n+4)(n+6)}{7!}t^7 + \cdots] .$$

The second set of solutions is, of course, linearly dependent upon the first.

It is not convenient to examine the asymptotic behaviour of the general solution of (1.12) from the asymptotic expansions of $u_1(t)$ and $u_2(t)$, but rather from those of the functions

$$D_n(t) = A\,u_1(t) + B\,u_2(t) ,$$

$$A = \frac{\Gamma(\frac{1}{2})2^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}-\frac{1}{2}n)} , \quad B = \frac{\Gamma(-\frac{1}{2})2^{\frac{1}{2}(n-1)}}{\Gamma(-\frac{1}{2}n)} ;$$

$$D_{-n-1}(it) = C\,v_1(t) + D\,v_2(t) ,$$

$$C = \frac{\Gamma(\frac{1}{2})2^{-\frac{1}{2}(n+1)}}{\Gamma(1+\frac{1}{2}n)} , \quad D = \frac{i\,\Gamma(-\frac{1}{2})2^{-\frac{1}{2}(n+1)}}{\Gamma(\frac{1}{2}+\frac{1}{2}n)} .$$

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*This equation is due to H. Weber: *Mathematische Annalen*, vol. 1 (1869), pp. 1-36; in part., p. 29. It has been standardized by E. T. Whittaker: *Proc. of the London Math. Soc.*, vol. 35 (1st series), (1903), p. 417-427. An extensive account of the equation is found in Whittaker and Watson: *Modern Analysis*, § § 16.5-16.7.

By the methods of section 4, chapter 5, it can be shown that $D_n(t)$ has the following asymptotic expansion, the first valid in the sector: $|\arg t| < 3\pi/4$, the second in the sector: $\pi/4 < \arg t < 5\pi/4$:

$$D_n(t) \backsim e^{-\frac{1}{4}t^2}\, t^n \left\{1 - \frac{n(n-1)}{2t^2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4\, t^4} - \cdots \right\},$$

$$D_n(t) \backsim e^{-\frac{1}{4}t^2}\, t^n \left\{1 - \frac{n(n-1)}{2t^2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4\, t^4} - \cdots \right\}$$
$$- \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-n)}\, e^{n\pi i}\, e^{\frac{1}{4}t^2}\, t^{-n-1} \left\{ 1 + \frac{(n+1)(n+2)}{2t^2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4\, t^4} + \cdots \right\}. \tag{1.13}$$

Making the substitution: it for t and $-n-1$ for n, we obtain the corresponding expansions for $D_{-n-1}(it)$, the first valid in the sector: $5\pi/4 < \arg t < -\pi/4$, the second valid in the sector: $7\pi/4 < \arg t < 3\pi/4$.

$$D_{-n-1}(it) \backsim e^{\frac{1}{4}t^2}\, e^{\frac{1}{2}n\pi i}\, t^n \left\{1 + \frac{(n+1)(n+2)}{2t^2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4\, t^4} + \cdots \right\},$$

$$D_{-n-1}(it) \backsim e^{\frac{1}{4}t^2}\, e^{\frac{1}{2}n\pi i}\, t^n \left\{1 + \frac{(n+1)(n+2)}{2t^2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4\, t^4} + \cdots \right\}$$
$$+ \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(n+1)}\, e^{-\frac{1}{2}n\pi i}\, e^{-\frac{1}{4}t^2} \left\{1 - \frac{n(n-1)}{2t^2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4\, t^4} - \cdots \right\}.$$

From these explicit formulas, it is clear that the general solution of equation (1.12),

$$u(t) = a\, D_n(t) + b\, D_{-n-1}(it),$$

cannot fulfill the boundary conditions

$$\lim_{t=-\infty} u(t) = \lim_{t=+\infty} u(t) = 0,$$

unless $b = 0$. The first form of (1.13), valid on the positive axis of

t, clearly fulfills the first condition of the problem. The second form of (1.13), valid on the negative axis of t, does not fulfill the second condition of the problem unless the second term in the expansion is identically zero. This can happen only if $1/\Gamma(-n) \equiv 0$, which means that n must be zero or a positive integer. Hence *the proper values are positive integers or zero.*

Returning to equation (1.11), we see that permissible values for W are given by

$$W = \frac{h}{2\pi}\sqrt{\frac{a}{m}}\,(n+\tfrac{1}{2}) \quad , \quad n = 0, 1, 2, \cdots .$$

We can now construct a normalized set of proper functions by writing

$$\psi_n(x) = K\, D_n(x/k) \quad ,$$

where K is to be so determined that

$$\int_{-\infty}^{\infty} \psi_n^2(x)\, dx = 1 \ .$$

We note that $D_n(t) = e^{-\frac{1}{4}t^2} h_n(t)$, where $h_n(t)$ are the Hermite polynomials. The first six of these are given explicitly as follows:

$$h_0(t) = 1\,,\ h_1(t) = t\,,\ h_2(t) = t^2 - 1\,,\ h_3(t) = t^3 - 3t \ ,$$

$$h_4(t) = t^4 - 6t^2 + 3\,,\ h_5(t) = t^5 - 10t^3 + 15t \ ,$$

and those of higher degree may be computed from the recurrence formula

$$h_{n+1}(t) - t\, h_n(t) + n\, h_{n-1}(t) = 0 \ .$$

It is well known* that the functions $D_n(t)$ satisfy the following orthogonality conditions

$$\int_{-\infty}^{\infty} D_m(t)\, D_n(t)\, dt = \begin{cases} 0 & , \quad m \neq n \\ (2\pi)^{\frac{1}{2}}\, n! \ , & \quad m = n \ . \end{cases}$$

Hence we obtain

$$\int_{-\infty}^{\infty} \psi_n^2(x)\, dx = K^2 \int_{-\infty}^{\infty} [D_n(x/k)]^2\, dx = k\, K^2 (2\pi)^{\frac{1}{2}}\, n! = 1 \ ,$$

from which we immediately compute

$$K^2 = 1/[k(2\pi)^{\frac{1}{2}}\, n!] \ .$$

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*See Whittaker and Watson: *Modern Analysis*, 3rd edition (1920), p. 351.

Example 4. We have already cited an example of a *continuous spectrum* in section 10, chapter 1. This example, due to E. Picard, is closely related to the following integral equation

$$u(x) = \lambda \int_0^\infty e^{-|x-t|}\, u(t)\, dt \ , \qquad (1.14)$$

discussed originally by T. Lalesco.* It is essentially the same as the equation of Picard, which includes the entire real axis as its domain of integration.

We shall discuss (1.14) by means of operators. Thus replacing $u(t)$ by $e^{(t-x)z} \to u(x)$, let us consider the operator

$$\begin{aligned} I(z) &= 1 - \lambda \int_0^\infty e^{-|x-t|}\, e^{(t-x)z}\, dt \\ &= 1 - \lambda \int_0^x e^{-(x-t)}\, e^{(t-x)z}\, dt - \lambda \int_x^\infty e^{-(t-x)}\, e^{(t-x)z}\, dt \\ &= 1 - \lambda \int_{-x}^0 e^{s(z+1)}\, ds - \lambda \int_0^\infty e^{s(z-1)}\, ds \\ &= 1 + \frac{2\lambda}{z^2-1} + \frac{\lambda\, e^{-x}}{z+1}\, e^{-xz} \ . \end{aligned}$$

Now let $I(z)$ operate upon $A\, e^{ax} + B\, e^{-ax}$. We thus obtain

$$\begin{aligned} I(z) &\to (A\, e^{ax} + B\, e^{-ax}) \\ &= (A\, e^{ax} + B\, e^{-ax})\,(1 + \frac{2\lambda}{a^2-1}) + (\frac{A\,\lambda}{a+1} + \frac{B\,\lambda}{-a+1})\, e^{-x} \ . \end{aligned}$$

Obviously the right hand member will vanish identically provided

$$1 + \frac{2\lambda}{a^2-1} = 0 \ , \quad \frac{A}{1+a} + \frac{B}{1-a} = 0 \ ;$$

that is to say, provided

$$2\lambda - 1 = -a^2 \ , \quad B/A = (a-1)/(a+1) \ .$$

Now consider the two cases: (1) $a^2 < 0$; (2) $a^2 > 0$. In the first case we have the solution

$$u(x) = \sin px + p \cos px \ , \quad p = ai \ , \quad 2\lambda - 1 = p^2 \ , \quad \tfrac{1}{2} < \lambda < \infty \ .$$

In the second case we have

$$u(x) = \sinh px + p \cosh px \ , p = a \ , \ 2\lambda - 1 = -p^2 \ , \ 0 < p < 1 \ , \qquad 0 < \lambda < \tfrac{1}{2} \ .$$

**Théorie des équations intégrales.* Paris, (1912), pp. 121-124.

Hence equation (1.14) possesses a unique solution for each value of λ between 0 and ∞, that is to say, *it possesses the positive axis of λ as a continuous spectrum.*

Example 5. The spectrum, in contrast to the one discussed in the preceding example, may consist of a single value, with which, however, an infinite number of proper functions may be associated. Such a value is called a *proper value of infinite multiplicity.* An example of such a spectrum was furnished by H. Weyl,* who considered the equation

$$u(x) - \lambda \int_0^\infty \sin(xt)\, u(t)\, dt = 0 \; . \qquad (1.15)$$

Thus noting the integrals,

$$\int_0^\infty e^{-at} \sin xt \, dt = \frac{x}{a^2+x^2} \; , \qquad \int_0^\infty \frac{t \sin xt}{a^2+t^2}\, dt = \tfrac{1}{2}\pi\, e^{-ax} \; ,$$

we see that the function

$$u(x) = \sqrt{\tfrac{1}{2}\pi}\; e^{-ax} + \frac{x}{a^2+x^2}$$

reduces the left hand member of (1.15) to

$$\left(\sqrt{\tfrac{1}{2}\pi}\; e^{-ax} + \frac{x}{a^2+x^2}\right)\left(1 - \lambda\sqrt{\tfrac{1}{2}\pi}\right) \; .$$

Hence for all positive values of a, $u(x)$ furnishes a solution of (1.15) provided $\lambda = \sqrt{2/\pi}$.

2. Some Theorems on Matrix Transformations. In order to establish a general basis for the study of spectra contemplated in this chapter, let us first consider the problem of maximizing (or minimizing) the quadratic form

$$F = \sum_{i\,j=1}^{n} a_{ij}\, x_i\, x_j \; , \quad (a_{ij} = a_{ji}) \qquad (2.1)$$

where the variables x_i are subject to the condition

$$L \equiv x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \; .$$

The determination of this extremum leads to the Lagrange parametric problem of maximizing (or minimizing) the function†

$$F - \mu L \; .$$

**Singuläre Integralgleichungen.* (Dissertation) Göttingen (1908).

†See E. Goursat (Hedrick translation): *Mathematical Analysis,* Boston, (1904), p. 129. See also section 4, chapter 3, second proof of Hadamard's theorem.

This leads in turn to the system of equation

$$\frac{\partial F}{\partial x_i} - 2\mu x_i = 0 , \quad i = 1, 2, \cdots, n$$

or explicitly

$$\begin{array}{lllll} (a_{11}-\mu) x_1 + & a_{12} x_2 + a_{13} x_3 + \cdots + & & a_{1n} x_n = 0 , & \\ a_{21} x_1 + & (a_{22}-\mu) x_2 + a_{23} x_3 + \cdots + & & a_{2n} x_n = 0 , & \\ \cdots & \cdots & \cdots & \cdots & (2.2) \\ a_{n1} x_1 + & a_{n2} x_2 + a_{n3} x_3 + \cdots + & (a_{nn}-\mu) x_n = 0 . & & \end{array}$$

Admissible values of the parameter μ are then obtained from the equation

$$\Delta(\mu) \equiv \begin{vmatrix} a_{11} - \mu , & a_{12} , a_{13} , \cdots & a_{1n} \\ a_{21} , & a_{22} - \mu , a_{23} , \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} , & a_{n2} , a_{n3} , \cdots & a_{nn} - \mu \end{vmatrix} = 0 ,$$

which we can more conveniently abbreviate in the form

$$\Delta(\mu) \equiv | A - \mu I | = 0 , \tag{2.3}$$

where A and I are respectively the matrices

$$A = \| a_{ij} \| , \quad I = \| \delta_{ij} \| , \quad \delta_{ii} = 1 , \quad \delta_{ij} = 0 , \quad i \neq j .$$

The roots, μ_1, μ_2, $\cdots$, μ_n, of (2.3) are called the *characteristic numbers* of the quadratic form $F(x,x)$, and their reciprocals, $\lambda = 1/\mu_1$, $\lambda_2 = 1/\mu_2, \cdots , \lambda_n = 1/\mu_n$, are called the *proper values* of the form.

The reciprocal of $\| \Delta(\mu) \|$ can be written

$$B = (A - \mu I)^{-1} = \| B_{ij}(\mu) \| \tag{2.4}$$

where we abbreviate

$$B_{ij}(\mu) = \frac{\beta_{ij}(\mu)}{\Delta(\mu)} . \tag{2.5}$$

The functions $\beta_{ij}(\mu)$ are the cofactors of $a_{ij} - \delta_{ij}\mu$ in $\Delta(\mu)$ and hence polynomials of degrees $\leqq n-1$. Thus (2.5) can be expanded in a series of the form

$$B_{ij}(\mu) = \sum_{m=1}^{n} \frac{\sigma_{ij}{}^{(m)}}{\mu - \mu_i} , \tag{2.6}$$

where the μ_i are the characteristic numbers.

Since for $\mu \to \infty$, $B_{ij}(\mu) \to 0$, it is possible also to expand $B_{ij}(\mu)$ in the Laurent series

$$B_{ij}(\mu) = \sum_{m=1}^{\infty} \frac{p_{ij}^{(m)}}{\mu^m} . \tag{2.7}$$

We next consider the transformation of (2.1) to a new set of variables, a transformation which we may designate by T as follows:

$$T: \qquad x_i = \sum_{j=1}^{n} c_{ij} y_j . \tag{2.8}$$

If we denote the matrix of this transformation by C, then we can write

$$T \to F(x,x) = G(y,y) = \sum_{ij=1}^{n} b_{ij} y_i y_j , \tag{2.9}$$

where the matrix $B = \| b_{ij} \|$ is given by

$$B = C' A C . \tag{2.10}$$

As usual C' represents the *conjugate* of the matrix C, that is $C' = \| c_{ji} \|$.

It is now possible to choose the matrix C in such a way that

$$T \to F = G = \sum_{i=1}^{n} \mu_i y_i^2 , \tag{2.11}$$

where the μ_i are the characteristic numbers which we have previously defined. This special definition of C is called the matrix of a *normalized orthogonal transformation* and will be designated by U. It may be described as follows:

Let β: $\beta_1, \beta_2, \cdots, \beta_n$ be a vector whose components form a solution of the system of equations (2.2) which correspond to the characteristic number μ, and let γ: $\gamma_1, \gamma_2, \cdots, \gamma_n$ be a second vector whose components yield a solution corresponding to the characteristic number ν, where we assume that $\mu \neq \nu$. Then we shall have

$$\sum_{j=1}^{n} a_{ij} \beta_j = \mu \beta_i , \quad \sum_{j=1}^{n} a_{ij} \gamma_j = \nu \gamma_i ,$$

from which it follows that

$$\sum_{ij=1}^{n} a_{ij} \gamma_i \beta_j = \mu \sum_{i=1}^{n} \gamma_i \beta_i , \tag{2.12}$$

$$\sum_{ij=1}^{n} a_{ij} \beta_i \gamma_j = \nu \sum_{i=1}^{n} \beta_i \gamma_i .$$

Since, however, $a_{ij} = a_{ji}$, the second equation can be written

$$\sum_{ij=1}^{n} a_{ij}\, \gamma_i\, \beta_j = \nu \sum_{i=1}^{n} \gamma_i\, \beta_i \ . \tag{2.13}$$

Subtracting (2.13) from (2.12), we get

$$(\mu - \nu) \sum_{i=1}^{n} \gamma_i\, \beta_i = 0 \quad ,$$

or $\sum_{i=1}^{n} \beta_i\, \gamma_i = 0$, since $\mu \neq \nu$ by assumption. Let us now divide β by $\sqrt{(\beta\,\beta)}$ and γ by $\sqrt{(\gamma\,\gamma)}$, where we employ the customary notation of the *inner product* of vector analysis:

$$(\beta\,\gamma) = \beta_1\,\gamma_1 + \beta_2\,\gamma_2 + \beta_3\,\gamma_3 + \cdots + \beta_n\,\gamma_n \ .$$

If we call these normalized vectors u_1 and u_2 respectively, it is clear that we shall have

$$(u_1\, u_2) = 0\ , \quad (u_1\, u_1) = (u_2\, u_2) = 1 \ . \tag{2.14}$$

Assuming for the moment that the characteristic numbers of (2.1) are all different, we may proceed in the manner indicated and obtain n solutions of (2.2) which satisfy the criteria (2.14) of normalized orthogonality. These solutions,

$$u_i\colon\ u_{i1}, u_{i2}, u_{i3}, \cdots, u_{in} \quad ,$$

form the matrix of a normalized orthogonal transformation:

$$U = \| u_{ij} \| \ ,$$

that is to say, a matrix which has the property,

$$U\, U' = U'\, U = I \ . \tag{2.15}$$

If we now effect the transformation

$$T\colon \qquad x_i = \sum_{j=1}^{n} u_{ji}\, y_j \tag{2.16}$$

upon $F(x,x)$, we shall obtain for the matrix of the transformation

$$B = U(A\, U') = U \, \Big\| \sum_{k=1}^{n} a_{ik}\, u_{jk} \Big\| \ .$$

From the definitive relation

$$\sum_{k=1}^{n} a_{ik}\, u_{jk} = \mu_j\, u_{ij} \quad ,$$

we then obtain

$$\begin{aligned} B &= U \, \| \mu_i u_{ij} \| = U \, \| \mu_i \delta_{ij} \| \, U' \\ &= U(\mu\, I)\, U' = U\, U'(\mu\, I) = (\mu\, I) \ . \end{aligned}$$

Hence, by the transformation T, we have reduced (2.1) to the normal form

$$T \rightarrow F(x,x) = G(y,y) = \sum_{i=1}^{n} \mu_i \, y_i^2 \ . \tag{2.17}$$

We next consider the case where $\Delta(\mu) = 0$ has a root of multiplicity p. In this case there will exist in general p linearly independent solutions of (2.2). Let us designate these as the components of the vectors $a, b, c, \cdots, p$:

$$\begin{aligned}
a&: \ a_1 \, , \ a_2 \, , \ a_3 \, , \ \cdots \, , \ a_n \ , \\
b&: \ b_1 \, , \ b_2 \, , \ b_3 \, , \ \cdots \, , \ b_n \ , \\
c&: \ c_1 \, , \ c_2 \, , \ c_3 \, , \ \cdots \, , \ c_n \ , \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
p&: \ p_1 \, , \ p_2 \, , \ p_3 \, , \ \cdots \, , \ p_n \ .
\end{aligned}$$

Let us now construct the new vectors, $B, C, D, \cdots, P$, where we define:

$$\begin{aligned}
B &= b + r_1 \, a \ , \\
C &= c + r_2 \, B + s_2 \, a \ , \\
D &= d + r_3 \, C + s_3 \, B + t_3 \, a \ , \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

Employing the notation for the *inner product* as defined above, we may then write the defining equations:

$$\begin{aligned}
(aB) &= (ab) + r_1(aa) = 0 \ ; & (aD) &= (ad) + t_3(aa) = 0 \ , \\
(aC) &= (ac) + s_2(aa) = 0 \ , & (BD) &= (Bd) + s_3(BB) = 0 \ , \\
(BC) &= (Bc) + r_2(BB) = 0 \ ; & (CD) &= (Cd) + r_3(CC) = 0 \ ; \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

Since (aa), (BB), (CC), the *norms* of the vectors, are different from zero, the above equations are sufficient to determine r_1, r_2, r_3, s_2, s_3, t_3, etc. Hence $a, B, C, D, \cdots, P$ form a set of orthogonal vectors, the components of which are solutions of the equations (2.2). These vectors may be normalized provided they are divided respectively by $\sqrt{(aa)}$, $\sqrt{(BB)}$, $\sqrt{(CC)}$, etc.

This analysis is sufficient to show that the assumption previously made that the characteristic numbers were different from one another was an unessential restriction. Hence, collecting these results, we have established the following general proposition:

To every real quadratic form

$$F(x,x) = \sum_{ij=1}^{n} a_{ij}\, x_i\, x_j \qquad (a_{ij} = a_{ji})$$

there corresponds a transformation T

$$T: \qquad x_i = \sum_{j=1}^{n} u_{ji}\, y_j \ ,$$

where $U = \| u_{ij} \|$ is a normal orthogonal matrix, such that

$$T \to F(x,x) = G(y,y) = \sum_{i=1}^{n} \mu_i\, y_i^2 \ , \tag{2.18}$$

in which the coefficients $\mu_1, \mu_2, \cdots, \mu_n$ are the roots of $\Delta(\mu) = 0$. Since $U\, U' = I$, that is, since $U' = U^{-1}$, the inverse of T is given by

$$T^{-1}: \qquad y_i = \sum_{j=1}^{n} u_{ij}\, x_j \ .$$

Moreover, we have

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} u_{ji}\, y_j \Big)^2 = \sum_{j=1}^{n} y_i^2 \ , \tag{2.19}$$

and the expansion

$$R(x,x;\mu) = \sum_{i=1}^{n} \frac{\Big(\sum_{j=1}^{n} u_{ij}\, x_j \Big)^2}{\mu - \mu_i} \tag{2.20}$$

is called the resolvent quadratic form associated with $F(x,x)$. It is defined by the reciprocal transformation associated with the matrix

$$R = (\mu I - A)^{-1} \ . \tag{2.21}$$

Let us now return to the extremal problem with which we started, namely, the determination of the maxima and minima of the form $F(x,x)$ where the variables are subject to the condition

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \ .$$

For this purpose we shall now assume that $F(x,x)$ is positive, definite, that is to say, does not vanish or become negative for any set of values of the variables. Under this assumption all the characteristic values are positive real numbers.

Hence we derive immediately from (2.18) and (2.19) the conclusion that

$$\operatorname{Max} F(x,x) = M \ , \quad \operatorname{Min} F(x,x) = m \ , \tag{2.22}$$

where M and m are respectively the largest and smallest of the characteristic numbers.

Let us also note that $-R(x,x;0)$ is the reciprocal of the form $F(x,x)$. If $F(x,x)$ is positive definite, then $-R(x,x;0)$ is also positive definite and its characteristic numbers are the reciprocals of the characteristic numbers of $F(x,x)$. Hence we derive the conclusion that

$$\operatorname{Max} F^{-1}(x,x) = 1/m, \qquad \operatorname{Min} F^{-1}(x,x) = 1/M \ , \quad (2.23)$$

where $F^{-1}(x,x)$ is used to designate the reciprocal of $F(x,x)$.

3. Hermitian Matrices. The theory which we have sketched above can be extended *mutatis mutandis* to a hermitian form, which we have already defined in chapter 3. It will be recalled that a form

$$H = \sum_{ij=1}^{n} a_{ij}\, x_i\, \bar{x}_j \ ,$$

is *hermitian* provided $a_{ij} = \bar{a}_{ji}$. The bars over the a and the x denote the complex conjugates of a_{ji} and x_j respectively.

Precisely as before, we define a matrix

$$U = \| u_{ij} \| \ ,$$

which, however, now has the property that

$$U\, U' = I \ .$$

It is customary to call this matrix *unitary,* rather than orthogonal.

The transformation

$$S: \qquad x_i = \sum_{j=1}^{n} \bar{u}_{ji}\, y_j \ , \quad \bar{x}_i = \sum_{j=1}^{n} u_{ji}\, \bar{y}_j \ ,$$

yields the new hermitian form

$$S \to H = \sum_{i=1}^{n} \mu_i y_i\, \bar{y}_i \ ,$$

where the μ_i are the roots of the equation,

$$\Delta(\mu) = 0 \ . \quad (a_{ij} = \bar{a}_{ji}) \ .$$

The transformation inverse to S is

$$S^{-1}: \qquad y_i = \sum_{j=1}^{n} u_{ij}\, x_j \ , \quad \bar{y}_i = \sum_{j=1}^{n} \bar{u}_{ij}\, \bar{x}_j \ .$$

4. Some Identities in Matrices. In section 2 it has been shown that the quadratic form

$$F(x,x) = \sum_{ij=1}^{n} a_{ij}\, x_i\, x_j \quad (a_{ij} = a_{ji}) \qquad (4.1)$$

can be reduced to the normal form

$$T \to F(x,x) = \sum_{i=1}^{n} \mu_i \, y_i^2 \tag{4.2}$$

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2 \tag{4.3}$$

where the μ_i are the roots of the equation $\Delta(\mu) = 0$ and T and T^{-1} are the transformations:

$$T: \quad x_i = \sum_{j=1}^{n} u_{ji} \, y_j \; ; \quad T^{-1}: \quad y_i = \sum_{j=1}^{n} u_{ij} \, x_j \; . \tag{4.4}$$

Moreover, the resolvent of (4.1) was written in the form

$$R(x,x) = \sum_{i=1}^{n} \frac{\left(\sum_{j=1}^{n} u_{ij} \, x_j\right)^2}{\mu - \mu_i} \tag{4.5}$$

We propose now to derive some identities in matrices suggested by these propositions. Let us first designate by U_i the matrix of the substitution which corresponds to the quadratic form

$$R_i(x,x) = \left(\sum_{j=1}^{n} u_{ij} \, x_j\right)^2 \; . \tag{4.6}$$

Explicitly this matrix is given by

$$U_i = \| u_{ik} \, u_{im} \| \; , \quad k, m = 1, 2, \cdots, n$$

and hence, since $U = \| u_{ij} \|$ is an orthogonal matrix, we have

$$U_i^2 = U_i \, , \text{ and } U_i \, U_j = 0 \; , \quad i \neq j \; . \tag{4.7}$$

It is also obvious from (4.3) that

$$U_1 + U_2 + U_3 + \cdots + U_n = I \; , \tag{4.8}$$

and from (4.2) that

$$A = \mu_1 \, U_1 + \mu_2 \, U_2 + \cdots + \mu_n \, U_n \; . \tag{4.9}$$

Hence we have

$$\mu I - A = (\mu - \mu_1) U_1 + (\mu - \mu_2) U_2 + \cdots + (\mu - \mu_n) U_n \; ,$$

from which it follows that

$$(\mu I - A) \left[\frac{U_1}{\mu - \mu_1} + \frac{U_2}{\mu - \mu_2} + \cdots + \frac{U_n}{\mu - \mu_n} \right]$$
$$= U_1 + U_2 + \cdots + U_n = I \; .$$

This is merely a recapitulation of the proposition that $R(x,x)$ is the resolvent of (4.1).

Let us now designate by $A(\mu)$ a matrix defined by the following equation:

$$\mu I - A = [\mu I + A(\mu)]^{-1} \mu^2 . \tag{4.10}$$

Multiplying this equation first to the right and then to the left by $\mu I + A(\mu)$, we obtain

$$\mu^2 I - \mu A + \mu A(\mu) - A A(\mu) = \mu^2 I ,$$

$$\mu^2 I - \mu A + \mu A(\mu) - A(\mu) A = \mu^2 I ,$$

from which it follows that

$$\mu[A - A(\mu)] = - A A(\mu) ,$$

$$\mu[A - A(\mu)] = - A(\mu) A . \tag{4.11}$$

If we now replace μ by ν in (4.10) and subtract the resulting equation from (4.10), we get

$$(\mu - \nu) I = \mu^2 [\mu I + A(\mu)]^{-1} - \nu^2 [\nu I + A(\nu)]^{-1} .$$

Now multiplying on the left by $\mu I + A(\mu)$ and on the right by $\nu I + A(\nu)$, we obtain

$$(\mu - \nu)[\mu I + A(\mu)][\nu I + A(\nu)]$$
$$= \mu^2 [\nu I + A(\nu)] - \nu^2 [\mu I + A(\mu)]) ,$$

that is,

$$(\mu - \nu)[\mu \nu I + \mu A(\nu) + \nu A(\mu) + A(\mu) A(\nu)]$$
$$= (\mu^2 \nu - \nu^2 \mu) I + \mu^2 A(\nu) - \nu^2 A(\mu) .$$

This equation reduces to

$$- \mu \nu A(\nu) + \mu \nu A(\mu) + (\mu - \nu) A(\mu) A(\nu) = 0 ,$$

that is, to

$$A(\nu) - A(\mu) + \left(\frac{1}{\mu} - \frac{1}{\nu}\right) A(\mu) A(\nu) = 0 . \tag{4.12a}$$

If μ and ν are interchanged in this equation, we obtain the second relation

$$A(\nu) - A(\mu) + \left(\frac{1}{\mu} - \frac{1}{\nu}\right) A(\nu) A(\mu) = 0 . \tag{4.12b}$$

From these results and from (4.11) we have established the two identities:

$$A A(\mu) = A(\mu) A , \quad A(\mu) A(\nu) = A(\nu) A(\mu) . \tag{4.13}$$

Let us also note that (4.10) can be written

$$I-\lambda A=[I+\lambda A(\mu)]^{-1}\ ,\quad \lambda=1/\mu\ .$$

Hence, since

$$I+\lambda A(\mu)=[I-\lambda A]^{-1}\ ,$$

we get

$$I+\lambda A(\mu)=I-\lambda A+\lambda^2 A^2-\lambda^3 A^3+\cdots\ ,$$

or

$$-A(\mu)=A-\lambda A^2+\lambda^2 A^3-\cdots\ , \tag{4.14}$$

this development being contingent upon the convergence of the series.

5. *Minimizing by the Method of Least Squares.* Before considering the applications contemplated by the preceding development, it will be useful to obtain some formal results from the method of minimization employed in the *method of least squares.*

Let us consider the following Stieltjes integral

$$I=\int_a^b [u(x)-f(x;a_1,a_2,\cdots,a_n)]^2 F(x)\,dv(x)\ ,$$

where $F(x)$ is a weight function, positive in the interval (a,b), $f(x;a_1,a_2,\cdots,a_p)$ is a real, continuous function of x and the n parameters $a_1, a_2, \cdots, a_n$ for x in the interval (a,b) and the parameters throughout some domain D, and $v(x)$ is either x or a step function of positive saltus equal to 1 at the points a, $a+(b-a)/p$, $a+2(b-a)/p$, etc.

Equating to zero the partial derivatives of I with respect to the parameters, we obtain the following set of equations for the determination of them:

$$\int_a^b f\frac{\partial f}{\partial a_i}F(x)\,dv-\int_a^b u(x)\frac{\partial f}{\partial a_i}F(x)\,dv=0\ ,\quad i=1,2,\cdots,n\ . \tag{5.1}$$

Let us now assume that $f(x)$ is of the form

$$f(x)=a_1u_1(x)+a_2u_2(x)+\cdots+a_nu_n(x)\ .$$

Under this assumption (5.1) becomes

$$\begin{array}{l}
a_1\int_a^b u_1^2F(x)\,dv+a_2\int_a^b u_1u_2F(x)\,dv+\cdots+a_n\int_a^b u_1u_nF(x)\,dv=\int_a^b u_1(x)F(x)u(x)\,dv\\
a_1\int_a^b u_2u_1F(x)\,dv+a_2\int_a^b u_2^2F(x)\,dv+\cdots+a_n\int_a^b u_2u_nF(x)\,dv=\int_a^b u_2(x)F(x)u(x)\,dv\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
a_1\int_a^b u_nu_1F(x)\,dv+a_2\int_a^b u_nu_2F(x)\,dv+\cdots+a_n\int_a^b u_n^2F(x)\,dv=\int_a^b u_n(x)F(x)u(x)\,dv
\end{array} \tag{5.2}$$

If the functions $u_1(x), u_2(x), \cdots, u_n(x)$ form a set of functions orthogonal with respect to $F(x)$ over the range (a,b), that is to say, if we have

$$\int_a^b F(x)u_i(x)u_j(x)\,dv = 0 \ , \quad i \neq j \ ,$$

then equations (5.2) assume the form

$$a_i \int_a^b F(x)u_i^2\,dv = \int_a^b u_i(x)F(x)u(x)\,dv \ .$$

Employing the abbreviation

$$\lambda_i = \int_a^b F(x)u_i^2(x)\,dv \ ,$$

we can write the approximating equation for $u(x)$ in the simple form

$$u(x) \backsim \int_a^b K(x,t)u(t)\,dv \ , \qquad (5.3)$$

where we adopt the notation

$$K(x,t) = F(t)\ [\frac{u_1(x)u_1(t)}{\lambda_1} + \frac{u_2(x)u_2(t)}{\lambda_2} + \cdots + \frac{u_n(x)u_n(t)}{\lambda_n}] \ .$$

If we introduce $f(x) = \Sigma\, a_i\, u_i(x)$ into the integral I with which we started and note that

$$\int_a^b f^2(x)F(x)\,dv = a_1^2\,\lambda_1 + a_2^2\,\lambda_2 + \cdots + a_n^2\,\lambda_n \ ,$$

we obtain

$$\begin{aligned} I &= \int_a^b u^2(x)F(x)\,dv - 2\int_a^b u(x)f(x)F(x)\,dv + \int_a^b f^2(x)F(x)\,dv \\ &= \int_a^b u^2(x)F(x)\,dv - 2(a_1^2\,\lambda_1 + a_2^2\,\lambda_2 + \cdots) + \int_a^b f^2(x)F(x)\,dv \\ &= \int_a^b u^2(x)F(x)\,dv - \int_a^b f^2(x)F(x)\,dv \geqq 0 \ . \end{aligned}$$

We thus establish the *Bessel inequality*

$$a_1^2\,\lambda_1 + a_2^2\,\lambda_2 + \cdots + a_n^2\,\lambda_n \leqq \int_a^b u^2(x)F(x)\,dv \ . \qquad (5.4)$$

If the set of orthogonal functions is an infinite set and *closed,* that is to say, if there exists no other function outside the set which is orthogonal with the set, then the equal sign holds and the approximation sign in equation (5.3) is replaced by an equal sign.

We should note in passing that the value

$$\sigma^2 = I^2/L^2 \ ,$$

where $L = \int_a^b F(x)\,dv$, is the familiar *variance* of the theory of statistics.

From the fact that

$$\frac{\partial^2 I}{\partial a_i^2} = \lambda_i > 0 \ , \quad \frac{\partial^2 I}{\partial a_i \, \partial a_j} = 0 \ , \quad i \neq j \ ,$$

it is obvious that (5.3) furnishes a minimum for the integral I.

Let us now assume for convenience that $F(x) \equiv 1$. No loss of generality is implied by this since we may replace $u_i(x)$ by $U_i(x) = \sqrt{F(x)}\ u_i(x)$. These new functions again form an orthogonal set. We may also assume without loss of generality that the functions $\{u_i(x)\}$ are members of a normal set, that is to say, $\lambda_i = 1$, since we may replace $u_i(x)$ by $u_i(x)/\sqrt{\lambda_i}$.

Bessel's inequality (5.4) then takes the normal form

$$a_1^2 + a_2^2 + \cdots + a_n^2 \leqq \int_a^b u^2(x)\,dx \ . \tag{5.5}$$

If the set of normalized orthogonal functions is *closed,* that is to say, if there exists no other function outside the set $u_i(x)$, which is orthogonal with the set, then the equal sign holds in (5.5) *for all functions of summable square.* A closed set is also called *complete.* The numbers $a_1, a_2, a_3, \cdots$ are called the *Fourier coefficients* of $u(x)$ with respect to the given orthogonal set.

We may now propose the following question: Given a closed set of orthogonal functions and a set of numbers $a_1, a_2, a_3 \cdots, a_n, \cdots$ such that the sum

$$a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2 + \cdots \tag{5.6}$$

converges, does there exist a unique function $u(x)$ of summable square for which the set forms the Fourier coefficients relative to the given system of orthogonal functions? The answer to this question is called the *theorem of Fischer-Riesz,* which may be stated as follows:

If the series (5.6) *converges, then there exists a function* $u(x)$ *of summable square, unique to within a null function,* for which the set* $a_1, a_2, a_3, \cdots$ *is the set of Fourier coefficients corresponding to the given closed orthogonal system.*

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*By a null function we mean a bounded function which is everywhere zero except over a set of points of measure zero.

Proof: Consider the series

$$U_n(x) = \sum_{i=1}^{n} a_i \, u_i(x)$$

where $u_1(x)$, $u_2(x)$, $\cdots$, $u_n(x)$ are members of a closed system of normalized orthogonal functions, and in connection with the series consider the integral

$$I_{mn} = \int_a^b [U_{n+m} - U_m]^2 \, dx$$

$$= a^2_{n+1} + a^2_{n+2} + \cdots + a^2_{n+m} \; .$$

Since (5.6) converges we know that

$$\lim I_{mn} = 0 \text{ as } m \text{ and } n \to \infty \; .$$

Hence we know that the set of functions $U_n(x)$ *converges in the mean* to a limit function $U(x)$. (See section 2, chapter 2).

In order finally to identify $U(x)$ with the desired function $u(x)$ we must show that

$$a_i = \int_a^b U(x) u_i(x) \, dx \; .$$

For this purpose we employ the Schwarz inequality (see section 9, chapter 3) and thus obtain

$$[\int_a^b |U(x) - U_n(x)| \, u_i(x) \, dx]^2$$

$$\leqq \int_a^b [U(x) - U_n(x)]^2 \, dx \int_a^b u_i^2(x) \, dx < \varepsilon \; .$$

Hence we obtain the desired result

$$\int_a^b U(x) u_i(x) \, dx = \lim_{n=\infty} \int_a^b U_n(x) u_i(x) \, dx = a_i \; ,$$

which establishes the theorem.

We note finally that any set of linearly independent functions

$$f_1(x), f_2(x), \cdots, f_n(x)$$

can be used to construct a set of orthogonal functions necessary in the above development. For convenience we define

$$(f_i, f_j) = \int_a^b f_i(x) f_j(x) \, dv \; . \tag{5.7}$$

Since the functions are linearly independent, we can find constants c_i such that

$$(f_i - c_i f_1, f_1) = 0 \;, \quad i = 2, 3, \cdots, n \; .$$

We thus see that the functions $g_i(x) = f_i(x) - c_i\, f_1(x)$ form a set orthogonal to $f_1(x)$.

Proceeding similarly we next compute a set of constants d_i such that

$$(g_i - d_i\, g_2, g_2) = 0 \ , \quad i = 3, 4, \cdots, n \ .$$

Hence the functions $h_i(x) = g_i(x) - d_i\, g_2(x)$ are orthogonal both to $f_1(x)$ and $g_2(x)$.

In this manner it is possible to construct a set of orthogonal functions

$$f_1(x), g_2(x), h_3(x), \cdots, p_n(x) \ ,$$

which may be normalized by dividing the members respectively by $(f_1,f_1)^{\frac{1}{2}}$, $(g_2,g_2)^{\frac{1}{2}}$, $(h_3,h_3)^{\frac{1}{2}}$, etc.

Since the definitive integral (5.7) is a Stieltjes integral, which by means of step functions may be reduced to a series over a set of integers 1, 2, $\cdots$, p, it is clear that the method of orthogonalization just given covers not only sets of continuous functions, but also the types of discontinuous functions useful in statistical applications.

PROBLEMS

1. From the functions, 1, x, x^2, x^3, etc. construct a set of orthogonal polynomials over the range -1 to $+1$. Show that they are proportional to the Legendre polynomials:

$$P_0(x) = 1,\ P_1(x) = x\ ,\ P_2(x) = \tfrac{1}{2}(3x^2-1)\ ,\ P_3(x) = \tfrac{1}{2}(5x^3-3x)\ , \text{ etc.}$$

Show that these latter are normalized by multiplication by $[\tfrac{1}{2}(2n+1)]^{\frac{1}{2}}$.

2. Derive the first five Hermite polynomials (see example 3, section 1) by constructing an orthogonal set of functions over the range $-\infty$ to $+\infty$ from the functions

$$\phi(x), x\,\phi(x), x^2\,\phi(x), x^3\,\phi(x), \text{ etc.},$$

where $\phi(x) = e^{-1/4x^2}$.

3. From the functions, 1, x, x^2, x^3, etc. derive a set of polynomials orthogonal over the discrete range from $-p$ to $+p$, that is, which satisfy the equations

$$\sum_{n=-p}^{p} \phi_m(x)\,\phi_n(x) = 0\ , \quad m \neq n\ .$$

Show that the first six are the following:

$$\phi_0(x) = 1\ ;\ \phi_1(x) = x\ ;\ \phi_2(x) = x^2 - p(p+1)/3\ ;$$

$$\phi_3(x) = x^3 - (3p^2 + 3p - 1)x/5\ ;$$

$$\phi_4(x) = x^4 - (6p^2 + 6p - 5)x^2/7 + 3p(p^2-1)(p+2)/35\ ;$$

$$\phi_5(x) = x^5 - 5(2p^2 + 2p - 3)x^3/9 + (15p^4 + 30p^3 - 35p^2 - 50p + 12)x/63\ .$$

Prove that these polynomials satisfy the recurrence formula

$$4(4n^2-1)\,\phi_{n+1}(x) - 4(4n^2-1)x\,\phi_n(x) + n^2(2p-n+1)(2p+n+1)\,\phi_{n-1}(x) = 0$$

and that they are solutions of the difference equation

$$[(p-1)(p+2)-3x-x^2]\,\Delta^2\phi_n(x) + [(n-1)(n-2)-2x]\,\Delta\phi_n(x)+n(n+1)\,\phi_n(x)=0\ .$$

4. Show that the functions

$$\psi_0(x)=1\,,\ 0\leqq x\leqq 1\,,\ \psi_1(x)=\begin{cases}-1\,, & 0\leqq x<\tfrac12\,,\\ +1\,, & \tfrac12\leqq x\leqq 1\,,\end{cases}$$

$$\psi_n(x)=\begin{cases}0 & ,\ 0\leqq x<1-1/2^{n-1}\,,\\ -2^{\frac12(n-1)} & ,\ 1-1/2^{n-1}\leqq x<1-1/2^n\,,\\ +2^{\frac12(n-1)} & ,\ 1-1/2^n\leqq x\leqq 1\,,\end{cases}\quad n=2,3,\cdots\,,$$

form an orthogonal system.

5. Show that the sum $\displaystyle\sum_{p=0}^{\infty}A_p\,\psi_p(x)$

is identically zero in $0\leqq x<1$, where $A=C$, $A=C\,2^{\frac12(p-1)}$, $p\geqq 1$, and the functions $\psi_p(x)$ are defined as in problem 4.

[The results stated in problems 4 and 5 are attributed to A. Haar by M. Plancherel: Les problèmes de Cantor et du Bois-Reymond. *Annales de l'école normale,* vol. 31 (3rd ser.) (1914), pp. 223-262; in particular p. 224. See also T. Carleman (*Bibliography*), pp. 62-64.]

6. From the functions 1, x, x^2, x^3, etc. construct a set of polynomials $L_n(x)$ (the Laguerre polynomials) such that

$$\int_0^{\infty}e^{-x}\,L_m(x)\,L_n(x)\,dx=\delta_{mn}\ .$$

Show in particular that

$$L_0(x)=1\,,\ L_1(x)=(-x+1)/1!\,,\ L_2(x)=(x^2-4x+2)/2!\,,$$

$$L_3(x)=(-x^3+9x^2-18x+6)/3!\,,$$

$$L_4(x)=(x^4-16x^3+72x^2-96x+24)/4!\ .$$

7. Construct a set of polynomials such that

$$\sum_{x=-p}^{p}C_x\,\Psi_m(x)\,\Psi_n(x)=0\,,\quad m\neq n\,,$$

where we define

$$C_x=\frac{(2p)!}{(p+x)!\,(p-x)!}\,(\tfrac12)^{p+x}\,(\tfrac12)^{p-x}\ .$$

Show that the first six are the following:

$$\Psi_0(x)=1\ ;\ \Psi_1(x)=x\ ;\ \Psi_2(x)=x^2-p/2\ ;$$

$$\Psi_3(x)=x^3-\tfrac12(3p-1)x\ ;$$

$$\Psi_4(x)=x^4-(3p-2)x^2+3p(p-1)/4\ ;$$

$$\Psi_5(x)=x^5-5(p-1)x^3+(15p^2-25p+6)x/4\ .$$

Show that these polynomials satisfy the recurrence formula

$$4\Psi_{n+1}(x)-4x\,\Psi_n(x)+n(2p+1-n)\,\Psi_{n-1}(x)=0\ ,$$

and that they are solutions of the difference equation

$$(p-x-1)\Delta^2\Psi_n(x) + 2(n-x-1)\Delta\,\Psi_n(x) + 2n\,\Psi_n(x) = 0 \ .$$

[These results, extending the theory of Hermite polynomials to discrete summation, are due to H. E. H. Greenleaf: Curve Approximation by Means of Functions Analogous to the Hermite Polynomials. *Annals of Mathematical Statistics*, vol. 3 (1932), pp. 204-255.]

8 Show that for $n > -1$,

$$\int_0^1 x\,J_n(\lambda_i x)\,J_n(\lambda_j x)\,dx = \begin{cases} 0 \ , & i \neq j \ , \\ \tfrac{1}{2}\{J_{n+1}(\lambda_i)\}^2 \ , & i = j \ , \end{cases}$$

where $J_n(x)$ is the nth Bessel function satisfying the equation

$$x^2\,J_n''(x) + x\,J_n'(x) + (x^2 - n^2)J_n(x) = 0 \ ,$$

and where the quantities λ_i are the roots of the equation

$$J_n(\lambda) = 0 \ .$$

9. Prove that if $u(t)$ is any function analytic in the interval (a,b) such that

$$\int_a^b t^n\,u(t)\,dt = 0$$

for $n = 0, 1, 2, \cdots$, then $u(t)$ must be identically zero. [J. Liouville: Solution d'un probleme d'analyse. *Journal de Mathématique*, vol. 2 (1837), pp. 1-2.]

10. Show that if

$$\int_a^b t^m\,u(t)\,dt = 0 \ , \quad m = 0, 1, 2, \cdots, n-1 \ ,$$

then a solution is obtained in the form

$$u_n(x) = \frac{d^n}{dx^n}\left[(x-a)^n(x-b)^n\,v_n(x)\right] \ ,$$

where $v_n(a)$ and $v_n(b)$ are finite.

If $v_n(x) = 1/[n!\,(b-a)^n]$, show that

$$\int_a^b u_m(x)\,u_n(x)\,dx = \begin{cases} 0 \ , & m \neq n \ , \\ (b-a)/(2n+1) \ , & m = n \ . \end{cases}$$

Show also that $u_n(x)$ satisfies the equation

$$(x-a)(x-b)\,u_n''(x) + (2x-a-b)\,u_n'(x) - n(n+1)u_n(x) = 0 \ .$$

[H. Laurent: Sur le calcul inverse des intégrales definies. *Journal de Math.*, vol. 4 (3rd ser.) (1878), pp. 225-246.]

11. Let $f(x)$ and $g(x)$ be functions of integrable square in the interval (a,b). Then if f_i are the Fourier coefficients of $f(x)$ with respect to a closed set of functions orthogonal and normalized for the interval (a,b) and if g_i are the corresponding Fourier coefficients of $g(x)$, prove that

$$\int_a^b f(x)\,g(x)\,dx = \sum_{i=1}^{\infty} f_i\,g_i \ .$$

[This is the generalized *Parseval's theorem* stated in 1806 by M. A. Parseval for the special case of a Fourier series but with rather stringent conditions imposed upon the functions $f(x)$ and $g(x)$. See *Paris mém. prés.*, vol. 1 (1806), p. 639.]

6. The Spectral Theory of Integral Equations. In the first section of this chapter we defined the principal values and the principal solutions of the Fredholm integral equation

$$u(x) = \lambda \int_a^b K(x,t)\, u(t)\, dt \ . \tag{6.1}$$

Again in section 5 there were developed some properties of kernels expressed as finite bilinear forms, the elements of which were the members of a set of orthogonal functions.

Since the theory of integral equations and the theory of quadratic forms in infinitely many variables possess a striking similarity to one another, amounting, in fact, essentially to a complete dualism, it will be illuminating to generalize the investigations of the preceding section.

These generalizations we shall set forth in a series of theorems.

Theorem 1. The integral equation conjugate to (6.1), *namely,*

$$v(x) = \lambda \int_a^b K(t,x)\, v(t)\, dt \ , \tag{6.2}$$

has the same set of principal values as (6.1).

Proof: The proof is immediately obtained from the observation that $D(\lambda)$ is the same for the kernel $K(t,x)$ as for the kernel $K(x,t)$. This fact is obvious from an inspection of the explicit expansion of $D(\lambda)$ given in equation (6.7) of chapter 11.

Theorem 2. The principal functions, $u_i(x)$, *of* (6.1) *and the principal functions,* $v_i(x)$, *of* (6.2) *form a system of biorthogonal functions.*

Proof: Let $u_i(x)$ belong to the principal value λ_i and $v_j(x)$ to the principal value λ_j, which we shall assume to be different from λ_i. Now substitute $u_i(x)$ and λ_i in (6.1) for $u(x)$ and λ respectively, multiply by $v_j(x)$, and integrate the product between a and b. We then obtain

$$\int_a^b u_i(x)\, v_j(x)\, dx = \lambda_i\, I \ , \tag{6.3}$$

where we abbreviate

$$I = \int_a^b \int_a^b K(x,t)\, u_i(t)\, v_j(x)\, dt\, dx \ .$$

Similarly we substitute $v_j(x)$ and λ_j for $v(x)$ and λ respectively in (6.2), multiply by $u_i(x)$, and integrate between a and b. We thus obtain

$$\int_a^b u_i(x) v_j(x)\, dx = \lambda_j I \ . \tag{6.4}$$

Multiplying (6.3) by λ_j, (6.4) by λ_i and subtracting the first equation from the second, we get

$$(\lambda_i - \lambda_j) \int_a^b u_i(x) v_j(x)\, dx = 0 \ .$$

Since by assumption $\lambda_i \neq \lambda_j$, the integral must vanish and hence we have established the biorthogonal character of the principal functions.

Theorem 3. The Fredholm minor, $D(x,t;\lambda_i)$ is a solution of equation (6.1) *and the adjoint minor, $D(t,x;\lambda_i)$ is a solution of equation* (6.2).

Proof: Referring to equations (7.13) of chapter 11, we obtain this result as an obvious consequence of the fact that $D(\lambda_i) = 0$. We should observe that the theorem does not exclude the possibility that the solutions may be the trivial ones $u(x) \equiv 0$ and $v(x) \equiv 0$.

Theorem 4. If $\lambda = \lambda_i$, then the Fredholm minor factors into the product

$$D(x,t;\lambda_i) = u_i(x) v_i(t) \ .$$

where $u_i(x)$ is a solution of equation (6.1) *and $v_i(x)$ is a solution of* (6.2).

Proof: In establishing this theorem we shall make use of the following lemma:

If $D(x,y)$ is a function of two variables and possesses first and second derivatives with respect to both, then a necessary and sufficient condition that it be factorable into a function of x and a function of y, $D(x,y) = u(x)v(y)$, is that it satisfy the following equation:

$$\Delta[D] \equiv D(x,y) \frac{\partial^2 D}{\partial x\, \partial y} - \frac{\partial D}{\partial x} \frac{\partial D}{\partial y} = 0 \ . \tag{6.5}$$

The proof of the lemma is left to the reader.

If we now compute $\Delta[D(x,t;\lambda)]$, we find after some computation that

$$\Delta[D] \equiv D(\lambda) \sum_{m=0}^{\infty} (-1)^m \frac{\lambda^m}{m!} \int_a^b \cdots \int_a^b S_m \left(\begin{matrix} x \\ y \end{matrix} \right) dt_1\, dt_2 \cdots dt_m \ , \tag{6.6}$$

where we abbreviate

$$S_m\begin{pmatrix} x_1\, x_2 \cdots x_p \\ y_1\, y_2 \cdots y_p \end{pmatrix} =$$

$$\begin{vmatrix} K''_{xy}(x_1,y_1) & K'_x(x_1,y_1) & \cdots & K'_x(x_1,y_p) & K'_x(x_1,t_1) & \cdots & K'_x(x_1,t_m) \\ K'_y(x_1,y_1) & K(x_1,y_1) & \cdots & K(x_1,y_p) & K(x_1,t_1) & \cdots & K(x_1,t_m) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ K'_y(x_p,y_1) & K(x_p,y_1) & \cdots & K(x_p,y_p) & K(x_p,t_1) & \cdots & K(x_p,t_m) \\ K'_y(t_1,y_1) & K(t_1,y_1) & \cdots & K(t_1,y_p) & K(t_1,t_1) & \cdots & K(t_1,t_m) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ K'_y(t_m,y_1) & K(t_m,y_1) & \cdots & K(t_m,y_p) & K(t_m,t_1) & \cdots & K(t_m,t_m) \end{vmatrix} .$$

If we denote by M a number greater than or equal to the upper bounds of $|K''_{xy}(x,y)|$, $|K'_x(x,y)|$, $|K'_y(x,y)|$, and $|K(x,y)|$ over the rectangle of definition, then it follows from Hadamard's theorem (see section 4, chapter 3) that

$$\left| S_m\begin{pmatrix} x \\ y \end{pmatrix}\right| \leqq (m+2)^{(m+2)/2} M^{m+2} .$$

Since, moreover, we have

$$\lim_{m=\infty} \sqrt[m]{(m+2)^{(m+2)/2} M^{m+2}/m!} = 0 ,$$

the second factor of (6.6) is seen to converge for all values of λ.

From this it follows that $\Delta[D] = 0$, when $\lambda = \lambda_i$, and the theorem is thus established.

Proofs of theorem 4, which differ essentially from the one just given, have been made by the following: L. Tocchi: Sopra una classe d'equazioni integrali. *Per. di Mat.*, vol. 12 (3rd ser.) (1915), pp. 253-261; Due teoremi sulle serie di Fredholm. *Giornale di Mat.*, vol. 54 (1916), pp. 141-150; G. Landsberg: Theorie der Elementarteilen linearer Integralgleichungen. *Math. Annalen*, vol. 69 (1910), pp. 227-265; and G. Kowalewski: *Integralgleichungen.* Berlin and Leipzig (1930), pp. 199-204.

It will be observed that this is the extension of the algebraic theory of elementary divisors to the transcendental case.

Theorem 5. If for some value $\lambda = \lambda_i$ it happens that

$$D\begin{pmatrix} x_1\, x_2 \cdots x_{i-1}\, x_{i+1} \cdots x_p \\ y_1\, y_2 \cdots y_{i-1}\, y_{i+1} \cdots y_p \end{pmatrix} = 0 ,$$

then the Fredholm minor

$$D\begin{pmatrix} x_1\, x_2 \cdots x_p \\ y_1\, y_2 \cdots y_p \end{pmatrix}$$

will factor into a function of x_i by a function of y_i.

Proof: This theorem is a generalization of theorem 4. It is an obvious deduction from the following identity where the derivatives in the symbol Δ apply to x_i and y_i:

$$\Delta\left\{D\begin{pmatrix} x_1\, x_2 \cdots x_p \\ y_1\, y_2 \cdots y_p \end{pmatrix}\right\} = D\begin{pmatrix} x_1\, x_2 \cdots x_{i-1}\, x_{i+1} \cdots x_p \\ y_1\, y_2 \cdots y_{i-1}\, y_{i+1} \cdots y_p \end{pmatrix}$$

$$\times\left\{\sum_{m=0}^{\infty} (-1)^m \frac{\lambda^m}{m!} \int_a^b \cdots \int_a^b S_m\begin{pmatrix} x_1\, x_2 \cdots x_p \\ y_1\, y_2 \cdots y_p \end{pmatrix} dt_1\, dt_2 \cdots dt_m\right\}.$$

Theorem 6. The series $S_p(\lambda)$ defined in section 1 *converges when* $p > 2$.

Proof: This theorem is immediately derived from the fact that $D(\lambda)$ is an entire function of genus not greater than 2. (See section 7, chapter 11).

Theorem 7. If all the principal values are simple zeros of $D(\lambda)$, then the resolvent kernel can be expressed in the form

$$k(x,t;\lambda) = \sum_{i=1}^{n} \frac{u_i(x)\, v_i(t)}{\lambda_i - \lambda} + R(x,t;\lambda) \ , \qquad (6.7)$$

where $u_i(x)$ *and* $v_i(x)$ *are solutions respectively of equations* (6.1) *and* (6.2) *and $R(x,t;\lambda)$ is a function which is finite for* $\lambda = \lambda_1, \lambda_2 \cdots, \lambda_n$.

Proof: Since $D(\lambda)$ is an entire function with a simple zero at $\lambda = \lambda_i$ it can be written

$$D(\lambda) = (\lambda - \lambda_i)\,[d_0 + d_1(\lambda - \lambda_i) + \cdots] \ . \qquad (6.8)$$

Similarly $D(x,t;\lambda)$ can be expanded

$$D(x,t;\lambda) = (\lambda - \lambda_i)^m [D_1(x,t) + (\lambda - \lambda_i) D_2(x,t) + \cdots] \ . \qquad (6.9)$$

But from the equation [see (7.2), chapter 11]

$$\frac{dD}{d\lambda} = -\int_a^b D(t,t;\lambda)\, dt \qquad (6.10)$$

we see that m must be equal to zero.

Substituting (6.8) and (6.9) in the equation

$$D(x,t;\lambda) = D(\lambda) K(x,t) + \lambda \int_a^b K(x,s) D(s,t;\lambda)\, ds \qquad (6.11)$$

and letting $\lambda \to \lambda_i$, we get

$$D_1(x,t) = \lambda_i \int_a^b K(x,s) D_1(s,t)\, ds \ . \qquad (6.12)$$

Moreover since $D_1(x,t) = D(x,t;\lambda_i)$ and since by theorem 4 $D(x,t;\lambda_i)$ factors into the product of a solution of (6.1) by a solution of (6.2), we at once obtain as the residue of

$$k(x,t;\lambda) = D(x,t;\lambda)/D(\lambda)$$

at $\lambda = \lambda_i$, the value $u_i(x)\ v_i(t)/(\lambda_i - \lambda)$. The argument is obviously extensible to any finite number of principal values.

Theorem 8. If all the principal values of $K(x,t)$ are simple zeros of $D(\lambda)$, then the kernel can be expanded in the form

$$K(x,t) = \sum_{i=1}^{n} \frac{u_i(x)v_i(t)}{\lambda_i} + S(x,t;\lambda) \ , \tag{6.13}$$

where

$$S(x,t;\lambda) = R(x,t;\lambda) - \lambda \int_a^b K(x,s)R(s,t;\lambda)\,ds \ .$$

Proof: Substituting the value of $k(x,t;\lambda)$ given by (6.7) in equation (7.5) of chapter 11, we obtain

$$\begin{aligned} K(x, t) &= k(x,t;\lambda) - \lambda \int_a^b K(x,s)k(s,t;\lambda)\,ds \\ &= \sum_{i=1}^{n} \frac{u_i(x)v_i(t)}{\lambda_i - \lambda} - \sum_{i=1}^{n} \frac{\lambda}{\lambda_i} \frac{u_i(x)v_i(t)}{\lambda_i - \lambda} \\ &\qquad + R(x,t;\lambda) - \lambda \int_a^b K(x,s)R(s,t;\lambda)\,dt \\ &= \sum_{i=1}^{n} \frac{u_i(x)v_i(t)}{\lambda_i} + S(x,t;\lambda) \ . \end{aligned}$$

The arguments employed in the proofs of theorems 7 and 8 can be extended without difficulty to the case where $D(\lambda)$ has a zero of order p. We may then replace equation (6.8) by

$$D(\lambda) = (\lambda - \lambda_i)^p[d_0 + d_1(\lambda - \lambda_i) + \cdots] \ , \tag{6.14}$$

and hence obtain from equation (6.10) that $m \leqq p - 1$.

If then both $D(\lambda)$ and $D(x,t;\lambda)$ be substituted in equation (6.11), the factor $(\lambda - \lambda_i)^m$ removed, and λ allowed to approach λ_i, equation (6.12) is again obtained.

In the neighborhood of the principal value $\lambda = \lambda_i$, the resolvent kernel now has the expansion

$$k(x,t;\lambda) = \frac{k_p(x,t)}{(\lambda_i - \lambda)^p} + \frac{k_{p-1}(x,t)}{(\lambda_i - \lambda)^{p-1}} + \cdots + \frac{k_1(x,t)}{(\lambda_i - \lambda)} + R(x,t;\lambda) \ , \tag{6.15}$$

where $R(x,t;\lambda)$ remains finite for $\lambda = \lambda_i$.

Since $K(x,t) = k(x,t;0)$, the expansion of the kernel correspond- to (6.13) is immediately written down from (6.15).

PROBLEMS

1. Show that if

$$K(x,y) = X_1(x)Y_1(y) + X_2(x)Y_2(y) + \dots + X_n(x)Y_n(y) \quad ,$$

where $X_i(x)$ and $Y_i(y)$ are integrable over the fundamental square, then $D(\lambda)$ is a polynomial of degree n.

Show in particular that

$$D(\lambda) = |a_{ij}| \quad ,$$

and that

$$D(x,t) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & X_1(x) \\ a_{21} & a_{22} & \cdots & a_{2n} & X_2(x) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} & X_n(x) \\ Y_1(t) & Y_2(t) & \cdots & Y_n(t) & 0 \end{vmatrix}$$

where we write

$$a_{ij} = \delta_{ij} - \lambda \int_a^b X_i(t)\, X_j(t)\, dt \; .$$

[*Bilinear kernels* were first studied by E. Goursat: Sur un cas élémentaire de l'équation de Fredholm. *Bull. de la soc. math. de France*, vol. 35 (1907), pp. 163-173. The extension of the theory to bilinear kernels in an infinite number of variables was made by H. Lebesgue: Sur la méthode de Goursat pour la résolution de l'équation de Fredholm. *Bull. de la soc. math. de France*, vol. 36 (1908), pp. 3-19.].

2. Show that if the functions $X_i(x)Y_i(y)$ of problem 1 possess first derivatives, then

$$D(x,y) = -D(\lambda)\, D\begin{pmatrix} X & X' \\ Y & Y' \end{pmatrix} \quad ,$$

where we abbreviate

$$D\begin{pmatrix} X & X' \\ Y & Y' \end{pmatrix} = \begin{vmatrix} a_{11}\, a_{12} & \cdots & a_{1n} & X_1 & X'_1 \\ a_{21}\, a_{22} & \cdots & a_{2n} & X_2 & X'_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1}\, a_{n2} & \cdots & a_{nn} & X_n & X'_n \\ Y_1\; Y_2 & \cdots & Y_n & 0 & 0 \\ Y_1'\; Y_2' & \cdots & Y'_n & 0 & 0 \end{vmatrix}$$

3. If a kernel has a finite number (m) of proper value, prove that the traces $(p > 2)$, S_p, S_{p+1}, S_{p+m}, must be linearly dependent.

4. Prove that if S_p, S_{p+1}, $\cdots$, S_{p+m}, $(p > 2)$, are linearly dependent, then the kernel to which they belong can have only a finite number (m) of principal values.

5. Prove that for the existence of a solution $u(x)$ of the equation

$$u(x) = f(x) + \lambda_i \int_a^b K(x,t)\,u(t)\,dt \ ,$$

where λ_i is a principal value, it is both necessary and sufficient that the function $f(x)$ satisfy the equation

$$\int_a^b f(x)\,v_i(x)\,dx = 0 \ ,$$

where $v_i(x)$ is the solution of the adjoint

$$v_i(x) = \lambda_i \int_a^b K(t,x)\,v_i(t)\,dt \ .$$

(This result is referred to in the literature as *Fredholm's third theorem.*)

6. Show that the solution of the equation

$$u(x) = 6x^2 - 6x + 1 + \lambda^* \int_0^1 (x+t)\,u(t)\,dt \ ,$$

where $D(\lambda^*) = 0$, is

$$u(x) = 6x^2 - 6x + 1 + c(\sqrt{3}x + 1) \ .$$

7. The Spectral Theory Associated with Symmetric Kernels. Results of great elegance can be obtained when the theory of the preceding section has been specialized so as to pertain to the symmetric kernel. Moreover, the spectral theory of the Fredholm integral equation in this specialized form is so closely analogous to the spectral theory of quadratic forms as set forth in section 2 that one can without difficulty state the theorems in one discipline by an almost obvious translation of the theorems in the other. Since, however, there is illumination in the proofs, we shall derive some of the principal results in the spectral theory of the symmetric kernel.

From the roots of the Fredholm equation

$$D(\lambda) = 0$$

we obtain a set of principal values, λ_1, λ_2, λ_3, $\cdots$, λ_n corresponding to which there exists a set of principal functions, $u_1(x)$, $u_2(x)$, $\cdots$, $u_n(x)$, which satisfy the equation

$$u_i(x) = \lambda_i \int_a^b K(x,t)\,u_i(t)\,dt \ .$$

Theorem 9. Every symmetric, quadratic integrable kernel, which is not a null function, has at least one principal value.

Proof: This theorem, first established by D. Hilbert in his *Grundzüge* (p. 22), has been proved in several ways. One of the most notable is due to E. Schmidt, whose papers on integral equations, particularly as they pertain to the properties of the symmetric kernel, are now classic. (See *Bibliography*). The following proof, however, is taken from A. Kneser.*

From section 7, chapter 11 we have the equation

$$\frac{D'(\lambda)}{D(\lambda)} = -(S_1 + \lambda S_2 + \lambda^2 S_3 + \lambda^3 S_4 + \cdots) \tag{7.1}$$

where the coefficients

$$S_p = \int_a^b K_p(t,t)\,dt$$

are traces of the kernel.

Since

$$K_{m+n}(x,y) = \int_a^b K_m(x,t)\,K_n(t,y)\,dt \;,$$

we see that

$$S_{m+n} = \int_a^b K_m(s,t)\,K_n(t,s)\,dt\,ds \;,$$

and hence by the Schwarz inequality (see section 9, chapter 3)

$$S^2_{m+n} \leqq S_{2m}\,S_{2n} \;. \tag{7.2}$$

We now observe that since $D'(\lambda)$ and $D(\lambda)$ are both entire functions of λ, series (7.1) will converge only within a circle of radius $|\lambda_1|$, where λ_1 is the first principal value. It is now our purpose to establish an upper value for this radius.

If in (7.2) we set $m = p+1$, $n = p-1$, we obtain

$$S^2_{2p} \leqq S_{2p+2}\,S_{2p-2} \;,$$

an equality which may be written in the form

$$\frac{S_{2p-2}}{S_{2p}} \geqq \frac{S_{2p}}{S_{2p+2}} \;,$$

provided none of the denominators is zero.

Letting p assume successively the values 2, 3, 4, $\cdots$, p, we obtain

$$\frac{S_2}{S_4} \geqq \frac{S_4}{S_6} \geqq \frac{S_6}{S_8} \geqq \cdots \geqq \frac{S_{2p}}{S_{2p+2}} \;. \tag{7.3}$$

*Ein Beitrag zur Theorie der Integralgleichungen. *Rendiconti di Palermo*, vol. 22 (1906), pp. 233-240.

We next note that

$$S_{2n} = \int_a^b \int_a^b [K_n(s,t)]^2 \, ds \, dt$$

from which we see that S_{2n} cannot be zero unless $K_n(x,t)$ is a null function.

In order to show that $K_n(x,t)$ is not a null function, we need the lemma: *If $K(x,t)$ is symmetric and is not a null function, then the iterated kernels are symmetric and not null functions.*

The symmetry of $K_n(x,t)$ is proved by induction. For this purpose we first assume that $K_{n-1}(x,t)$ is symmetric, and then compute

$$\begin{aligned} K_n(x,t) &= \int_a^b K_{n-1}(x,s) K(s,t) \, ds \\ &= \int_a^b K_{n-1}(s,x) K(t,s) \, ds \qquad (7.4) \\ &= K_n(t,x) \ . \end{aligned}$$

Since, moreover, by the same argument, which makes use only of the known symmetry of $K(x,t)$, we know that $K_2(x,t) = K_2(t,x)$. Hence by induction we prove that all the iterated kernels are symmetric.

Let us now assume that of the kernels $K_2(x,t)$, $K_3(x,t)$, $\cdots$, the kernel $K_r(x,t)$, where $r \neq 1$, is the first null function. Then by the first equation in (7.4), $K_{r+1}(x,t)$ must be a null function, and hence by iteration all the other kernels must be null functions. But since either r, or $r+1$ must be an even number, $2m$, we shall have

$$K_{2m}(x,x) = \int_a^b K_m(x,s) K_m(s,x) \, ds = \int_a^b [K_m(x,s)]^2 \, ds = 0 \ ,$$

except over a set of points of measure zero, since $K_{2m}(x,t)$ is a null function. But this is impossible since $K_m(x,s)$ is not a null function. The contradiction proves the lemma.

Now returning to the sequence (7.3), we see that all the fractions exist and are not zero. In particular we have

$$\frac{S_2}{S_4} > 0 \ .$$

Since, moreover, the square of the radius of convergence of (7.1) is given by

$$\lambda_1^2 = \lim_{p=\infty} \frac{S_{2p}}{S_{2p+2}} \leqq \frac{S_2}{S_4} \ ,$$

we have proved that $|\lambda_1| \leqq \sqrt{S_2/S_4}$, which establishes the theorem.

Example. As an example, consider the kernel

$$K(x,t) = \begin{cases} x(1-t) \ , & x \leqq t \ , \\ t(1-x) \ , & x \geqq t \ . \end{cases}$$

We can show by direct calculation that $S_2 = 1/90$, $S_4 = 1/9450$, and hence that $|\lambda_1| \leqq \sqrt{(9450/90)} = \sqrt{105} = 10.2470$.

Since $D(\lambda) = (\sin \sqrt{\lambda}) / \sqrt{\lambda}$, it is clear that $\lambda_1 = \pi^2 = 9.8696$.

Theorem 10. All the principal values of a symmetric kernel are real.

Proof: Since the coefficients of $D(\lambda)$ are real, the conjugate, $\mu = a - bi$, of a complex root, $\lambda = a + bi$, must also be a principal value. Hence if $u = u_1 + i\,u_2$ is the principal function corresponding to λ, then $v = u_1 - i\,u_2$ is the principal function corresponding to μ.

Then from the orthogonality property we get

$$\int_a^b u(x)\,v(x)\,dx = \int_a^b [u_1^2(x) + u_2^2(x)]\,dx = 0 \ ,$$

which is impossible. From this contradiction we derive the proof of the theorem.

Theorem 11. If μ is the principal value for a symmetric kernel and $v(x)$ is the corresponding principal function, then μ^n and $v(x)$ are the corresponding principal value and principal function for the iterated kernel $K_n(x,t)$.

Proof: By hypothesis

$$v(s) = \mu \int_a^b K(s,t)\,v(t)\,dt \ ;$$

from which we derive

$$\int_a^b K(x,s)\,v(s)\,ds = \mu \int_a^b \int_a^b K(x,s)\,K(s,t)\,v(t)\,dt\,ds$$

$$= \mu \int_a^b K_2(x,t)\,v(t)\,dt = \frac{1}{\mu}\,v(x) \ ,$$

that is to say,

$$v(x) = \mu^2 \int_a^b K_2(x,t)\,v(t)\,dt \ .$$

Since this argument may be continued to higher kernels, we finally obtain

$$v(x) = \mu^n \int_a^b K_n(x,t)\,v(t)\,dt \ .$$

Theorem 12. If $K(x,t)$ is symmetric, then the poles of its resolvent are simple.

Proof: Let us assume the contrary, and for simplicity of argument, let us assume that the order of the zero $\lambda = \lambda_i$ of $D(\lambda)$ is 2. Then from equation (6.14) in which $p = 2$, equation (6.9) in which $m = 0$, and from (6.11) and its derivative with respect to λ, we readily obtain

$$D_1(x,t) = \lambda_i \int_a^b K(x,s) D_1(s,t)\, ds \ ,$$

$$\begin{aligned} D_2(x,t) &= \lambda_i \int_a^b K(x,s) D_2(s,t)\, ds + \int_a^b K(x,s) D_1(s,t)\, ds \\ &= \lambda_i \int_a^b K(x,s) D_2(s,t)\, ds + \frac{1}{\lambda_i} D_1(x,t) \ . \end{aligned}$$

Multiplying the first of these equations by $D_2(x,t)$ and the second by $D_1(x,t)$, subtracting the first product from the second, integrating with respect to t, and taking account of the symmetry of the kernel, we can show without difficulty that

$$\int_a^b [D_1(x,t)]^2\, dt = 0 \ .$$

Since $D_1(x,t)$ is not a null function, the assumption that the resolvent has a pole of second order is false and the theorem is proved.

From this theorem it is evident that the expansions (6.7) and (6.13) for symmetric kernels are always respectively the following:

$$k(x,t;\lambda) = \sum_{i=1}^{n} \frac{u_i(x) u_i(t)}{\lambda_i - \lambda} + R(x,t;\lambda) \ ; \qquad (7.5)$$

$$K(x,t) = \sum_{i=1}^{n} \frac{u_i(x) u_i(t)}{\lambda_i} + S(x,t;\lambda) \ . \qquad (7.6)$$

Theorem 13. Every continuous function of the form

$$F(x) = \int_a^b K(x,t) h(t)\, dt \qquad (7.7)$$

where $h(t)$ is a function of integrable square and $K(x,t)$ is a symmetric kernel of integrable square, can be developed in a uniformly convergent series of the principal functions which belong to $K(x,t)$.

Proof: This theorem is due to D. Hilbert and E. Schmidt and is generally referred to as the *Hilbert-Schmidt theorem.** We begin by assuming the expansion

*Hilbert: *Grundzüge*, pp. 24 et seq.; Schmidt: See *Bibliography.*

$$F(x)=\sum_{i=1}^{\infty} f_i\, u_i(x) \ , \tag{7.8}$$

from which we derive obviously

$$f_i=\int_a^b F(x)\, u_i(x)\, dx=\int_a^b \int_a^b K(x,t)\, h(t)\, u_i(x)\, dx\, dt=\frac{(u_i,h)}{\lambda_i} \ ,$$

where we use the abbreviation of (5.7).

Employing the Schwarz inequality (see section 9, chapter 3), we have

$$P_n^2 \equiv [f_n u_n(x)+\cdots+f_m u_m(x)]^2 \leqq [(u_n,h)^2+\cdots+(u_m,h)^2]$$
$$\times \left[\frac{u_n^2(x)}{\lambda_n^2}+\cdots+\frac{u_m^2(x)}{\lambda_m^2}\right] .$$

Since the quantities (u_r,h) are the Fourier coefficients of a function of integrable square, the sum of their squares converges by Bessel's inequality [see equation (5.5)]. Hence the first sum on the right side of the inequality is less than some arbitrarily small positive constant ε when n and m exceed M. Moreover, the quantities $u_r(x)/\lambda_r$ are the Fourier coefficients of the function $K(x, s)$ and since the integral of its square exists, the second sum on the right side of the inequality also exists by virtue of Bessel's inequality and is inferior to some arbitrarily small positive constant ε'. Hence P_n^2 is less than $\varepsilon\,\varepsilon'$ and from this we infer that series (7.8) converges uniformly and absolutely in (a, b).

In order to identify $F(x)$ with the series which forms the right member of (7.8), we designate the series by $S(x)$ and consider the difference

$$P(x)=F(x)-S(x) \ .$$

Since $P(x)$ is orthogonal to all the principal functions and hence to $K(x, t)$ we shall have

$$\begin{aligned}\int_a^b P^2(x)\, dx &= \int_a^b F(x)\, P(x)\, dx-\int_a^b S(x)\, P(x)\, dx\\ &= \int_a^b F(x)\, P(x)\, dx=\int_a^b P(x) \int_a^b K(x,t)\, h(t)\, dt\, dx\\ &= \int_a^b h(t) \int_a^b K(x,t)\, P(x)\, dx\, dt=0 \ .\end{aligned}$$

From this we infer that $P(x)$ is zero and hence that $F(x)$ is equivalent to $S(x)$.

Theorem 14. The principal values of a positive definite symmetric kernel are positive; conversely, a symmetric kernel is positive definite provided the principal values are positive.

Proof: Consider the integral

$$I(h) = \int_a^b \int_a^b K(x, t)\ h(x)\ h(t)\ dx\, dt\ .$$

We know from theorem 13 that

$$F(x) = \int_a^b K(x,t) h(t) dt = \sum_{i=1}^{\infty} f_i\, u_i(x)\ ,$$

from which we immediately infer that

$$I(h) = \int_a^b F(x) dx = \sum_{i=1}^{\infty} \frac{(u_i,h)^2}{\lambda_i}\ . \tag{7.9}$$

Hence, since $I(h) > 0$ by hypothesis, the principal values must be positive.

Conversely, if the principal values are positive then the kernel is positive definite. That the condition is sufficient is at once seen from (7.9). That the condition is also necessary is observed from the case $I(u_i) = 1/\lambda_i$.

Theorem 15. If $K(x, t)$ is a positive definite, continuous, symmetric kernel, then it may be represented as follows:

$$K(x,t) = \sum_{i=1}^{\infty} \frac{u_i(x) u_i(t)}{\lambda_i}\ , \tag{7.10}$$

where the series converges uniformly and absolutely.

Proof: This theorem is due to J. Mercer and is usually referred to as *Mercer's theorem.** We shall first represent the series comprising the right hand member of (7.10) by $S(x,t)$ and the first n terms by $S_n(x,t)$. Assuming then that $S(x,t)$ is uniformly convergent in the fundamental square, we shall begin by proving that $K(x,t) = S(x,t)$.

For this purpose let us consider the function

$$H(x,t) = K(x,t) - S(x,t)\ ,$$

*Functions of Positive and Negative Type and their connection with the Theory of Integral Equations. *Trans. of the London Phil. Soc.* (A), vol. 209 (1909), pp. 415-446. Other references to the subject of positive kernels include the following: H. Bateman: On Essentially Positive Double Integrals and the Part which they play in the Theory of Integral Equations. *British Assn. Report (Leicester)*, vol. 77 (1908), pp. 447-449; On Definite Functions. *Messenger of Math.*, vol. 37 (1907), pp. 91-95. H. S. Carslaw: Functions of Positive Type and their Application to the Determination of Green's Functions. *Messenger of Math.*, vol. 42 (1913), pp. 135-140. W. H. Young: A Note on a Class of Symmetric Functions and on a Theorem Required in the Theory of Integral Equations. *Messenger of Math.*, vol. 40 (1910-1911), pp. 37-43.

which is obviously symmetric. If it is not identically zero it will possess at least one principal value, λ, and a corresponding principal function, $u(x)$, which is not identically zero.

We then form the equation

$$\int_a^b u(x)\, u_i(x)\, dx = \lambda \int_a^b \int_a^b H(x,t)\, u(t)\, u_i(x)\, dx\, dt \ ,$$

the right hand member of which, from the uniform convergence of $S(x,t)$, can be integrated term by term. We thus obtain

$$\int_a^b u(x)\, u_i(x)\, dx = \frac{\lambda}{\lambda_i} \int_a^b u(t)\, u_i(t)\, dt - \frac{\lambda}{\lambda_i} \int_a^b u_i(t)\, u(t)\, dt = 0 \ .$$

Hence $u(x)$ is orthogonal to all the principal functions and thus

$$u(x) = \lambda \int_a^b H(x,t)\, u(t)\, dt = \lambda \int_a^b K(x,t)\, u(t)\, dt \ ,$$

that is to say, $u(x)$ is a principal function and λ a principal value of the kernel. From this one immediately concludes that $u(x)$ is either identically zero or else a linear combination of the members of the original set of principal functions. But the fact that $u(x)$ is orthogonal to all the members of the set proves that the second possibility is untenable and hence $u(x) \equiv 0$. This contradiction proves that $H(x,y) \equiv 0$.

We shall now show that $S(x,t)$ is uniformly and absolutely convergent in both variables, provided $K(x,t)$ satisfies the conditions of the theorem. Since the kernel is positive definite, all the principal values are positive; hence we obtain from the Schwarz inequality (see section 9, chapter 3):

$$[S_m(x,t) - S_n(x,t)]^2 = \left[\sum_{i=n}^{m} \frac{u_i(x)}{\sqrt{\lambda_i}} \frac{u_i(t)}{\sqrt{\lambda_i}} \right]^2 \leqq \sum_{i=n}^{m} \frac{u_i^2(x)}{\lambda_i} \sum_{i=n}^{m} \frac{u_i^2(t)}{\lambda_i} \ .$$

We now observe the inequality

$$K(x,x) - \sum_{i=1}^{n} \frac{u_i^2(x)}{\lambda_i} \geqq 0 \ , \qquad (7,11)$$

and in connection with it the following lemma due to U. Dini:*

If a series of positive continuous functions of a single variable converges to a continuous function, then the series converges uniformly in its region of definition.

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Since from (7.11) the series $\sum u_i^2(x)/\lambda_i$ is seen to conform to the conditions of the Dini lemma, it converges uniformly in (a, b). From this it follows that there exist positive quantities ε and M independent of the variables, such that

$$[S_m(x,t) - S_n(x,t)]^2 < \varepsilon ,$$

when $m, n > M$, and from this we conclude that $S(x,t)$ converges uniformly in the fundamental square.

Corollary: If $K(x,t)$ is a symmetric kernel, then $K_2(x,t)$ is given by

$$K_2(x,t) = \frac{u_i(x)\, u_i(t)}{\lambda_i^2} ,$$

and the series converges absolutely and uniformly in the fundamental square.

The proof is derived from the observation that the principal values of $K_2(x,t)$ are positive. Hence by theorem 14 the kernel is positive definite. Since it is also continuous we may apply the results of theorem 15.

A very important application of the results discussed above is found in the theory of the following Sturmian system:

$$\frac{d}{dx}(p\,u') - q\,u + \lambda\,k\,u = 0 ,$$

$$u'(a) = h\,u(a) ,$$

$$u'(b) = -H\,u(b) , \tag{7.12}$$

where $p(x)$, $q(x)$, and $k(x)$ are positive and continuous and $p'(x)$ is continuous in the interval (a,b). The constants h and H are positive.

Referring to the theory developed in section 5, chapter 11, it is at once seen that $\Gamma(x,t)$, the Green's function for the equation

$$L(u) \equiv \frac{d}{dx}(p\,u') - q\,u = 0 , \tag{7.13}$$

subject to the boundary conditions of (7.12), is symmetric since the system is self-adjoint.

Moreover, system (7.12) is equivalent to the integral equation

$$u(x) = \lambda \int_a^b \Gamma(x,t) k(t) u(t)\, dt . \tag{7.14}$$

If we write $w(x) = \sqrt{k(x)}\, u(x)$, then this equation can be put in the symmetric form

$$w(x) = \lambda \int_a^b K(x,t) w(t)\, dt , \tag{7.15}$$

where we abbreviate

$$K(x,t) = \Gamma(x,t)\, [k(x) k(t)]^{\frac{1}{2}} . \tag{7.16}$$

We shall now prove that *the kernel* $K(x,t)$ *is positive definite.* For this purpose consider the following identity:

$$\int_a^b u(x)L(u)\,dx = -\int_a^b \{p[u'(x)]^2 + q\,u^2\}\,dx + [p\,u\,u']_a^b \qquad (7.17)$$

which is easily established from the identities of section 2, chapter 11.

If now we let $\lambda = \lambda_n$, where λ_n is a principal value of (7.14) [and also of (7.15)], and if $u(x) = u_n(x)$, the corresponding principal function normalized by the condition

$$\int_a^b k(x)\,u_n^2(x)\,dx = 1 \ ,$$

then (7.17) becomes

$$\lambda_n = \int_a^b \{p[u'_n(x)]^2 + q\,u_n^2\}\,dx + h\,p(a)\,[u_n(a)]^2 + H\,p(b)\,[u_n(b)]^2 \ .$$

From the conditions on the functions and the two constants, it is obvious that λ_n is a positive number. Hence from theorem 14 the kernel $K(x,t)$ must be positive definite.

We are thus able to deduce that the principal functions of the differential system (7.12) form a closed, and hence infinite, set and that an arbitary function, $f(x)$, of integrable square can be expanded in terms of them. The series thus obtained is referred to as a *Sturm-Liouville series.*

We have already discussed at length in example 2, section 1, a special case. The reason why such elegant results were obtained is now seen to be essentially the positive definite character of the kernel of the equivalent integral equation.

PROBLEMS

1. Prove that for a symmetric kernel

$$\lambda_p = \lim_{n\to\infty} (k_p)^{1/n}\,(Q_n)^{-1/n} \ ,$$

where $k_1, k_2, \cdots, k_{p-1}$ are the multiplicities of the roots $\lambda_1, \lambda_2, \cdots, \lambda_{p-1}$, and where we abbreviate

$$Q_n = S_n - \frac{k_1}{\lambda_1^n} - \frac{k_2}{\lambda_2^n} - \cdots - \frac{k_{p-1}}{\lambda^n_{p-1}} \ .$$

Hint: Consider the development of

$$-\sum_{m=0}^{\infty} S_m\,\lambda^m - \frac{k_1}{\lambda - \lambda_1} - \frac{k_2}{\lambda - \lambda_2} \ .$$

Then set $\lambda = \lambda_2 + \mu$, and show that

$$-S_m + k_1/\lambda_1^{m+1} + k_2/\lambda_2^{m+1} = \varepsilon_m(\lambda_2 + \mu)^{-m} \ ,$$

where $\varepsilon_m \to 0$. [A. Kneser: Ein Beitrag zur Theorie der Integralgleichungen. *Rendiconti di Palermo,* vol. 22 (1906), pp. 233-240.]

2. Show that for a symmetric kernel

$$\lambda_n{}^2 = \lim_{m=\infty} \frac{S^{(n-1)}_{2m+2}\ S^{(n)}_{2m}}{S^{(n-1)}_{2m}\ S^{(n)}_{2m+2}},$$

where we abbreviate

$$S_m{}^{(n)} = \int_a^b \cdots \int_a^b K_m\begin{pmatrix} t_1\, t_2 \cdots t_n \\ t_1\, t_2 \cdots t_n \end{pmatrix} dt_1\, dt_2 \cdots dt_n\ .$$

From the expansion

$$S_m{}^{(n)} = \sum_{s_1, s_2, s_n} (-1)^{s_1 + s_2 + \cdots} \left(\frac{S_m}{1}\right)^{s_1} \left(\frac{S_{2m}}{2}\right)^{s_2} \cdots \left(\frac{S_{nm}}{n}\right)^{s_n}$$

where $s_1 + 2s_2 + \cdots + n\, s_n = n$, prove that

$$\lambda_2{}^2 = \lim_{m=\infty} \frac{S_{2m+2}(S^2{}_{2m} - S_{4m})}{S_{2m}(S^2{}_{2m+2} - S_{2m+4})},$$

$$\lambda_3{}^2 = \lim_{m=\infty} \frac{(S^2{}_{2m+2} - S_{4m+4})(S^3{}_{2m} - 3S_{2m}S_{4m} + 2S_{6m})}{(S^2{}_{2m} - S_{4m})(S^3{}_{2m+3} - 3S_{2m+2}S_{4m+4} + 2S_{6m+6})}.$$

[I. Schur: Zur Theorie der linearen homogene Integralgleichungen. *Mathematische Annalen*, vol. 67 (1909), pp. 306-339.]

The theory of the skew-symmetric kernel is similar in many respects to the theory of the symmetric kernel. The following theorems contain the essential characteristics of this kernel:

3. Show that a skew-symmetric kernel has at least two principal values.

4. Prove that the principal values of a real skew-symmetric kernel are pure imaginaries.

5. Show that the poles of the resolvent of a skew-symmetric kernel are simple.

6. Prove that if $u(x)$ is the principal function of a skew-symmetric kernel, then $\bar{u}(x)$, the conjugate imaginary of $u(x)$, is the principal function of the associated kernel.

8. The Equivalence of the Theories of Quadratic Forms and Integral Equations. We have already commented in section 6 about the remarkable dualism which exists between the theory of quadratic forms and the theory of integral equations. The reason for this striking similarity is easily exhibited by means of the properties of the symmetric kernel which were developed in the preceding section.

Let $x(t)$ represent a function of integrable square and let us consider the following expansion valid in an interval $a \leqq t \leqq b$:

$$x(t) = x_1\varphi_1(t) + x_2\varphi_2(t) + x_3\varphi_3(t) + \cdots\ . \qquad (8.1)$$

The functions $\varphi_i(t)$ are the members of a closed set of normalized, orthogonal functions in (a, b), and hence we have

$$x_i = \int_a^b x(t)\,\varphi_i(t)\,dt\,. \tag{8.2}$$

Let us now introduce a symmetric kernel, $K(s,t)$, to which belongs the normalized set of principal functions: $u_1(s)$, $u_2(s)$, $u_3(s), \cdots$; we shall assume that the set $\{u_i(s)\}$ is closed over the interval (a, b).

Then if we define

$$\int_a^b \int_a^b K(s,t)\,\varphi_p(s)\,\varphi_q(t)\,ds\,dt = a_{pq} = a_{qp}\,,$$

we shall attain immediately from (8.1) and (8.2) the symmetric quadratic form

$$F(x,x) = \int_a^b \int_a^b K(s,t)\,x(s)\,x(t)\,ds\,dt = \sum_{pq=1}^{\infty} a_{pq}\,x_p\,x_q\,.$$

Since by assumption both the set $\{\varphi_i(t)\}$ and the set $\{u_j(s)\}$ are closed, we can form the series

$$x(t) = \sum_{i=1}^{\infty} y_i\,u_i(t)\,, \qquad y_i = \int_a^b x(t)\,u_i(t)\,dt\,;$$

$$\varphi_i(t) = \sum_{j-1}^{\infty} \varphi_{ij}\,u_j(t)\,, \qquad \varphi_{ij} = \int_a^b \varphi_i(t)\,u_j(t)\,dt\,;$$

$$u_j(t) = \sum_{i=1}^{\infty} \varphi_{ij}\,\varphi_i(t)\,.$$

Moreover, from the equation

$$\int_a^b \varphi_i(t)\,\varphi_j(t)\,dt = \sum_{k=1}^{\infty} \varphi_{ik}\varphi_{jk} = \delta_{ij}\,,$$

it is seen that the matrix $||\varphi_{ij}||$ is orthogonal.

From the equations

$$\begin{aligned} x(t) &= \sum_{ij=1}^{\infty} x_i\,\varphi_{ij}\,u_j(t) = \sum_{j=1}^{\infty} y_j\,u_j(t) \\ &= \sum_{ij=1}^{\infty} y_j\,\varphi_{ij}\,\varphi_j(t) = \sum_{i=1}^{\infty} x_i\,\varphi_i(t)\,, \end{aligned}$$

we at once obtain the orthogonal transformations:

$$y_j = \sum_{i=1}^{\infty} x_i\,\varphi_{ij}\,, \qquad x_i = \sum_{j=1}^{\infty} y_j\,\varphi_{ij}\,.$$

Since the functions $u_i(t)$ belong to the kernel $K(s,t)$, it is clear that the form $F(x,x)$ can be normalized in terms of the y_i as follows:

$$F(x,x) = \int_a^b x(s) \int_a^b K(s,t) \sum_{p=1}^{\infty} y_p u_p(t)\, dt\, ds$$

$$= \int_a^b x(s) \sum_{p=1}^{\infty} y_p \frac{u_p(s)}{\lambda_p}\, ds = \sum_{p=1}^{\infty} \frac{y_p^2}{\lambda_p}$$

Moreover, from the integration of the equation

$$x(s) = \lambda \int_a^b K(s,t)\, x(t)\, dt \ ,$$

we obtain

$$\int_a^b x(s)\, ds = \lambda \int_a^b \int_a^b K(s,t)\, x(t)\, ds\, dt \ ,$$

from which we immediately derive the following infinite system:

$$x_p - \sum_{q=1}^{\infty} k_{pq} x_q = 0 \ , \quad p = 1, 2, 3, \cdots . \tag{8.3}$$

Conversely, from the solutions of (8.3) we derive

$$x(s) = \lambda \int_a^b K(s,t) \sum_{p=1}^{\infty} x_p \varphi_p(t)\, dt$$

$$= \lambda \sum_{p=1}^{\infty} x_p \int_a^b K(s,t)\, \varphi_p(t)\, dt \ .$$

Having thus obtained the main features of the theory developed in section 2, we see that the construction of the resolvent of the form $F(x,x)$ may be obtained in an obvious manner.

9. The Continuous Spectrum of a Quadratic Form — Hilbert's Example. In section 2 there was developed a normalization theory of quadratic forms which centered around the properties of the characteristic numbers of the form. The formal development of this theory, however, assumed the existence of a discrete spectrum and any condensation of the members of the spectrum in the finite plane was excluded. That this assumption is a severe restriction upon the theory of infinite quadratic forms was pointed out by Hilbert, who gave a simple, but elegant example, which it will be profitable for us to discuss at this point.

Let us consider the following quadratic form:

$$F(x,x) = 2(x_1 x_2 + x_2 x_3 + x_3 x_4 + \cdots) \ , \tag{9.1}$$

which has for characteristic numbers the roots of the equation

$$\Delta(\mu) \equiv \begin{vmatrix} \mu & -1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & \mu & -1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & \mu & -1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & \mu & -1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0 \ . \qquad (9.2)$$

In order to determine explicitly the nature and distribution of these roots let us represent the mth principal minor of $\Delta(\mu)$ by $\Delta_m(\mu)$. Expanding this determinant by the elements of the first column, we immediately obtain the recurrence relation

$$\Delta_m(\mu) = \mu\Delta_{m-1}(\mu) - \Delta_{m-2}(\mu) \ .$$

We compare this with the similar formula for the Legendre polynomials, namely,

$$P_m(\tfrac{1}{2}\mu) = (1-1/2m)\,\mu P_{m-1}(\tfrac{1}{2}\mu) - (1-1/m)P_{m-2}(\tfrac{1}{2}\mu) \ ,$$

and thus observe that when $m \to \infty$, we have

$$\Delta_m(\mu) \to k\,P_m(\tfrac{1}{2}\mu) \ .$$

Noting the asymptotic formula*

$$P_n(\cos\vartheta) \sim \left(\frac{2}{\pi\sin\vartheta}\right)^{\frac{1}{2}} \frac{\Gamma(n+1)}{\Gamma(n+3/2)} \sin\left[(n+\tfrac{1}{2})\vartheta + \tfrac{1}{4}\pi\right] + O(1/n^{3/2}) \ ,$$

where ϑ is between 0 and π, we see that the zeros of $P_n(\tfrac{1}{2}\mu)$ are confined to the interval between -2 and $+2$, and tend to a continuous set as $n \to \infty$.

We thus infer that *the characteristic numbers associated with the quadratic form* (9.1) *form a continuous spectrum between* -2 *and* $+2$.

In order to bring this conclusion within the scope of the theory developed in section 2, we now replace the transformation $T \to F$ [equation (2.18)] by the integral

$$T \to F = \int_{-2}^{2} \mu\, y^2(\mu)\, d\mu \ . \qquad (9.3)$$

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*See E. W. Hobson: *The Theory of Spherical and Ellipsoidal Harmonics.* Cambridge, 1931, p. 297.

Since $y(\mu)$ is an orthogonal linear form in the variables x_i, we define it as follows:

$$y(\mu) = \sum_{i=1}^{\infty} u_i(\mu) x_i \ , \tag{9.4}$$

where $\{u_i(\mu)\}$ is a normalized set of functions orthogonal in the interval between -2 and $+2$.

Such a set is furnished by

$$u_p(\mu) = \frac{1}{(\pi)^{\frac{1}{2}}} \frac{\sin pt}{(\sin t)^{\frac{1}{2}}} \ , \qquad \cos t = \tfrac{1}{2}\mu \ .$$

From the fact that

$$\int_0^\pi \cos t \sin mt \sin nt \, dt = \begin{cases} \tfrac{1}{2}\pi \ , & \text{when } n = m+1 \text{ or } m-1 \ , \\ 0 \ , & \text{when } n \neq m+1 \text{ or } m-1 \ , \end{cases}$$

we immediately derive

$$\int_{-2}^{2} y^2(\mu) \, d\mu = \sum_{i=1}^{\infty} x_i^2 \ ,$$

$$\int_{-2}^{2} \mu \, y^2(u) \, d\mu = F(x,x) \ .$$

We next note that the reciprocal of the form, $\lambda I(x,x) - F(x,x)$, is given by

$$R(x,x;\lambda) = \int_{-2}^{2} \frac{y^2(\mu) \, d\mu}{\lambda - \mu} \tag{9.5}$$

$$= \sum_{pq=1}^{\infty} R_{pq}(\lambda) \, x_p x_q \ ,$$

where we abbreviate

$$R_{pq}(\lambda) = \frac{2}{\pi} \int_0^\pi \frac{\sin pt \sin qt}{\lambda - 2 \cos t} \, dt$$

$$= \frac{1}{\pi} \int_0^\pi \frac{\cos(p-q)t - \cos(p+q)t}{\lambda - 2\cos t} \, dt \ .$$

From the integral

$$\int_0^\pi \frac{\cos ax \, dx}{\lambda - 2\cos x} = \frac{\pi}{(\lambda^2-4)^{\frac{1}{2}}} \left[\frac{\lambda - (\lambda^2-4)^{\frac{1}{2}}}{2} \right]^a ,$$

where a is a positive integer and $\lambda > 2$, we derive

$$R_{pq}(\lambda) = -\frac{2}{(\lambda^2-4)^{\frac{1}{2}}}\begin{cases} e^{pL}\sinh qL\ , & p > q\ , \\ e^{pL}\sinh pL\ , & p = q\ , \\ e^{qL}\sinh pL\ , & p < q\ , \end{cases}$$

where we employ the abbreviation

$$L = \log[\tfrac{1}{2}\{\lambda - (\lambda^2-4)^{\frac{1}{2}}\}]\ .$$

It is instructive to verify this result for the cases $p = q = 1$, and $p = 2,\ q = 1$.

Since $\lambda I(x,x) - F(x,x)$ is a Jacobi form (see chapter 3), we can investigate explicitly the solution of the associated system of linear equations, that is to say, the system

$$\begin{aligned} \lambda x_1 - x_2 &= 1\ , \\ -x_1 + \lambda x_2 - x_3 &= 0\ , \\ -x_2 - \lambda x_3 - x_4 &= 0\ , \\ -x_3 - \lambda x_4 - x_5 &= 0\ , \\ \cdots\cdots&\cdots\cdots \end{aligned} \tag{9.6}$$

It is clear that the general solution of this system is given by

$$x_p = R_{p1}(\lambda)\ ,$$

and hence we immediately derive

$$x_1 = -\frac{2}{(\lambda^2-4)^{\frac{1}{2}}}\, e^L \sinh L = \tfrac{1}{2}\lambda - \tfrac{1}{2}(\lambda^2-4)^{\frac{1}{2}}\ ,$$

$$x_2 = -\frac{2}{(\lambda^2-4)^{\frac{1}{2}}}\, e^{2L} \sinh L = e^L x_1 = \tfrac{1}{2}\lambda^2 - \tfrac{1}{2}\lambda(\lambda^2-4)^{\frac{1}{2}} - 1\ .$$

Referring now to the explicit solution for x_1 given in section 2, chapter 3, we can compute x_1 directly as follows:

$$x_1 = \cfrac{1}{\lambda - \cfrac{1}{\lambda - \cfrac{1}{\lambda - \cfrac{1}{\lambda - \cdots}}}}\ ,$$

that is,

$$x_1 = \frac{1}{\lambda - x_1}\ .$$

From this equation we derive immediately

$$x_1 = \tfrac{1}{2}\lambda - \tfrac{1}{2}(\lambda^2-4)^{\frac{1}{2}}\ ,$$

and from the first equation of system (9.6),

$$x_2 = \lambda x_1 - 1 = \tfrac{1}{2}\lambda^2 - \tfrac{1}{2}\lambda(\lambda^2 - 4)^{\frac{1}{2}} - 1 = x_1^2 \ ,$$

results which agree with those obtained above.

We now inquire whether the solutions of (9.6) belong to Hilbert space. From the explicit values of $R_{p1}(\lambda)$, it is immediately seen that

$$x_p^2 = R^2{}_{p1}(\lambda) = x_1^{2p} \ ,$$

and hence the convergence of the series

$$\sum_{p=1}^{\infty} x_p^2 \ ,$$

is dependent upon the inequality $x_1^2 < 1$. This leads to the definition $|\lambda| > 2$, and hence *the solutions of* (8.6) *belong to Hilbert space for all values of* λ *not included in the spectrum of* $F(x,x)$.

E. Hellinger in a notable paper: Neue Begründung der Theorie quadratischer Formen von unendlichvielen Veränderlichen, *Journal für Mathematik,* vol. 136 (1909), pp. 210-271, took his departure from a consideration of the solutions of the homogeneous system corresponding to (9.6), that is to say, of the system

$$\begin{aligned} \lambda x_1 - \quad x_2 &= 0 \ , \\ -x_1 + \lambda x_2 - \quad x_3 &= 0 \ , \\ -x_2 - \lambda x_3 - x_4 &= 0 \ , \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{9.7}$$

If we replace λ by $2 \cos \mu$ and assume that $x_1 = a \sin \mu$, we may then compute

$$x_2 = \lambda x_1 = a\, 2 \sin \mu \cos \mu = a \sin 2\mu \ ,$$

$$x_3 = -x_1 + \lambda x_2 = -a \sin \mu + a\, 2 \cos \mu \sin 2\mu = a \sin 3\mu \ ,$$

and in general,

$$x_n = a \sin n\mu \ .$$

It is now observed that this solution does not belong to Hilbert space since the series

$$\sum_{m=1}^{n} x_m^2 = a^2 \sum_{m=1}^{n} \sin^2 m\mu = a^2 [\tfrac{1}{2}n + \tfrac{1}{4} - \tfrac{1}{4}\{\sin(2n+1)\mu / \sin \mu\}]$$

has no limit as $n \to \infty$.

Hellinger observed, however, that the functions

$$y_n(\mu) = \int_0^{\mu} x_n(\mu)\, d\mu$$

belong to Hilbert space and hence he was led to broaden the field of permissible solutions to include those whose *integrals* belong to Hilbert space.

PROBLEMS

1. Show that the spectrum of the form

$$F(x,x) = 2(a_1x_1x_2 + a_2x_2x_3 + a_3x_3x_4 + \cdots) ,$$

where $a_m = m/(4m^2 - 1)^{\frac{1}{2}}$, is the totality of values between $+1$ and -1. Show that the generating set of normalized orthogonal functions is given by

$$u_{p+1}(\mu) = \sqrt{\tfrac{1}{2}(2p+1)}\; P_p(\mu) \;, \quad p = 0, 1, 2, \cdots .$$

where $P_n(x)$ is the nth Legendrian polynomial satisfying the equations

$$(n+1)P_{n+1}(x) - (2n+1)x\,P_n(x) + n\,P_{n-1}(x) = 0 \;,$$

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & m \neq n \;, \\ 2/(2n+1) \;, & m = n \;. \end{cases} \qquad \text{(Hilbert).}$$

2. Show that the reciprocal form $[\mu I(x,x) - F(x,x)]^{-1}$, where $F(x,x)$ is the quadratic form of problem 1, is equal to

$$R(x,x;\lambda) = K\sum_{pq=1}^{\infty} \sqrt{(2p-1)(2q-1)}\; P_{p-1}(\lambda)\,Q_{q-1}(\lambda)\,x_p\,x_q \;, \quad q \geqq p \;,$$

in which $Q_q(\lambda)$ is the Legendrian function of second kind and K is 1 or 2 as q is equal to or greater than p.* (Hilbert).

3. Solve the system of equations

$$\begin{aligned} \lambda\,x_1 - a_1\,x_2 &= 1 \;, \\ -a_1\,x_1 + \lambda\,x_2 - a_2\,x_3 &= 0 \;, \\ -a_2\,x_2 + \lambda\,x_3 - a_3\,x_4 &= 0 \;, \\ \cdots\cdots\cdots \end{aligned}$$

where the values of the coefficients are those given in the first problem. (See section 2, chapter 3).

10. Some General Considerations Concerning the Continuous Spectra of Quadratic Forms. The special problem which we have discussed at length in the preceding section may be generalized in the following manner.

Let us assume that we have a quadratic form

$$F(x,x) = \sum_{ij=1}^{\infty} a_{ij}\,x_i\,x_j \;, \quad (a_{ij} = a_{ji}) \tag{10.1}$$

where the variables are assumed to belong to Hilbert space. The form may have both a discrete spectrum, $\mu_1, \mu_2, \mu_3, \cdots$ and a continuous

*This depends upon establishing the identity

$$\int_{-1}^{1} \frac{P_p(\mu)P_q(\mu)}{\lambda - \mu}\,d\mu = 2\,P_p(\lambda)\,Q_q(\lambda) \;, \quad q \geqq p \;.$$

For $p = 0$, this reduces to what is called *Neumann's formula* [*Journal für Math.*, vol. 37 (1848), p. 24.]

spectrum over a set of intervals, M_1, M_2, M_3, $\cdots$ in the infinite range from $-\infty$ to $+\infty$.

Under these conditions we may assume that (10.1) can be written in the following way:

$$F(x,x) = \sum_{i=1}^{\infty} \mu_i \left(\sum_{j=1}^{\infty} u_{ij}\, x\right)^2 + \int_{-\infty}^{\infty} \mu\, d\sigma(x,x;\mu) \ , \qquad (10.2)$$

where the matrix $||u_{ij}||$ is orthogonal and the quantity $\sigma(x,x;\mu)$ is a quadratic form which depends upon the continuous variable μ. This form is assumed to be identically zero except over the range, M_1, M_2, M_3, $\cdots$ of the continuous spectrum.

Similarly the inverse form

$$R(x,x;\mu) = [\mu I(x,x) - F(x,x)]^{-1}$$

may be written

$$R(x,x;\mu) = \sum_{i=1}^{\infty} \frac{\left(\sum\limits_{j=1} u_{ij}\, x_j\right)^2}{\mu - \mu_i} + \int_{-\infty}^{\infty} \frac{d\sigma(x,x;\lambda)}{\mu - \lambda} \ . \qquad (10.3)$$

The second term of the right hand member of this equation has been called by Hilbert the *spectral form* associated with $F(x,x)$.

A special case of particular importance is found in the theory of Laurent forms, which we have defined in chapter 3. The general *Laurent* or *L-form* may be written

$$C(x,x) = \sum_{pq=-\infty}^{\infty} c_{q-p}\, x_p\, x_q \ , \qquad (10.4)$$

which, for convenience, we shall assume is Hermitian.

Now consider the *associated function,*

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots$$
$$\qquad + c_{-1} z^{-1} + c_{-2} z^{-2} + \cdots \ ,$$

the expansion being assumed convergent within an annulus which includes the unit circle in its interior.

We shall then have

$$f(e^{\varphi i}) = c_0 + c_1(\cos\varphi + i\sin\varphi) + c_2(\cos 2\varphi + i \sin 2\varphi) + \cdots$$
$$+ c_{-1}(\cos\varphi - i\sin\varphi) + c_{-2}(\cos 2\varphi - i\sin 2\varphi) + \cdots \ .$$

If we denote $f(e^{\varphi i})$ by $F(\varphi)$, we may compute

$$\int_0^{2\pi} \sin n\varphi\, F(\varphi)\, d\varphi = i\pi(c_n - c_{-n}) \ , \int_0^{2\pi} \cos n\varphi\, F(\varphi)\, d\varphi = \pi(c_n + c_{-n}) .$$

Representing $e^{n\varphi i}$ by φ_n and $e^{-n\varphi i}$ by $\overline{\varphi}_n$, we obtain

$$\int_0^{2\pi} \varphi_n F(\varphi)\, d\varphi = 2\pi c_{-n} \,, \quad \int_0^{2\pi} \overline{\varphi}_n F(\varphi)\, d\varphi = 2\pi c_n \,,$$

Then if we define the two functions

$$x(\varphi) = \sum_{i=1}^{\infty} x_i\,(\varphi_i)\,, \qquad \overline{x}(\varphi) = \sum_{i=1}^{\infty} \overline{x}_i\, \overline{\varphi}_i\,,$$

and observe the orthogonality condition

$$\int_0^{2\pi} \varphi_m \overline{\varphi}_n\, d\varphi = \delta_{mn}\,,$$

it is clear that we shall have

$$\int_0^{2\pi} x(\varphi)\, \overline{x}(\varphi)\, d\varphi = 2\pi(x_1 \overline{x}_1 + x_2 \overline{x}_2 + \cdots)$$

$$\int_0^{2\pi} F(\varphi)\, x(\varphi)\, \overline{x}(\varphi)\, d\varphi = 2\pi \sum_{pq=-\infty}^{\infty} c_{q-p}\, x_p\, x_q = 2\pi\, C(x,x)\,.$$

In order to determine the spectrum of the form we make the transformation

$$\mu = F(\varphi)\,.$$

Hence, since μ varies from $\mu_1 = F(0)$ to $\mu_2 = \max F(\varphi)$ and from μ_2 to $\mu_1 = F(2\pi)$ as φ varies from 0 to 2π, we obtain the following theorem proved by O. Toeplitz:*

Theorem 16. The spectrum of a regular L-form contains the totality of values which the associated analytic function assumes upon the unit circle.

An example of particular interest is furnished by the L-form whose associated function is $f(z) = \log z$. Then we have $F(\varphi) = \varphi i$ and hence

$$C(x,x) = \frac{1}{2\pi} \int_0^{2\pi} \varphi\, i\, x(\varphi)\, \overline{x}(\varphi)\, d\varphi = \sum_{pq=-\infty}^{\infty}{}' \frac{x_p\, \overline{x}_q}{p-q}\,,$$

where the dash on the symbol of summation denotes that the terms for which $p \neq q$ are to be excluded.

The spectrum is obviously the totality of values from 0 to 2π.

The theory of the L-form may be generalized in the following manner: Let us suppose that $u_1(t)$, $u_2(t)$, $u_3(t)$, $\cdots$ is a normalized, orthogonal set of functions which is closed over the interval (a, b).

*Zur Theorie der quadratischen und bilinearen Formen von unendlichvielen Veränderlichen. I. Teil: Theorie der L-Formen. *Math. Annalen*, vol. 70 (1911), pp. 351-376.

If we then define two functions of integrable square

$$x(t) = \sum_{i=1}^{\infty} x_i u_i(t) \quad , \quad y(t) = \sum_{i=1}^{\infty} y_i u_i(t) \quad ,$$

we shall have

$$\int_a^b x^2(t)\, dt = \sum_{i=1}^{\infty} x_i^2 \ , \ \int_a^b x(t)\, y(t)\, dt = \sum_{i=1}^{\infty} x_i y_i \ , \ \int_a^b y^2(t) dt = \sum_{i=1}^{\infty} y_i^2 .$$

Next let us define a third function $F(t)$ in terms of which we obtain the coefficients of the form,

$$a_{ij} = \int_a^b F(t)\, u_i(t)\, u_j(t)\, dt \ .$$

Thus we construct the bilinear form

$$F(x,y) \equiv \int_a^b F(t)\, x(t)\, y(t) = \sum_{ij=1}^{\infty} a_{ij}\, x_i\, y_j \ . \quad (10.5)$$

If now we set $F(t) = \mu$, $t = \varphi(\mu)$, then (10.5) can be put in normal form

$$F(x,y) = \int_{F(a)}^{F(b)} \mu \frac{x^*(\mu)}{\sqrt{\psi(\mu)}} \frac{y^*(\mu)}{\sqrt{\psi(\mu)}}\, d\mu \ ,$$

where $x^*(\mu)$ and $y^*(\mu)$ designate $x(t)$ and $y(t)$ after the transformation. We also use the abbreviation $\psi(\mu) = 1/\varphi'(\mu)$.

Since also

$$\int_a^b u_i(t)\, u_j(t)\, dt = \int_{F(a)}^{F(b)} U_i(\mu)\, U_j(\mu)\, d\mu = \delta_{ij} \ ,$$

where we write $U_i(\mu) = u_i[\varphi(\mu)]/\sqrt{\varphi(\mu)}$, we may define the functions $U_i(\mu)$ as the basic set of the bilinear form. The spectrum is obviously the continuous interval from $\mu_1 = F(a)$ to $\mu_2 = F(b)$.

This theory will be seen to include the Hilbert example discussed in the previous section.

PROBLEMS

1. Show that the Jacobi form

$$J \equiv a_1 x_1^2 + a_2 x_2^2 + \cdots - 2\, b_1 x_1 x_2 - 2\, b_2 x_2 x_3 - \cdots$$

in which we set $a_1 = a_2 \cdots = 0$, $b_1 = b_3 = b_5 = \cdots = a$, $b_2 = b_4 = b_6 = \cdots = b$, $(ab = 1)$, has a continuous spectrum from $-a - b$ to $-a + b$, and from $a - b$ to $a + b$. (Toeplitz).

2. Show that for the Jacobi form of problem 1

$$R_{11}(\lambda) = (1/2\lambda)[a^2\lambda^2 - (a^4 - 1)$$

$$\pm \sqrt{(a\lambda + a^2 + 1)(a\lambda - a^2 + 1)(a\lambda + a^2 - 1)(a\lambda - a^2 - 1)}\,]$$

(Toeplitz).

3. Given the series

$$f(x) = a_0 + 2\sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx) \ ,$$

and the determinant

$$D_n(f-\lambda) = \begin{vmatrix} a_0 - \lambda \ , & a_1 + i\,b_1 \ , & a_2 + i\,b_2 \ , & \cdots, & a_n + i\,b_n \\ a_1 - i\,b_1 \ , & a_0 - \lambda \ , & a_1 + i\,b_1 \ , & \cdots, & a_{n-1} + i\,b_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n - i\,b_n \ , & a_{n-1} - i\,b_{n-1} \ , & a_{n-2} - i\,b_{n-2} \ , & \cdots, & a_0 - \lambda \end{vmatrix} \ ,$$

show that

$$\lim_{n=\infty} (\lambda_0^{(n)} \lambda_1^{(n)} \cdots \lambda_n^{(n)})^{1/n} = \lim_{n=\infty} [D_n(f)]^{1/n} = \exp\left[(1/2\pi)\int_0^{2\pi} \log f(x)\,dx\right] \ .$$

[This problem is due to G. Pólya: *L'intermédiare des mathématiciens*, vol. 21 (1914), p. 27, question 4340. For a discussion of it see G. Szegö: *Mathematische Annalen*, vol. 76 (1915), pp. 490-503.]

4. Apply the theorem of problem 3 to the Poisson series

$$\frac{1 - r^2}{1 - 2r\cos x + r^2} = 1 + 2\sum_{n=1}^{\infty} r^n \cos nx \ , \quad |r| < 1 \ .$$

5. Show that the function

$$u(x) = e^{-x^2/4}\, h_n(x) \ ,$$

where $h_n(x)$ is the nth Hermite polynomial, satisfies the integral equation

$$u(x) = \lambda \int_{-\infty}^{\infty} e^{\frac{1}{2}itx}\, u(t)\, dt \ ,$$

when $\lambda_n^2 = (-1)^n/4\pi$. [See A. Milne: On the Equation of the Parabolic Cylinder Function. *Proc. of the Edinburgh Math Soc.*, vol. 32 (1914), pp. 2-14.]

6. Show that the integral equation

$$u(x) = \lambda \int_{-\pi}^{\pi} e^{k \cos x \cos t}\, u(t)\, dt$$

is satisfied by even periodic solutions of period 2π of the differential equation

$$u''(x) + (a^2 + k^2 \cos^2 x)\, u(x) = 0 \ .$$

[This problem is due to E. T. Whittaker. See Whittaker and Watson: *Modern Analysis.* (3rd. ed.) Cambridge (1920), pp. 407-411.]

7. A function $f(x)$ is said to be self-reciprocal with respect to a transformation T, provided

$$f(x) = T[f(t)] \ .$$

Prove that if

$$T[f(t)] = \sqrt{2/\pi} \int_0^{\infty} \cos xt\, f(t)\, dt \ ,$$

then $f(x) = x^{-\frac{1}{2}} e^{-\frac{1}{2}x^2}$, $\operatorname{sech}(x\sqrt{\tfrac{1}{2}\pi})$, $x^{-\frac{1}{2}} J_{-1/4}(\tfrac{1}{2}x^2)$, are self-reciprocal.

Also, given

$$T[f(t)] = \sqrt{2/\pi} \int_0^\infty \sin xt\, f(t)\, dt \ ,$$

show that $f(x) = x^{-\frac{1}{2}}$, $x\, e^{-\frac{1}{2}x^2}$, $1/[\exp(x\sqrt{2\pi}) - 1] - 1/(x\sqrt{2\pi})$, are self-reciprocal functions.

Show also that $f(x) = x^{-\frac{1}{2}}$, $x^{\nu+\frac{1}{2}} e^{-\frac{1}{2}x^2}$, $x^{\frac{1}{2}} J_{\nu/2}(x^2/2)$, are self-reciprocal functions with respect to the transformation

$$T[f(t)] = \int_0^\infty \sqrt{xt}\, J_\nu(xt)\, f(t)\, dt \ , \quad \nu \geqq -\tfrac{1}{2} \ .$$

[See G. H. Hardy and E. C. Titchmarsh: Self-reciprocal Functions. *Quarterly Journal of Math.* (Oxford series), vol. 1 (1930), pp. 196-231.]

11. Historical Note on Infinite Quadratic Forms. As we have already stated in section 10, chapter 1, the study of infinite quadratic and bilinear forms was inaugurated by D. Hilbert in his now classical *Grundzüge einer allgemeinen Theorie der Linearen Integralgleichungen,* which was first published in the *Göttinger Nachrichten* from 1904 to 1910, and appeared in completed form in 1912. This powerful work immediately stimulated an intensive study of such forms and numerous memoirs were produced, mainly in Germany, in connection with the intimately related theory of integral equations, which by that time was stirring the imagination of the mathematical world.

In 1913 F. Riesz published a comprehensive account of the general theory of systems in infinitely many variables (see *Bibliography*) and this work probably more than any other has served to introduce the theory to mathematicians and to suggest the generalizations which have followed. It well deserves to rank as a classic in this subject. More recently A. Wintner in his *Spektraltheorie der unendlichen Matrizen,* which appeared in 1929, has brought together and correlated the principal contributions and generalizations which the subject has stimulated. His work may be regarded as definitive of the present status of the theory.

To enumerate in any complete way the contributions to the subject of quadratic and bilinear forms in infinitely many variables would be a large task and cannot be undertaken here. However, the reader will find a masterful survey of the literature by E. Hellinger and O. Toeplitz in their *Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten* (see *Bibliography*), published in 1928.

It may be useful, however, to enumerate a few of the classical papers from which the general theory has been largely evolved and a brief bibliography is accordingly appended.

E. Hellinger: Neue Begründung der Theorie quadratischer Formen von unendlichvielen Veränderlichen. *Journal für Math.,* vol. 136 (1909), pp. 210-271.

E. Hellinger and O. Toeplitz: (1) Grundlagen für eine Theorie der unendlichen Matrizen. *Mathematische Annalen*, vol. 69 (1910), pp. 289-330.

(2) Zur Einordnung der Kettenbruchtheorie in die Theorie der quadratischen Formen von unendlichvielen Veränderlichen. *Journal für Math.*, vol. 144 (1914), pp. 212-238.

E. Hilb: (1) Über Integraldarstellungen willkürlicher Funktionen. *Mathematische Annalen*, vol. 66 (1909), pp. 1-66.

(2) Über die Auflösung unendichvieler Linearer Gleichungen mit unendlichvielen Unbekannten. *Mathematische Annalen*, vol. 70 (1911), pp. 79-86.

Anna Johnson Pell: Biorthogonal Systems of Functions. *Trans. of the American Math. Soc.*, vol. 12 (1911), pp. 135-164.

(2) Applications of Biorthogonal Systems of Functions to the Theory of Integral Equations. *Trans. of the American Math. Soc.*, vol. 12 (1911), pp. 165-180.

F. Riesz: Über quadratische Formen von unendlichvielen Veränderlichen. *Göttinger Nachrichten*, (1910), pp. 190-195.

E. Schmidt: Über die Auflösung linearer Gleichungen mit unendlichvielen Unbekannten. *Rendiconti di Palermo*, vol. 25 (1908), pp. 53-77.

I. Schur: (1) Über die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen. *Mathematische Annalen*, vol. 66 (1909), pp. 488-510.

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O. Toeplitz: (1) Die Jacobische Transformation der quadratischen Formen von unendlichvielen Veränderlichen. *Gottinger Nachrichten* (1907), pp. 101-109.

(2) Zur Theorie der quadratischen Formen von unendlichvielen Veränderlichen. *Göttinger Nachrichten* (1910), pp. 489-502.

(3) Zur Theorie der quadratischen und bilinearen Formen von unendlichvielen Veränderlichen. I Teil: Theorie der L-Formen. *Mathematische Annalen*, vol. 70 (1911), pp. 351-376.

12. Generalizations. In the preceding sections of this chapter we have traced the interconnections between the theory of Fredholm integral equations and the theory of quadratic forms in infinitely many variables. As we have already indicated elsewhere, this intriguing dualism has exerted an appeal to numerous mathematicians, who by various generalizations have endeavored to unify the essential parts of the two theories and thus to reduce them to a single general symbolism. In all of this the concept of linear operator has been the dominating idea. The present status of the general problem has been ably epitomized by T. H. Hildebrandt in a paper published in 1931 under the title *Linear Functional Transformations in General Spaces*, from which we quote:*

"An abstract theory of linear functional transformations has as guide linear transformations in a finite or denumerably infinite set of variables, linear integral transformations and equations associated with these. The desire to proceed symbolically and replace details by general procedure seems to be inherent in the situation. Pincherle is perhaps one of the great exponents, so that he even seems to have anticipated some of the famous results of integral equations by a number of years. E. H. Moore set himself the task of unifying the Fred-

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holm theory of integral equations and algebraic equations in finitely and infinitely many variables, and has succeeded in setting up a system which indicates in a host of special cases a valid and elegant method of procedure analogous to the Fredholm integral equation theory. Volterra has devised an elegant theory of linear integral and associated operations based on the notion of permutability or commutativity of operations."

The most comprehensive sources of the general theory as it is contemplated in its abstract form by Hildebrand are the volumes by S. Banach (1932), M. Fréchet (1928), P. Lévy, (1922), F. Riesz (1913), M. H. Stone (1932), A. Wintner (1929), and a series of voluminous memoirs by J. von Neumann, the first of which appeared in 1927. The reader is referred to the *Bibliography* for these contributions. Of somewhat more limited scope, but notable because of its penetration into the problem of the continuous spectra of integral equations, is the volume by T. Carleman (1923) on the theory of singular integral equations with a real and symmetric kernel.

The general problem begins naturally with the definition of the space fundamental to the operators. The postulates for this space have generally assumed commutative and associative addition, and commutative, associative and distributive multiplication. A large amount of freedom has been permitted in the definitions of the *norm* of the space and the concept of *completeness* with respect to this norm. In the main the notion of *convergence in the mean* has been assumed, but a number of different spaces have been defined by variations of this basic concept.

Proceeding from the fundamental space, the various authors have found considerable freedom in their definitions of the operators on the elements of the space. The postulates of *linearity* have been generally assumed and most writers have made use of the Stieltjes-Lebesgue integral as the most general tool for their investigations. We have already indicated the nature of this problem in the brief description of the operational symbol of von Neumann and Stone given in section 4, chapter 2.

The third part of the problem is associated with *transformations* on the space and with general definitions for the *inverse* of a given transformation. The basic thread which runs through all these investigations is that of the matrix calculus. One is upon the threshold of these modern abstract theories when he has mastered such details as we have set forth in the preceding pages of this chapter.

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It will be obvious that the attainment of a reasonably complete bibliography for the calculus of linear operators would be an immense undertaking. It would involve the compilation of several essentially separate bibliographies, one on linear differential equations, another on difference equations, a third on integral equations, a fourth on functionals and general abstract operators. The history of this subject is also extensive in time, its origin being found in the foundations of the integral and differential calculus and its development extending with ever widening horizons down to the present time.

The most adequate bibliography of difference equations is found in N. E. Nörlund's *Differenzenrechnung*, Berlin (1924), where in 68 pages 1427 references, the work of 540 authors, are to be found. The bibliography of Nörlund has been supplemented by more than 300 additional titles listed at the end of an important summary: *Linear q-Difference Equations*, by C. R. Adams, Bulletin of the American Math. Society, vol. 37 (1931), pp. 361-400.

A comprehensive bibliography of functionals is given at the end of the chapters in V. Volterra's *Theory of Functionals*, London and Glasgow (1930), where 539 references are cited. The literature of integral equations prior to 1910 is conveniently reached through H. Bateman's *Report on the History and Present State of the Theory of Integral Equations*, made to the British Association for the Advancement of Science at the Sheffield meeting (1910), pp. 345-424; prior to 1912 through T. Lalesco's *Introduction à la théorie des équations intégrales*, Paris (1912), 152 p.; prior to 1926 through the author's monograph: *The Present Status of Integral Equations*, Indiana University Studies, where 495 titles, the work of 213 authors, are classified; and prior to 1927 through the extensive account of the subject prepared by E. Hellinger and O. Toeplitz: *Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten*, Encyklopädie der mathematischen Wissenschaften, vol. 2, part 3, hefte 13, pp. 1335-1616. In this work the researches of 340 authors are cited.

No adequate bibliography of differential equations has ever been prepared so far as the author is aware. For the development of the subject prior to 1906, the six volume work of A. R. Forsyth: *The Theory of Differential Equations*, Cambridge (1890-1906) gives numerous bibliographical references. Although containing no extensive bibliography, the work of E. L. Ince: *Ordinary Differential Equations*, London (1927), viii + 558 p., refers to the researches of 165 authors. Some idea of the magnitude of the subject may be gained from a casual survey of the 48 pages of condensed bibliography devoted to it

in the *Catalogue of Science Papers* 1800-1900, vol. 1, Cambridge (1908), prepared by the Royal Society of London. The most extensive bibliography of linear differential equations was published in 1909 by L. Schlesinger: *Bericht über die Entwickelung der Theorie der Linearen Differentialgleichungen seit* 1865. Leipzig und Berlin (1909), 133 p. In this bibliography about 1750 titles, the work of 469 authors, are cited.

The present bibliography contains references to the formal theory of operators, differential and integral equations of infinite order, the Heaviside calculus, fractional operators and the modern theory of functionals and abstract operations. No systematic attempt has been made to attain a complete bibliography, but the author hopes that sufficient reference has been made to exhibit the trend of the subject both in its theoretical development and in its application. Considerable help in the assembling of the items which refer to the formal theory of operators has been received from a short study published by E. Stephens in 1925. (See *Bibliography*). Also R. D. Carmichael kindly made accessible to the author an extensive list of titles, which pertained to the theory of differential equations of infinite order.

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